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Quasiconvex functions: how to separate, if you must!

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. Since quasiconvex functions have convex lower level sets it is possible to minimize them by means of separating hyperplanes. An example of such a procedure, well-known for convex functions, is the subgradient method. However, to find the normal vector of a separating hyperplane is in general not easy for the quasiconvex case. This paper attempts to gain some insight into the *computational* aspects of determining such a normal vector and the geometry of lower level sets of quasiconvex functions. In order to do so, the directional differentiability of quasiconvex functions is thoroughly studied. As a consequence of that study, it is shown that an important subset of quasiconvex functions belongs to the class of quasidifferentiable functions. The main emphasis is, however, on computing actual separators. Some important examples are worked out for illustration.

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1. Introduction

The backbone of every successful procedure to minimize a general nonsmooth convex function is *separation.* For example, so-called *subgradient methods* as discussed in [18], refinements of such methods with space dilation yielding the *ellipsoid algorithm*, [9, 22], use the important property of a finite-valued convex function that every nonoptimal point in its domain can be properly separated by an affine functional, or hyperplane, from the nonempty set of points with lower functional value, the so-called *lower level set.* Also, the important class of *bundle methods*, [13], is based on the construction of hyperplanes supporting the epigraph and so these methods can be seen as refinements of the *cutting plane* idea of Kelley, [14]. Since the epigraph of

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a convex function and its lower level sets are convex it is possible to separate both the epigraph and a lower level set from points outside their relative interiors and use the corresponding separating hyperplanes to minimize the function. Moreover, for finite-valued convex functions the normal vectors of both types of hyperplane are determined by elements of the nonempty subgradient set at the corresponding point. To extend the above results to a larger class of functions it is natural to consider quasiconvex functions. These functions, by definition, have convex lower level sets. We first observe that for an important subset of the quasiconvex functions, the so-called *lower subdifferentiable functions*, one can define the concept of a *lower subgradient*, [17]. This lower subgradient at some point satisfies the subgradient inequality on the corresponding lower level set (therefore its name!) and this enables us to apply the cutting plane approach of Kelley, [17]. Since this lower subgradient can be identified by means of a hyperplane separating the point from its convex lower level set it is important to be able to compute such a separating hyperplane. A similar observation holds for all quasiconvex functions and this paper addresses the question how to compute the normal vector of a hyperplane separating the lower level set of a quasiconvex function from any given nonminimal point on its domain. We try to keep the class of quasiconvex functions as general as possible by not assuming lower subdifferentiability. Unfortunately some results are only valid under some additional assumptions. These assumptions cease to hold for quasiconvex functions which are constant in some neighborhood of a nonminimal point. If this happens it seems impossible to compute a normal vector of a separating hyperplane using only local information. However, this does not imply that every algorithm based on the construction of separating hyperplanes will get trapped in such a "bad" point. In a pair of subsequent papers, [6, 8], an adaptation of the ellipsoid method is considered which keeps track of a hyperrectangle containing a minimal point. This hyperrectangle is in general much smaller than the current ellipsoid and can be constructed without increasing the complexity order of the algorithm. This gives the opportunity, in case the center of the current ellipsoid is such a "bad" point, to search this "easy" hyperrectangle in order to either prove optimality of the present point or find another point from where it is possible to proceed. In this paper we also show that every quasiconvex function with a Lipschitz continuous directional derivative is quasidifferentiable, [5]. This result relates these two function classes.

2. Quasiconvex functions

We recall that a function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is called *proper* if the *domain* of f, given by $\text{dom}(f) := \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) < \infty\}$, is nonempty and if $f(\boldsymbol{x}) > -\infty$ for every $\boldsymbol{x} \in \mathbb{R}^n$. Among the set of proper functions we will now concentrate on the so-called evenly quasiconvex functions defined below.

Definition 2.1. A function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is called quasiconvex if the lower level sets $\mathcal{L}_f^{\leq}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \leq \alpha\}$ are convex for every $\alpha \in \mathbb{R}$. The function is called evenly quasiconvex if its lower level sets are all evenly convex. Observe a set is called *evenly convex* if it can be represented by the intersection of open halfspaces.

Observe that every lower semicontinuous quasiconvex function is evenly quasiconvex, since it has closed convex (hence evenly convex) lower level sets. Moreover, it can also be shown (see [15]) that every upper semicontinuous quasiconvex function is evenly quasiconvex Clearly, for f quasiconvex, it is well known that $\mathrm{dom}(f) = \bigcup_{\alpha \in \mathbf{R}} \mathcal{L}_f^{\leq}(\alpha)$ is convex due to $\mathcal{L}_f^{\leq}(\alpha) \subseteq \mathcal{L}_f^{\leq}(\beta)$ for every $\alpha \leq \beta$. The following result lists some well-known equivalent characterizations of quasiconvexity, see [19].

Lemma 2.2. *The following conditions are equivalent.*

1. *The function* $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ *is quasiconvex.*
2. *The sets* $\mathcal{L}_f^{<}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) < \alpha\}$ *are convex for each* $\alpha \in \mathbb{R}$.
3. $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $0 < \lambda < 1$.

In the next section we will consider the special class of proper positively homogeneous evenly quasiconvex functions.

3. On properties of proper positively homogeneous evenly quasiconvex functions

This section mainly derives similar results as those obtained by Crouzeix in [2, 3, 4]. However, while Crouzeix considers proper, positively homogeneous, lower semicontinuous quasiconvex functions we replace lower semicontinuity and quasiconvexity by evenly quasiconvexity. Despite this weaker assumption it is possible to derive similar results by means of easier proofs. Since the main results in this section are a consequence of duality results for quasiconvex functions these simple proofs are possible using a more natural generalization, [16, 7], of the well-known biconjugate or Fenchel-Moreau theorem for convex functions, [12, 21]. It turns out that proper evenly quasiconvex functions originate a more symmetrical representation in the dual space than proper lower semicontinuous quasiconvex functions, [16, 7], and using this more suitable representation one can give simpler proofs. Moreover, since the definition of an evenly quasiconvex function already "includes" a separation result for convex sets it is also possible to give a very simple and easy proof for this dual representation of proper evenly quasiconvex functions. For a proof of the next result the reader should consult Theorem 1.16 and 1.18 of [6]. Observe that $\langle ., . \rangle$ denotes the well known innerproduct.

Lemma 3.1. *Let* $\varphi : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ *be a proper positively homogeneous evenly quasiconvex function satisfying* $\varphi(\mathbf{0}) = 0$. *For every* $\boldsymbol{x} \in \mathbb{R}^n$ *it follows*

$$\varphi(\boldsymbol{x}) = \sup\left\{\psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x} \rangle\Big) : \boldsymbol{x}^\star \in \mathbb{R}^n\right\} \tag{3.1}$$

with

$$\psi(\boldsymbol{x}^\star, r) := \inf\{\varphi(\boldsymbol{y}) : \langle \boldsymbol{x}^\star, \boldsymbol{y} \rangle \geq r,\ \boldsymbol{y} \in \mathbb{R}^n\}. \tag{3.2}$$

Moreover, for every $\boldsymbol{x}^\star \in \mathbb{R}^n$ *the function* $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ *is a nondecreasing positively homogeneous function.*

The above lemma is the alluded dual representation. Using it, the next results provide slight improvements over related results in [4, 2]. Recall that a convex positively homogeneous function is also called *sublinear*, see [12].

Lemma 3.2. *If* $\varphi : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ *is a proper positively homogeneous evenly quasiconvex nonnegative function satisfying* $\varphi(\mathbf{0}) = 0$ *then* φ *is a lower semicontinuous sublinear function with its subgradient set* $\partial\varphi(\mathbf{0})$ *at* $\mathbf{0}$ *nonempty.*

Proof. Since the function φ is assumed to be positively homogeneous it remains to prove that it is lower semicontinuous and convex. By (3.1) it is sufficient to prove that the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$, and hence the function $\boldsymbol{x} \longmapsto \psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle\Big)$, is lower semicontinuous and convex for every $\boldsymbol{x}^\star \in \mathbb{R}^n$ with ψ defined by (3.2) in Lemma 3.1. Clearly it follows by the nonnegativity of the function φ that $0 \leq \psi(\boldsymbol{x}^\star, r)$ for every $(\boldsymbol{x}^\star, r) \in \mathbb{R}^{n+1}$. Also, $\psi(\boldsymbol{x}^\star, 0) = \inf\{\varphi(\boldsymbol{y}) : \langle \boldsymbol{x}^\star, \boldsymbol{y}\rangle \geq 0,\ \boldsymbol{y} \in \mathbb{R}^n\} \leq \varphi(\mathbf{0}) = 0$ and so $\psi(\boldsymbol{x}^\star, 0) = 0$. Hence by the nonnegativity of ψ and the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ is nondecreasing we conclude that $\psi(\boldsymbol{x}^\star, r) = 0$ for every $r \leq 0$. Again using φ is a positively homogeneous function and hence the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ is also positively homogeneous it follows for $r > 0$ that $\psi(\boldsymbol{x}^\star, r) = r\psi(\boldsymbol{x}^\star, 1)$ with $\psi(\boldsymbol{x}^\star, 1) \geq 0$. This shows that the convexity and lower semicontinuity of the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ is established whether $\psi(\boldsymbol{x}^\star, 1)$ is finite or not. To prove the last part we observe, since the function φ is proper lower semicontinuous and sublinear, that by Theorem V.3.1.1 of [12] the function φ is the support function of the closed nonempty convex set $\mathcal{C} := \{\boldsymbol{x}^\star \in \mathbb{R}^n : \langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle \leq \varphi(\boldsymbol{x}) \text{ for every } \boldsymbol{x} \in \mathbb{R}^n\}$. Since $\varphi(\mathbf{0}) = 0$ we have $\mathcal{C} = \partial\varphi(\mathbf{0})$ and the proof is finished. ☐

Another consequence of Lemma 3.1 is given by the following result. Remember K° denotes the well known polar of the cone K given by $K^\circ = \{\boldsymbol{x}^* \in \mathbb{R}^n : \langle \boldsymbol{x}^*, \boldsymbol{x}\rangle \leq 0 \text{ for every } \boldsymbol{x} \in K\}$.

Lemma 3.3. *If* $\varphi : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ *is a proper positively homogeneous evenly quasiconvex function satisfying* $\varphi(\mathbf{0}) = 0$ *and* $\text{dom}(\varphi) \subseteq \text{cl}\left(\mathcal{L}^<_\varphi(0)\right)$ *then* φ *is a lower semicontinuous sublinear function with its subgradient set* $\partial\varphi(\mathbf{0})$ *at* $\mathbf{0}$ *nonempty.*

Proof. To ensure the first part of the result only the convexity and lower semicontinuity of the function φ require a proof. This will once again be based on analyzing the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ for each $\boldsymbol{x}^\star \in \mathbb{R}^n$. We discuss the following mutually exclusive cases for $\boldsymbol{x}^\star$.

1. Let $\boldsymbol{x}^\star$ not belong to $\left(\text{cl}\left(\mathcal{L}^<_\varphi(0)\right)\right)^\circ$. If this holds we can find some $\boldsymbol{x}_0$ satisfying $\varphi(\boldsymbol{x}_0) < 0$ and $\langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle > 0$. Since the function φ is proper the value $\varphi(\boldsymbol{x}_0)$ must be finite. Hence, for every $r > 0$ we obtain by Lemma 3.1 that

$$\begin{aligned}\psi(\boldsymbol{x}^\star, r) &= \langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle^{-1}\psi\Big(\boldsymbol{x}^\star, r\langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle\Big) \\ &\leq \langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle^{-1}\varphi(r\boldsymbol{x}_0) = r\langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle^{-1}\varphi(\boldsymbol{x}_0) < 0\end{aligned}$$

   and so $\lim_{r\uparrow\infty} \psi(\boldsymbol{x}^\star, r) = -\infty$. This yields using $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ is nondecreasing that $\psi(\boldsymbol{x}^\star, r) = -\infty$ for each $r \in \mathbb{R}$ and we obtain by relation (3.1) and the function φ is proper that

$$\varphi(\boldsymbol{x}) = \sup\{\psi(\boldsymbol{x}, \langle \boldsymbol{x}^*, \boldsymbol{x}\rangle) : \boldsymbol{x}^* \in \mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)\right)^\circ\}.$$

2. Let $\boldsymbol{x}^\star$ belong $\left(\mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)\right)^\circ$ and consider $\psi(\boldsymbol{x}^\star, r)$ for $r > 0$. If the vector $\boldsymbol{y} \in \mathbb{R}^n$ satisfies $\langle \boldsymbol{x}^\star, \boldsymbol{y}\rangle \geq r > 0$ then $\boldsymbol{y}$ does not belong to $\mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)$ and so $\boldsymbol{y}$ is not an element of $\mathrm{dom}(\varphi)$. This implies $\varphi(\boldsymbol{y}) = +\infty$ for each $\boldsymbol{y} \in \mathbb{R}^n$ satisfying $\langle \boldsymbol{x}^\star, \boldsymbol{y}\rangle \geq r > 0$ and by relation (3.2) we obtain $\psi(\boldsymbol{x}^\star, r) = +\infty$ for $r > 0$. To analyze $\psi(\boldsymbol{x}^\star, r)$ for $r \leq 0$ we consider the following two mutually exclusive cases.
   (a) There exists an $\boldsymbol{x}_0$ belonging to $\mathcal{L}_\varphi^<(0)$ such that $\langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle = 0$. If this holds we obtain by relation (3.2) for every $\alpha > 0$ that

   $$\psi(\boldsymbol{x}^\star, 0) = \psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle\Big) \leq \varphi(\alpha \boldsymbol{x}_0) = \alpha\varphi(\boldsymbol{x}_0) < 0$$

   and as in part 1 we obtain $\psi(\boldsymbol{x}^\star, 0) = -\infty$.This shows $\psi(\boldsymbol{x}^\star, r) = -\infty$ for every $r \leq 0$.
   (b) For every $\boldsymbol{x}$ belonging to $\mathcal{L}_\varphi^<(0)$ it follows that $\langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle < 0$. To compute $\psi(\boldsymbol{x}^\star, 0)$ we first observe for each $\boldsymbol{y}$ satisfying $\langle \boldsymbol{x}^\star, \boldsymbol{y}\rangle \geq 0$ that by our assumption the vector $\boldsymbol{y}$ does not belong to $\mathcal{L}_\varphi^<(0)$ and so $\varphi(\boldsymbol{y}) \geq 0$. Since $\mathbf{0}$ is one of those elements $\boldsymbol{y}$ and $\varphi(\mathbf{0}) = 0$ it follows from (3.2) that $\psi(\boldsymbol{x}^\star, 0) = 0$. Clearly, Lemma 3.1 yields for $r < 0$ that

   $$\psi(\boldsymbol{x}^\star, r) = -r\psi(\boldsymbol{x}^\star, -1)$$

   with $-\infty \leq \psi(\boldsymbol{x}^\star, -1) < 0$.

To finish the proof it follows by the above analysis that we must only concentrate on part 2b and verify that there exists some $\boldsymbol{x}^\star$ belonging to $\left(\mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)\right)^\circ$ satisfying $\psi(\boldsymbol{x}^\star, -1) > -\infty$. If such an $\boldsymbol{x}^\star$ does not exists then $\varphi(\boldsymbol{x}) = -\infty$ for every $\boldsymbol{x} \in \mathcal{L}_\varphi^<(0)$ and this contradicts that the function φ is proper. Hence, to represent the function φ as in relation (3.1) it is enough to consider elements of the set

$$\mathcal{S} := \left\{\boldsymbol{x}^\star \in \left(\mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)\right)^\circ : -\infty < \psi(\boldsymbol{x}^\star, -1) < +\infty\right\}$$

and we have verified that

$$\varphi(\boldsymbol{x}) = \sup\left\{\psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle\Big) : \boldsymbol{x}^\star \in \mathcal{S}\right\}. \tag{3.3}$$

Since for every $\boldsymbol{x}^\star \in \mathcal{S}$ the function $\boldsymbol{x} \longmapsto \psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle\Big)$ is convex and lower semicontinuous this shows by relation (3.3) that the function φ is lower semicontinuous and convex. The last part follows from similar arguments as used in the proof of Lemma 3.2. □

Observe for φ convex with $\varphi(\mathbf{0}) = 0$ that $\varphi(\boldsymbol{x}) \geq 0$ for every $\boldsymbol{x} \in \mathbb{R}^n$ is equivalent to $\mathbf{0} \in \partial\varphi(\mathbf{0})$. So, for φ satisfying the conditions of Lemma 3.2 we have $\mathbf{0} \in \partial\varphi(\mathbf{0})$ while for φ satisfying the conditions of Lemma 3.3 we have $\mathbf{0} \notin \partial\varphi(\mathbf{0})$.

An immediate consequence of the previous two lemmas is the following theorem, which improves a related result in [4, 2]. Before discussing this theorem we introduce for any function φ the related functions φ_- and φ_+ given by

$$\varphi_-(\boldsymbol{x}) := \begin{cases} \varphi(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \operatorname{cl}\left(\mathcal{L}_\varphi^<(0)\right) \\ +\infty & \text{otherwise} \end{cases} \tag{3.4}$$

and

$$\varphi_+(\boldsymbol{x}) := \begin{cases} 0 & \text{if } \boldsymbol{x} \in \operatorname{cl}\left(\mathcal{L}_\varphi^<(0)\right) \\ \varphi(\boldsymbol{x}) & \text{otherwise.} \end{cases} \tag{3.5}$$

Theorem 3.4. *Every proper, positively homogeneous evenly quasiconvex function φ satisfying $\varphi(\mathbf{0}) = 0$ is lower semicontinuous and is the minimum of two lower semicontinuous sublinear functions φ_- and φ_+.*

Proof. If $\varphi(\boldsymbol{x}) \geq 0$ for every $\boldsymbol{x}$ or equivalently $\mathcal{L}_\varphi^<(0)$ is empty we obtain by relation (3.4) and (3.5) that $\varphi_-(\boldsymbol{x}) = +\infty$ and $\varphi_+(\boldsymbol{x}) = \varphi(\boldsymbol{x})$ for every $\boldsymbol{x}$ and the desired result follows by Lemma 3.2. If $\mathcal{L}_\varphi^<(0)$ is nonempty it follows using φ is a proper positively homogeneous evenly quasiconvex function and $\operatorname{cl}\left(\mathcal{L}_\varphi^<(0)\right)$ a nonempty closed convex cone (hence evenly convex) that φ_+ satisfies the conditions of Lemma 3.2 and φ_- the conditions of Lemma 3.3. Hence the functions φ_- and φ_+ are lower semicontinuous and sublinear. This also implies by relation (3.4) that $\varphi_-(\boldsymbol{x}) \leq 0$ for every $\boldsymbol{x} \in \operatorname{cl}\left(\mathcal{L}_\varphi^<(0)\right)$ and by relations (3.4) and (3.5) we obtain $\varphi(\boldsymbol{x}) = \min\{\varphi_-(\boldsymbol{x}), \varphi_+(\boldsymbol{x})\}$ showing the desired result. □

By Theorem 3.4 every proper evenly quasiconvex positively homogeneous function which is finite at $\mathbf{0}$ must be lower semicontinuous. This is a rather remarkable result which does not hold in general for evenly quasiconvex functions. As an example we mention the evenly quasiconvex function

$$\operatorname{sign}(x) := \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

which is neither lower nor upper semicontinuous at 0.

If φ is a finite positively homogeneous evenly quasiconvex function one can show, under some additional condition, that φ is continuous on $\mathbb{R}^n$. To establish this result we need the following lemma.

Lemma 3.5. *If the function φ is proper positively homogeneous and evenly quasiconvex and its lower level set $\mathcal{L}_\varphi^<(0)$ is nonempty then the following conditions are equivalent.*

1. $\operatorname{rbd}(\mathcal{L}_\varphi^<(0)) \subseteq \mathcal{L}_\varphi^=(0)$.
2. $\mathcal{L}_\varphi^<(0)$ *is relatively open.*

Proof. To verify $1 \Rightarrow 2$ it is sufficient to prove that $\mathcal{L}_\varphi^<(0) \subseteq \operatorname{ri}(\mathcal{L}_\varphi^<(0))$. Let $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0) \subseteq \operatorname{cl}(\mathcal{L}_\varphi^<(0))$ and suppose that $\boldsymbol{d}_0$ does not belong to $\operatorname{ri}(\mathcal{L}_\varphi^<(0))$. Then $\boldsymbol{d}_0 \in \operatorname{rbd}(\mathcal{L}_\varphi^<(0))$ and hence by 1 we obtain that $\varphi(\boldsymbol{d}_0) = 0$. This contradicts $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$ and we have shown that $\mathcal{L}_\varphi^<(0) \subseteq \operatorname{ri}(\mathcal{L}_\varphi^<(0))$. To prove $2 \Rightarrow 1$ we observe for $\boldsymbol{d} \in \operatorname{rbd}(\mathcal{L}_\varphi^<(0))$

that $\boldsymbol{d}$ does not belong to $\mathrm{ri}(\mathcal{L}_{\varphi}^{<}(0)) = \mathcal{L}_{\varphi}^{<}(0)$. Hence, $\varphi(\boldsymbol{d}) \geq 0$ and since $\varphi_+(\boldsymbol{d}) = 0$ for every $\boldsymbol{d} \in \mathrm{cl}(\mathcal{L}_{\varphi}^{<}(0))$ with φ_+ defined as in relation (3.5), it follows by Theorem 3.4 that $0 \leq \varphi(\boldsymbol{d}) = \min\{\varphi_-(\boldsymbol{d}), \varphi_+(\boldsymbol{d})\} \leq 0$ or equivalently $\varphi(\boldsymbol{d}) = 0$. □

In the next result we show that under some additional condition a finite positively homogeneous evenly quasiconvex functions is actually continuous.

Lemma 3.6. *If the function φ is a finite positively homogeneous evenly quasiconvex function and the set $\mathcal{L}_{\varphi}^{<}(0)$ is relatively open then the function φ is continuous on $\mathbb{R}^n$.*

Proof. If $\mathcal{L}_{\varphi}^{<}(0)$ is empty then by Lemma 3.2 we obtain that $\varphi(\boldsymbol{d}) = \varphi_+(\boldsymbol{d})$ for every $\boldsymbol{d} \in \mathbb{R}^n$. Since $\mathrm{dom}(\varphi_+) = \mathbb{R}^n$ and by Lemma 3.2 the function φ_+ is convex it follows by Corollary 10.1.1 of [21] that φ is continuous. If $\mathcal{L}_{\varphi}^{<}(0)$ is nonempty then by Theorem 10.1 of [21], Lemma 3.3, Lemma 3.2 and Theorem 3.4 it is sufficient to prove that φ is upper semicontinuous on $\mathrm{rbd}(\mathcal{L}_{\varphi}^{<}(0))$. Since by assumption the set $\mathcal{L}_{\varphi}^{<}(0)$ is relatively open we obtain by Lemma 3.5 that $\varphi(\boldsymbol{d}) = 0$ for every $\boldsymbol{d} \in \mathrm{rbd}(\mathcal{L}_{\varphi}^{<}(0))$. Suppose now by contradiction that $\limsup_{\boldsymbol{d} \to \boldsymbol{d}_0} \varphi(\boldsymbol{d}) > \varphi(\boldsymbol{d}_0) = 0$ for some $\boldsymbol{d}_0 \in \mathrm{rbd}(\mathcal{L}_{\varphi}^{<}(0))$. Hence, there exists a sequence $\{\boldsymbol{d}_k : k \geq 1\}$ with $\lim_{k\uparrow\infty} \boldsymbol{d}_k = \boldsymbol{d}_0$ such that $\lim_{k\uparrow\infty} \varphi(\boldsymbol{d}_k) > 0$. Since by Lemma 3.4 the function φ_+ is sublinear and finite it follows as in the first part of this proof that φ_+ is continuous and we obtain by Theorem 3.4 that $\varphi_+(\boldsymbol{d}_0) = \lim_{k\uparrow\infty} \varphi_+(\boldsymbol{d}_k) \geq \lim_{k\uparrow\infty} \varphi(\boldsymbol{d}_k) > 0$. This implies using the definition of φ_+ in relation (3.5) that $\boldsymbol{d}_0 \notin \mathrm{cl}(\mathcal{L}_{\varphi}^{<}(0))$ contradicting $\boldsymbol{d}_0 \in \mathrm{rbd}(\mathcal{L}_{\varphi}^{<}(0))$. Therefore the function φ must be upper semicontinuous for every $\boldsymbol{d} \in \mathrm{rbd}(\mathcal{L}_{\varphi}^{<}(0))$ and this proves the desired result. □

It is now immediately clear for φ continuous on $\mathbb{R}^n$ that the set $\mathcal{L}_{\varphi}^{<}(0)$, if not empty, has full dimension n and so $\mathrm{ri}(\mathcal{L}_{\varphi}^{<}(0)) = \mathrm{int}(\mathcal{L}_{\varphi}^{<}(0)) = \mathcal{L}_{\varphi}^{<}(0)$.

The properties of the above special class of positively homogeneous evenly quasiconvex functions will be useful to study the local properties of more general quasiconvex functions. A way to do this is to look at directional derivatives of quasiconvex functions as functions of the direction. This will be discussed in the next section.

4. Directional derivatives of quasiconvex functions

Unlike convex functions, quasiconvex functions do not always have directional derivatives. An important generalization of directional derivatives is given by the *Dini upper derivative* of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{d}$. This generalization coincides with the definition of Dini upper derivative used within the theory of quasidifferentiable functions if f is locally Lipschitz around $\boldsymbol{x}_0$, see [5].

Definition 4.1. If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is some function with $f(\boldsymbol{x}_0)$ finite the Dini upper derivative of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{d}$ is given by

$$f'_+(\boldsymbol{x}_0; \boldsymbol{d}) := \limsup_{t \downarrow 0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{d}) - f(\boldsymbol{x}_0)}{t}.$$

We observe by the definition of lim sup that $f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ always exists, i.e. $-\infty \leq f'_+(\boldsymbol{x}_0;\boldsymbol{d}) \leq +\infty$, for any function $f:\mathbb{R}^n \longrightarrow [-\infty,+\infty]$ with $f(\boldsymbol{x}_0)$ finite. Moreover, it is easy to verify that $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ is positively homogeneous and that $f'_+(\boldsymbol{x}_0;\mathbf{0}) = \mathbf{0}$. If we know additionally that f is quasiconvex then the next result is easy to prove using Lemma 2.2, see [2].

Lemma 4.2. *Let $f:\mathbb{R}^n \longrightarrow [-\infty,+\infty]$ be a quasiconvex function with $f(\boldsymbol{x}_0)$ finite. Then the function $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ is positively homogeneous, quasiconvex and $f'_+(\boldsymbol{x}_0;\mathbf{0}) = \mathbf{0}$.*

In the remainder of this paper we will always assume that $f:\mathbb{R}^n \longrightarrow [-\infty,+\infty]$ is a quasiconvex function with $f(\boldsymbol{x}_0)$ finite and $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ is a proper evenly quasiconvex function. Introducing the function $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ we observe by Lemma 4.1 that this function satisfies the properties of the functions studied in Section 3. Although this function depends on $\boldsymbol{x}_0$, whenever no risk of confusion exists we do not refer to it for the sake of notation convenience.

Lemma 4.3. *If the function f is quasiconvex and finite at $\boldsymbol{x}_0$ and the function φ given by $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ is a finite evenly quasiconvex function then the function φ is continuous on $\mathbb{R}^n$.*

Proof. The function φ is positively homogeneous and satisfies $\varphi(\mathbf{0}) = 0$. Applying now Lemma 3.6 it is sufficient to verify that $\mathcal{L}^<_\varphi(0)$ is relatively open. By definition this holds for $\mathcal{L}^<_\varphi(0)$ empty. Hence assume that $\mathcal{L}^<_\varphi(0)$ nonempty and let $\boldsymbol{d} \in \mathcal{L}^<_\varphi(0)$. This implies that there exists some $t_0 > 0$ satisfying $\boldsymbol{x}_0 + t\boldsymbol{d} \in \mathrm{ri}(\mathcal{L}^<_f(f(\boldsymbol{x}_0))$. Hence by Theorem 6.8.2 of [21] we obtain that $\boldsymbol{d} \in t_0^{-1}(\mathrm{ri}(\mathcal{L}^<_f(f(\boldsymbol{x}_0))) - \boldsymbol{x}_0) \subseteq \mathrm{ri}(\mathrm{cone}(\mathcal{L}^<_f(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ and we obtain by Lemma 4.4 that $\boldsymbol{d} \in \mathrm{ri}(\mathcal{L}^<_\varphi(0))$. This shows the result. □

Notice that Crouzeix in [2, 3, 4] observed that φ might not be lower semicontinuous even if φ is a finite, positively homogeneous and quasiconvex function. Finally, if f is quasiconvex and additionally locally Lipschitz around $\boldsymbol{x}_0$, (see [1] fot the definition of locally Lipschitz), it is easy to show by a direct proof that φ is Lipschitz continuous (and hence continuous) on $\mathbb{R}^n$.

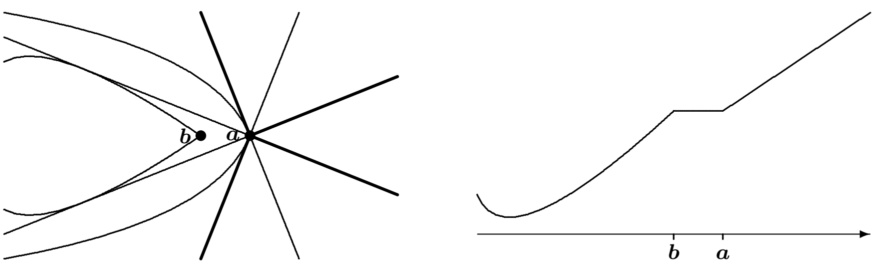

FIGURE 1. Interpretation of the partial description

As already pointed out in the introduction, it is crucial for many optimization methods to be able to compute an element of the normal cone of $\mathcal{L}^<_f(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$.

It is essential to consider the strict lower level set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ since, unlike for convex functions, a nonminimal point may be in the interior of its lower level set. In order to see that take any point in the segment connecting $\boldsymbol{a}$ and $\boldsymbol{b}$ in Figure 1. The first picture is drawn in the domain and shows two lower level sets. The one with a dashed boundary is $\mathcal{L}_f^<(f(\boldsymbol{b}))$ and the one with a full boundary is $\mathcal{L}_f^{\leq}(f(\boldsymbol{a})) = \mathcal{L}_f^{\leq}(f(\boldsymbol{b}))$. The second picture is drawn in the epigraph space and corresponds to slicing the graph of the function along the line going through $\boldsymbol{a}$ and $\boldsymbol{b}$.

Clearly, in order to seek a vector normal to $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ we must know that the set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty. A sufficient condition to ensure that this strict lower level set is nonempty is the nonemptiness of the set of *strict descent directions* at $\boldsymbol{x}_0$ defined as $\mathcal{L}_\varphi^<(0) := \{\boldsymbol{d} \in \mathbb{R}^n : f'_+(\boldsymbol{x}_0;\boldsymbol{d}) < 0\}$. Unfortunately, in the case of quasiconvex functions, contrary to convex functions (see [12]), the nonemptiness of the set of strict descent directions is not necessary as shown by $f(x) = x^3$ at 0.

This function is differentiable at 0 and its derivative at this point equals 0. Therefore $f'(0;d) = 0$ for every $d \in \mathbb{R}$, while $\mathcal{L}_f^<(0) = (-\infty, 0)$ is clearly nonempty.

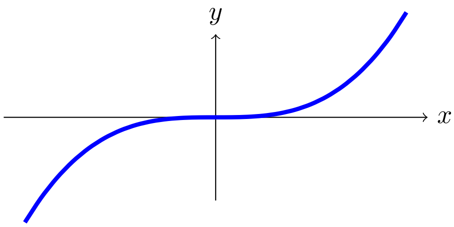

FIGURE 2. A simple but "nasty" quasiconvex function: x^3

For quasiconvex functions a necessary condition is given by the nonemptiness of the set $\mathcal{L}_\varphi^{\leq}(0) \setminus \{\boldsymbol{0}\}$ with $\mathcal{L}_\varphi^{\leq}(0) := \{\boldsymbol{d} \in \mathbb{R}^n : f'_+(\boldsymbol{x}_0;\boldsymbol{d}) \leq 0\}$ the set of *descent directions*. It turns out, see Section 4.2 ahead, that the function φ completely characterizes the normal cone of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ if $\mathcal{L}_\varphi^<(0)$ is nonempty. For $\mathcal{L}_\varphi^<(0)$ empty we also need global information to find out whether $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty or not and so the local information given by φ is insufficient even to decide whether $\boldsymbol{x}_0$ minimizes f or not.

To discuss the case with $\mathcal{L}_\varphi^<(0)$ nonempty we first observe that $\mathcal{L}_\varphi^<(0)$ is a convex subset of the nonempty convex cone $\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)$. For $\mathcal{L}_\varphi^<(0)$ nonempty it is shown in the next result that $\operatorname{ri}(\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ equals $\operatorname{ri}(\mathcal{L}_\varphi^<(0))$. The same result is proven by Crouzeix in [3] but for completeness we list a more detailed proof.

Lemma 4.4. *If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function with $f(\boldsymbol{x}_0)$ finite and $\mathcal{L}_\varphi^<(0)$ nonempty then $\operatorname{ri}(\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ equals $\operatorname{ri}(\mathcal{L}_\varphi^<(0))$.*

Proof. Since $\mathcal{L}_\varphi^<(0) \subseteq \operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)$ and $\mathcal{L}_\varphi^<(0)$ is nonempty it is sufficient by Theorem 6.3.1 of [21] to verify that $\operatorname{ri}(\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)) \subseteq \mathcal{L}_\varphi^<(0)$. Consider now some $\boldsymbol{d}_0 \in \operatorname{ri}(\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ and let

$$\boldsymbol{d} \in \mathcal{L}_\varphi^<(0) \subseteq \operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0).$$

By Theorem 6.4 of [21] there exists some $\mu < 0$ such that

$$\boldsymbol{d}_\mu := \boldsymbol{d}_0 + \mu(\boldsymbol{d} - \boldsymbol{d}_0) \in \mathrm{ri}(\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$$

and so $\boldsymbol{d}_0 = \frac{1}{1-\mu}\boldsymbol{d}_\mu - \frac{\mu}{1-\mu}\boldsymbol{d}$. Moreover, since $\boldsymbol{d}_\mu \in \mathrm{ri}(\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ and by Theorem 6.8.1 of [21] it follows that $\mathbf{0}$ does not belong to $\mathrm{ri}(\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ we can find some $t_0 > 0$ satisfying $f(\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu) < f(\boldsymbol{x}_0)$. Construct now for each $t > 0$ the line $\mathcal{L}_t$ going through $\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu$ and $\boldsymbol{m}_t := \boldsymbol{x}_0 + t\boldsymbol{d}_0$ and crossing $\boldsymbol{x}_0 + \alpha\boldsymbol{d}$ in $\boldsymbol{n}_t := \boldsymbol{x}_0 + \xi_t\boldsymbol{d}$, see Figure 3. By the quasiconvexity of f it follows that

$$f(\boldsymbol{m}_t) = f(\boldsymbol{x}_0 + t\boldsymbol{d}_0) \le \max\{f(\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu), f(\boldsymbol{x}_0 + \xi_t\boldsymbol{d})\}. \tag{4.1}$$

To compute ξ_t we intersect the line $\mathcal{L}_t$ with the line $\boldsymbol{x}_0 + \alpha\boldsymbol{d}$. After some computations we obtain

$$\xi_t = \frac{-\mu t t_0}{(1-\mu)t_0 - t}.$$

Substituting this into (4.1) yields

$$\frac{f(\boldsymbol{x}_0 + t\boldsymbol{d}_0) - f(\boldsymbol{x}_0)}{t} \le \max\left\{\frac{f(\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu) - f(\boldsymbol{x}_0)}{t}, \frac{f\left(\boldsymbol{x}_0 + \frac{-\mu t t_0}{(1-\mu)t_0 - t}\boldsymbol{d}\right) - f(\boldsymbol{x}_0)}{t}\right\}$$

and due to $\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu \in \mathcal{L}_f^<(f(\boldsymbol{x}_0))$ and $\lim_{t\downarrow 0}\frac{\xi_t}{t} = \frac{-\mu}{1-\mu} > 0$ we obtain

$$f'_+(\boldsymbol{x}_0; \boldsymbol{d}_0) \le \frac{-\mu}{1-\mu} f'_+(\boldsymbol{x}_0; \boldsymbol{d}) < 0.$$

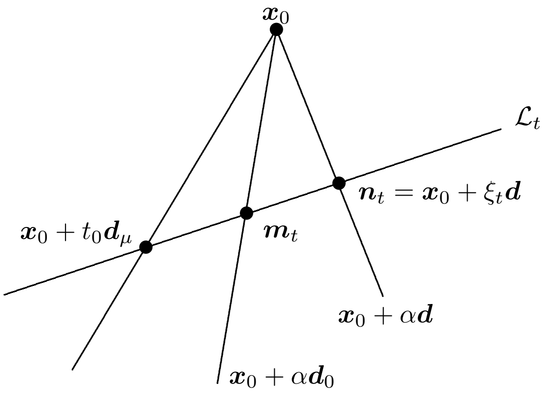

FIGURE 3. Construction of $\mathcal{L}_t$

Hence, $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$ and the desired result is proven. □

The previous result will play an important role in the sequel. Its main importance is to show that if the set $\mathcal{L}_\varphi^<(0)$ is nonempty then this set is indistinguishable by polarity from the set $\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)$.

4.1. Quasidifferentiability of quasiconvex functions

This section shows the important result that a quasiconvex function with a Lipschitz continuous directional derivative at $\boldsymbol{x}_0$ is quasidifferentiable at $\boldsymbol{x}_0$.

Definition 4.5 ([5]). If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is some function with $f(\boldsymbol{x}_0)$ finite the directional derivative of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{d}$ is given by

$$f'(\boldsymbol{x}_0; \boldsymbol{d}) := \lim_{t \downarrow 0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{d}) - f(\boldsymbol{x}_0)}{t}.$$

Moreover, f is said to be quasidifferentiable at $\boldsymbol{x}_0$ if $\boldsymbol{d} \longmapsto f'(\boldsymbol{x}_0; \boldsymbol{d})$ exists for every $\boldsymbol{d} \in \mathbb{R}^n$ and

$$f'(\boldsymbol{x}_0; \boldsymbol{d}) = \max_{\boldsymbol{y} \in \underline{\partial} f(\boldsymbol{x}_0)} \langle \boldsymbol{d}, \boldsymbol{y} \rangle + \min_{\boldsymbol{y} \in \overline{\partial} f(\boldsymbol{x}_0)} \langle \boldsymbol{d}, \boldsymbol{y} \rangle$$

with $\underline{\partial} f(\boldsymbol{x}_0)$, resp. $\overline{\partial} f(\boldsymbol{x}_0)$, compact convex subsets of $\mathbb{R}^n$. The sets $\underline{\partial} f(\boldsymbol{x}_0)$ and $\overline{\partial} f(\boldsymbol{x}_0)$ are called respectively the subdifferential and the superdifferential of f at $\boldsymbol{x}_0$ being $\mathcal{D} f(\boldsymbol{x}) := [\underline{\partial} f(\boldsymbol{x}_0), \overline{\partial} f(\boldsymbol{x}_0)] \subseteq \mathbb{R}^{2n}$ the quasidifferential of f at $\boldsymbol{x}_0$.

Observe that whenever $f'(\boldsymbol{x}_0; \boldsymbol{d})$ exists it equals $f'_+(\boldsymbol{x}_0; \boldsymbol{d})$. It is well-know that every finite convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasidifferentiable at every $\boldsymbol{x} \in \mathbb{R}^n$ with $\mathcal{D} f(\boldsymbol{x}) := [\partial f(\boldsymbol{x}), \{\mathbf{0}\}]$ and $\partial f(\boldsymbol{x})$ the nonempty subgradient set of f at $\boldsymbol{x}$. Moreover, it can be easily shown, [5], that the function $\boldsymbol{x} \longmapsto \min\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}$ is quasidifferentiable at $\boldsymbol{x}_0$ if f_i, $i = 1, 2$, is quasidifferentiable at $\boldsymbol{x}_0$. In general, the set of quasidifferentiable functions at $\boldsymbol{x}_0$ is a linear space closed with respect to all algebraic operations and, more importantly, to the operations of taking maxima and minima. Also for f quasidifferentiable at $\boldsymbol{x}_0$ it is easy to verify that $\boldsymbol{d} \longmapsto f'(\boldsymbol{x}_0; \boldsymbol{d})$ is Lipschitz continuous. To relate the previous results for quasiconvex functions to the above class of functions we observe for $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ quasiconvex, $\mathcal{L}^<_\varphi(0)$ empty with $\varphi(\boldsymbol{d}) := f'(\boldsymbol{x}_0; \boldsymbol{d})$ a finite continuous function that by Theorem (3.4) and Lemma 3.2 the function f is quasidifferentiable at $\boldsymbol{x}_0$ with $\mathcal{D} f(\boldsymbol{x}_0) = [\partial \varphi_+(\mathbf{0}), \{\mathbf{0}\}]$. If this holds the Lipschitz continuity of φ follows by the finiteness of φ_+. However, if $\mathcal{L}^<_\varphi(0)$ is nonempty we have to assume that φ is Lipschitz continuous (with Lipschitz constant $L > 0$) and by Theorem 3.4 this implies $\varphi(\boldsymbol{d}) = \min\{\varphi_-(\boldsymbol{d}), \varphi_+(\boldsymbol{d})\}$. Applying Lemma 3.3 and Lemma 3.2 we know that φ_+ is a finite positively homogeneous convex function and φ_- a proper lower semicontinuous positively homogeneous convex function. Hence to prove that f is quasidifferentiable at $\boldsymbol{x}_0$ it is sufficient to replace φ_- by a finite positively homogeneous convex function without destroying Theorem 3.4. Clearly for φ Lipschitz continuous it follows that $\mathcal{L}^<_\varphi(0)$ is open and hence $\operatorname{int}(\operatorname{dom}(\varphi_-)) = \mathcal{L}^<_\varphi(0)$. This implies by Theorem 23.4 of [21] that $\partial \varphi_-(\boldsymbol{d})$ is a nonempty compact convex set for every $\boldsymbol{d} \in \mathcal{D}^<_f(\boldsymbol{x}_0)$ and since φ is Lipschitz continuous with Lipschitz constant L it is easy to show by the subgradient inequality applied to φ_- that $\partial \varphi_-(\boldsymbol{d}) \subseteq L\mathcal{B}$ for every $\boldsymbol{d} \in \mathcal{L}^<_\varphi(0)$ with $\mathcal{B} := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq 1\}$ the closed unit Euclidean ball. On the other hand, since φ_- is positively homogeneous it follows that $\partial \varphi_-(\lambda \boldsymbol{d}) = \partial \varphi_-(\boldsymbol{d})$ for every $\lambda > 0$ and $\boldsymbol{d} \in \mathcal{L}^<_\varphi(0)$ and so by the previous observations one can pick for every $\boldsymbol{d}_0 \in \mathcal{L}^<_\varphi(0)$ and corresponding ray $\{\lambda \boldsymbol{d}_0 : \lambda > 0\} \subseteq \mathcal{L}^<_\varphi(0)$ a subgradient $\boldsymbol{\xi}(\boldsymbol{d}_0) \in \partial \varphi_-(\lambda \boldsymbol{d}_0)$, $\lambda > 0$, satisfying

$\|\boldsymbol{\xi}(\boldsymbol{d}_0)\| \leq L$. Consider now the function $\hat{\varphi}_- : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ given by

$$\hat{\varphi}_-(\boldsymbol{d}) := \sup_{\boldsymbol{d}_0 \in \mathcal{L}^{<}_{\varphi}(0)} \varphi_-(\boldsymbol{d}_0) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} - \boldsymbol{d}_0 \rangle. \tag{4.2}$$

For this function the following result holds.

Lemma 4.6. *If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function with $f(\boldsymbol{x}_0)$ finite, φ is Lipschitz continuous with $\varphi(\boldsymbol{d}) := f'(\boldsymbol{x}_0; \boldsymbol{d})$ and $\mathcal{L}^{<}_{\varphi}(0)$ is nonempty then the function $\hat{\varphi}_-$ given by (4.2) is a finite, positively homogeneous and convex function. Moreover, $\hat{\varphi}_-(\boldsymbol{d})$ equals $\varphi(\boldsymbol{d})$ for every $\boldsymbol{d} \in \mathrm{cl}(\mathcal{L}^{<}_{\varphi}(0))$ and $\hat{\varphi}_-(\boldsymbol{d}) > 0$ for every $\boldsymbol{d} \notin \mathrm{cl}(\mathcal{L}^{<}_{\varphi}(0))$.*

Proof. Clearly $\hat{\varphi}_-$ is a convex function. Since $\|\boldsymbol{\xi}(\boldsymbol{d})\| \leq L$ and $\boldsymbol{\xi}(\boldsymbol{d}) \in \partial\varphi_-(\boldsymbol{d})$ for every $\boldsymbol{d} \in \mathcal{L}^{<}_{\varphi}(0)$ we obtain for $\boldsymbol{d}_0 \in \mathcal{L}^{<}_{\varphi}(0)$ fixed and any $\boldsymbol{d} \in \mathbb{R}^n$ that

$$\varphi_-(\boldsymbol{d}_0) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} - \boldsymbol{d}_0 \rangle \leq \varphi_-(\mathbf{0}) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} \rangle \leq L\|\boldsymbol{d}\|$$

and so by (4.2) the function $\hat{\varphi}_-$ is finite. To prove that $\hat{\varphi}_-$ is positively homogeneous we first observe using $\boldsymbol{\xi}(\boldsymbol{d}_0) \in \partial\varphi_-(\lambda\boldsymbol{d}_0)$, $\lambda > 0$, for every $\boldsymbol{d}_0 \in \mathcal{L}^{<}_{\varphi}(0)$ that

$$\begin{aligned} \varphi_-(\boldsymbol{d}_0) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \lambda\boldsymbol{d} - \boldsymbol{d}_0 \rangle &= \lambda\varphi_-\left(\lambda^{-1}\boldsymbol{d}_0\right) + \lambda\left\langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} - \lambda^{-1}\boldsymbol{d}_0 \right\rangle \\ &= \lambda\varphi_-\left(\lambda^{-1}\boldsymbol{d}_0\right) + \lambda\left\langle \boldsymbol{\xi}\left(\lambda^{-1}\boldsymbol{d}_0\right), \boldsymbol{d} - \lambda^{-1}\boldsymbol{d}_0 \right\rangle \\ &\leq \lambda\hat{\varphi}_-(\boldsymbol{d}) \end{aligned}$$

for every $\boldsymbol{d} \in \mathbb{R}^n$. This yields by the definition of $\hat{\varphi}_-$ that $\hat{\varphi}_-(\lambda\boldsymbol{d}) \leq \lambda\hat{\varphi}_-(\boldsymbol{d})$ for every $\lambda > 0$ and hence $\hat{\varphi}_-(\lambda\boldsymbol{d}) \leq \lambda\hat{\varphi}_-(\boldsymbol{d}) = \lambda\hat{\varphi}_-\left(\lambda^{-1}\lambda\boldsymbol{d}\right) \leq \hat{\varphi}_-(\lambda\boldsymbol{d})$ implying $\hat{\varphi}_-$ is positively homogeneous. Also for every $\boldsymbol{d}_0 \in \mathcal{L}^{<}_{\varphi}(0)$ it follows that $\varphi_-(\boldsymbol{d}) \geq \varphi_-(\boldsymbol{d}_0) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} - \boldsymbol{d}_0 \rangle$ and so $\varphi_-(\boldsymbol{d}) \geq \hat{\varphi}_-(\boldsymbol{d})$. If $\boldsymbol{d} \in \mathcal{L}^{<}_{\varphi}(0)$ we obtain by (4.2) that $\hat{\varphi}_-(\boldsymbol{d}) \geq \varphi_-(\boldsymbol{d})$ and this yields that $\hat{\varphi}_-$ equals φ_- on $\mathcal{L}^{<}_{\varphi}(0)$. By the lower semicontinuity of φ_- and the continuity of $\hat{\varphi}_-$ the functions are equal on $\mathrm{cl}(\mathcal{L}^{<}_{\varphi}(0))$. Finally, assume that $\hat{\varphi}_-(\boldsymbol{d}_1) \leq 0$ for some $\boldsymbol{d}_1 \notin \mathrm{cl}(\mathcal{L}^{<}_{\varphi}(0))$ and consider a fixed $\boldsymbol{d}_0 \in \mathcal{L}^{<}_{\varphi}(0)$. Since $\mathcal{L}^{<}_{\varphi}(0)$ is open there exists some $0 < \mu < 1$ such that $\boldsymbol{d}_\mu := \mu\boldsymbol{d}_0 + (1-\mu)\boldsymbol{d}_1 \in \mathrm{rbd}(\mathcal{L}^{<}_{\varphi}(0))$. This implies by the convexity of $\hat{\varphi}_-$ that $\hat{\varphi}_-(\boldsymbol{d}_\mu) \leq \mu\hat{\varphi}_-(\boldsymbol{d}_0) + (1-\mu)\hat{\varphi}_-(\boldsymbol{d}_1) < 0$, and so $\varphi_-(\boldsymbol{d}_\mu) = \hat{\varphi}_-(\boldsymbol{d}_\mu) < 0$ contradicting Lemma 3.5. This yields $\hat{\varphi}_-(\boldsymbol{d}) > 0$ for every $\boldsymbol{d} \notin \mathrm{cl}(\mathcal{L}^{<}_{\varphi}(0))$ and the proof of the result is finished. □

Introduce now the function $\tilde{\varphi}_- : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\tilde{\varphi}_-(\boldsymbol{d}) := \varphi_+(\boldsymbol{d}) + \hat{\varphi}_-(\boldsymbol{d}). \tag{4.3}$$

Using this function one can show the following result.

Theorem 4.7. *If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function with $f(\boldsymbol{x}_0)$ finite and φ Lipschitz continuous then f is quasidifferentiable at $\boldsymbol{x}_0$.*

Proof. The result is already verified for $\mathcal{L}^{<}_{\varphi}(0)$ empty. Assume therefore that $\mathcal{L}^{<}_{\varphi}(0)$ is nonempty. If this holds, it follows by Lemma 4.6 and Lemma 3.2 that $\tilde{\varphi}_-$ given by relation (4.3) and φ_+ are quasidifferentiable. Moreover, it is easy to verify by Theorem 3.4 and again relation (4.3) using $\hat{\varphi}_-(\boldsymbol{d}) > 0$ for every $\boldsymbol{d} \notin \mathrm{cl}(\mathcal{L}^{<}_{\varphi}(0))$ that

$$f'(\boldsymbol{x}_0; \boldsymbol{d}) = \min\left\{\tilde{\varphi}_-(\boldsymbol{d}), \varphi_+(\boldsymbol{d})\right\}$$

for every $\boldsymbol{d} \in \mathbb{R}^n$ and hence the desired result is proved. □

4.2. Where are the separators?

This subsection, based on Lemma 4.4 and on the properties of the Dini upper derivative, characterizes the elements of the normal cone of the set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$. Introduce now the set

$$\Gamma_f := \{\boldsymbol{x} \in \mathbb{R}^n : f'_+(\boldsymbol{x}; \boldsymbol{d}) \geq 0 \text{ for every } \boldsymbol{d} \in \mathbb{R}^n\} \tag{4.4}$$

which is sometimes called the set of stationary points. For reasons to be soon clarified we call this the set of *"bad" points*. Before deriving the announced characterization, we observe by Theorem 11.3 of [21] that the normal cone

$$\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0) := \left\{\boldsymbol{x}^\star \in \mathbb{R}^n : \langle \boldsymbol{x}^\star, \boldsymbol{x} - \boldsymbol{x}_0 \rangle \leq 0 \text{ for every } \boldsymbol{x} \in \mathcal{L}_f^<(f(\boldsymbol{x}_0))\right\} \tag{4.5}$$

of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ is a proper nonempty convex cone of $\mathbb{R}^n$ if $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty. In the next lemma a partial description of $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ is given by means of the function φ if the set $\mathcal{L}_\varphi^<(0)$ is empty. Introducing the nonempty sets

$$\mathcal{L}_\varphi^=(0) := \{\boldsymbol{d} \in \mathbb{R}^n : f'_+(\boldsymbol{x}_0; \boldsymbol{d}) = 0\}$$

and

$$\mathcal{L}_\varphi^\leq(0) := \{\boldsymbol{d} \in \mathbb{R}^n : f'_+(\boldsymbol{x}_0; \boldsymbol{d}) \leq 0\}$$

it follows that $\mathcal{L}_\varphi^<(0)$ is empty if and only if $\mathcal{L}_\varphi^\leq(0) = \mathcal{L}_\varphi^=(0)$. A sufficient condition for $\mathcal{L}_\varphi^<(0)$ to be empty is given by $\boldsymbol{x}_0 \in \operatorname{int}(\mathcal{L}_f^\leq(f(\boldsymbol{x}_0)))$. The example $f(x) = x^3$ at $x = 0$, shows that this condition is not necessary. Although trivial the next result seems to be new.

Lemma 4.8. *If the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function satisfying $f(\boldsymbol{x}_0)$ finite and the set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty and the set $\mathcal{L}_\varphi^<(0)$ is empty or equivalently $\boldsymbol{x}_0 \in \Gamma_f$, then*

$$(\mathcal{L}_\varphi^=(0))^\circ \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$$

with $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ the normal cone of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ defined in relation (4.5).

Proof. Since $\mathcal{L}_\varphi^<(0)$ is empty and f is quasiconvex it must follow by Lemma 4.2 that $\mathcal{L}_\varphi^=(0)$ is a nonempty convex cone. Moreover, the nonemptyness of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ and the emptyness of $\mathcal{L}_\varphi^<(0)$ enable us to verify that $\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0 \subseteq \mathcal{L}_\varphi^=(0)$ and so $\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0) \subseteq \mathcal{L}_\varphi^=(0)$. This implies that

$$(\mathcal{L}_\varphi^=(0))^\circ \subseteq (\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))^\circ = \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$$

and hence the desired result is proven. □

Clearly, the above result reduces to a useless observation if $\mathcal{L}_\varphi^=(0)$ equals $\mathbb{R}^n$. In this case we obtain $(\mathcal{L}_\varphi^=(0))^\circ = \{\boldsymbol{0}\}$ and this happens for $f(x) = x^3$ at $x_0 = 0$.

Figure 1 provides an interpretation of Lemma 4.8. The first picture is drawn in the domain and shows two lower level sets. The one with a dashed boundary is $\mathcal{L}_f^<(f(\boldsymbol{b}))$ and the one with a full boundary is $\mathcal{L}_f^\leq(f(\boldsymbol{a})) = \mathcal{L}_f^\leq(f(\boldsymbol{b}))$. The second picture is drawn in the epigraph space and corresponds to slicing the graph of the function along the line going through $\boldsymbol{a}$ and $\boldsymbol{b}$. Observe first that if $\boldsymbol{x}_0 \in (\boldsymbol{a}, \boldsymbol{b})$ then

$\mathcal{L}_\varphi^=(0) = \mathbb{R}^n$ and so no useful information is provided. On the other hand, if $\boldsymbol{x}_0 = \boldsymbol{a}$ then $\mathcal{L}_\varphi^=(0) \neq \mathbb{R}^n$ and so $\left(\mathcal{L}_\varphi^=(0)\right)^\circ$ also contains nonzero elements.
Applying Lemma 4.4 and Theorem 6.3 of [21] it follows for $\mathcal{L}_\varphi^<(0)$ nonempty that

$$\mathrm{cl}(\mathcal{L}_\varphi^<(0)) = \mathrm{cl}(\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)).$$

Similar as for convex functions (see [12, 21]), this yields

$$\begin{aligned} \mathcal{N}_{\mathcal{L}_f^<}(\mathbf{x}_0) &= \{\mathbf{x}^* \in \mathbb{R}^n : \langle \mathbf{x}^*, \mathbf{x} - \mathbf{x}_0 \rangle \leq 0 \text{ for every } \mathbf{x} \in \mathcal{L}_f^<(f(\mathbf{x}_0)\} \\ &= \Big(\mathrm{cone}(\mathcal{L}_f^<(f(\mathbf{x}_0)) - \mathbf{x}_0)\Big)^\circ \\ &= \Big(cl(\mathrm{cone}(\mathcal{L}_f^<(f(\mathbf{x}_0)) - \mathbf{x}_0))\Big)^\circ \\ &= (cl(\mathcal{L}_\varphi(0)))^\circ \end{aligned} \tag{4.6}$$

with $\mathcal{K}^\circ$ denoting the polar cone of $\mathcal{K}$. Hence, to give an alternative description of the set $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$, it is sufficient by relation (4.6) or Lemma 4.8 to show that the set $\mathrm{cl}(\mathcal{L}_\varphi^<(0))$ or $\mathrm{cl}(\mathcal{L}_\varphi^=(0))$ is the polar cone of some other closed cone K and then apply the bipolar theorem.

Lemma 4.9. *If the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function satisfying $f(\boldsymbol{x}_0)$ finite and the function φ given by $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0; \boldsymbol{d})$ is a proper evenly quasiconvex function satisfying $\mathcal{L}_\varphi^<(0)$ is nonempty then*

$$\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0) = \mathrm{cl}(\mathrm{cone}(\partial\varphi_-(\mathbf{0})))$$

with $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ the normal cone of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ defined in relation (4.5).

Proof. By Theorem 3.4 and Proposition VI.1.3.3 of [12] we obtain that

$$\begin{aligned} \mathrm{cl}(\mathcal{L}_\varphi^<(0)) &= \{\boldsymbol{d} \in \mathbb{R}^n : \varphi_-(\boldsymbol{d}) \leq 0\} \\ &= \{\boldsymbol{d} \in \mathbb{R}^n : \langle \boldsymbol{x}^\star, \boldsymbol{d} \rangle \leq 0 \text{ for every } \boldsymbol{x}^\star \in \partial\varphi_-(\mathbf{0})\} \\ &= (\mathrm{cone}(\partial\varphi_-(\mathbf{0})))^\circ. \end{aligned}$$

Hence by (4.6) and Proposition III.4.2.7 of [12] it follows that

$$\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0) = (\mathrm{cl}(\mathcal{L}_\varphi^<(0))^\circ = (\mathrm{cone}(\partial\varphi_-(\mathbf{0})))^{\circ\circ} = \mathrm{cl}(\mathrm{cone}(\partial\varphi_-(\mathbf{0})))$$

and this shows the desired result. □

An interpretation of Lemma 4.9 is provided by Figure 4.

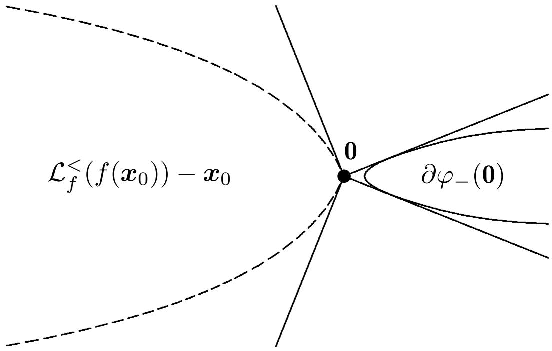

FIGURE 4. The normal cone to the strict lower level set in the favorable case

Compare now this figure with Figure 1, The situation described in Figure 4 corresponds to taking $\boldsymbol{x}_0 = \boldsymbol{b}$ in Figure 1.

Since φ_- is sublinear it follows that $\partial\varphi_-(\boldsymbol{d}) \subseteq \partial\varphi_-(\mathbf{0}) \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ for every $\boldsymbol{d} \in \mathrm{dom}(\varphi_-) = \mathrm{cl}(\mathcal{L}_\varphi^<(0))$. If additionally, $\mathcal{L}_\varphi^<(0)$ is a convex cone of dimension n this implies by Theorem IV.4.2.3 of [12] that φ_- is differentiable on a dense subset of $\mathrm{int}(\mathcal{L}_\varphi^<(0))$, and so we can conclude for a point belonging to this dense subset that $\nabla\varphi_-(\boldsymbol{d}) \in \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$.

An immediate consequence of Lemma 4.8 and Lemma 3.2 is given by the following result. Although this result is not difficult to prove it appears to be new and improves Lemma 4.8 for φ evenly quasiconvex.

Lemma 4.10. *If the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function satisfying $f(\boldsymbol{x}_0)$ finite and the set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ nonempty and the function φ given by $\varphi(\boldsymbol{d}) = f_+'(\boldsymbol{x}_0;\boldsymbol{d})$ is proper and evenly quasiconvex and the set $\mathcal{L}_\varphi^<(0)$ empty then*

$$\mathrm{cl}(\mathrm{cone}(\partial\varphi_+(\mathbf{0}))) \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$$

with $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ the normal cone of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ defined in relation (4.5).

Proof. By our assumptions it follows that $\mathcal{L}_\varphi^=(0)$ is nonempty and $\mathcal{L}_\varphi^\leq(0) = \mathcal{L}_\varphi^=(0)$. This implies by Lemma 3.2 and φ_+ being the support function of $\partial\varphi_+(\mathbf{0})$ that

$$\begin{aligned} \mathcal{L}_\varphi^=(0) &= \{\boldsymbol{d} \in \mathbb{R}^n : \varphi_+(\boldsymbol{d}) = 0\} \\ &= \{\boldsymbol{d} \in \mathbb{R}^n : \langle \boldsymbol{x}^\star, \boldsymbol{d}\rangle \leq 0 \text{ for every } \boldsymbol{d} \in \partial\varphi_+(\mathbf{0})\} \\ &= (\mathrm{cone}(\partial\varphi_+(\mathbf{0})))^\circ. \end{aligned}$$

Applying now Lemma 4.8 and Proposition III.4.2.7 of [12] yields

$$\mathrm{cl}(\mathrm{cone}(\partial\varphi_+(\mathbf{0}))) = (\mathrm{cone}(\partial\varphi_+(\mathbf{0})))^{\circ\circ} = (\mathcal{L}_\varphi^=(0))^\circ \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$$

and the desired result is proven. □

As already observed, if $\boldsymbol{d} \longmapsto f_+'(\boldsymbol{x}_0;\boldsymbol{d})$ is the zero functional or equivalently $\partial\varphi_+(\mathbf{0}) = \{\mathbf{0}\}$, the above result does not provide any useful information.

5. How to separate, if you must!

In this section we analyze the problem of computing an element of the normal cone $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ if $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty. As already observed, a sufficient condition for $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ to be nonempty is given by the nonemptiness of the set $\mathcal{L}_\varphi^<(0)$ with $\varphi(\boldsymbol{d}) := f_+'(\boldsymbol{x}_0;\boldsymbol{d})$ and so it is natural to consider the optimization problem

$$\vartheta(S) = \inf\{\varphi(\mathbf{d}) : \mathbf{d} \in \mathcal{C}\} \tag{S}$$

with $\mathcal{C}$ a compact convex set satisfying $\mathbf{0} \in \text{int}(\mathcal{C})$. Notice, since $\dim(\mathcal{C}) = n$ that the boundary $\text{bd}(\mathcal{C})$ of $\mathcal{C}$ is given by $\mathcal{C} \setminus \text{int}(\mathcal{C})$. In order to guarantee that the optimization problem (S) is solvable, i.e. there exists some $\boldsymbol{d}_0 \in \mathcal{C}$ satisfying $\varphi(\boldsymbol{d}_0) = \vartheta(S)$, it is sufficient by Theorem 3.4 to assume that φ is a proper evenly quasiconvex function. In the remainder of this section we always assume that $\boldsymbol{d} \longmapsto f_+'(\boldsymbol{x}_0;\boldsymbol{d})$ satisfies this property and so the set $\mathcal{S}$ of optimal solutions of optimization problem (S) is always nonempty. Clearly, one should choose the compact convex set $\mathcal{C}$ with $\mathbf{0} \in \text{int}(\mathcal{C})$ in such a way that optimization problem (S) is "easy" solvable. Since φ is a proper, positively homogeneous and evenly quasiconvex function the following result is easy to verify and so its proof is omitted.

Lemma 5.1. *It follows $\vartheta(S) < 0$ if and only if $\mathcal{L}_\varphi^<(0)$ is nonempty. If this holds then $\mathcal{S} \subseteq \text{bd}(\mathcal{C})$. Moreover, if $\vartheta(S) = 0$, i.e. $\boldsymbol{x}_0 \in \Gamma_f$, then either $\mathbf{0}$ is the unique solution of (S) or $\mathcal{S} \cap \text{bd}(\mathcal{C})$ is nonempty.*

Clearly, if $\mathbf{0}$ is the unique solution of optimization problem (S) then due to $\mathbf{0} \in \text{int}(\mathcal{C})$, $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ quasiconvex and $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ proper and positively homogeneous, it follows that $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is empty. Also, if $\vartheta(S) < 0$ we obtain by Lemma 4.9 that an optimal solution $\boldsymbol{d}_0$ of optimization problem (S) is also an optimal solution of the optimization problem

$$\vartheta(S') = \inf\{\varphi_-(\mathbf{d}) : \gamma_{\mathcal{C}}(\mathbf{d}) \leq 1, \mathbf{d} \in cl(\mathcal{L}_\varphi^<(0)\} \tag{S'}$$

with $\gamma_{\mathcal{C}}(\boldsymbol{d}) := \inf\{t > 0 : \boldsymbol{d} \in t\mathcal{C}\}$ the *gauge* of $\mathcal{C}$. Since by Lemma 3.3 the function φ_- is proper and convex with $\text{dom}(\varphi_-) = \text{cl}(\mathcal{L}_\varphi^<(0))$ and the function $\gamma_{\mathcal{C}}$ is finite and convex due to $\mathcal{C}$ compact, convex and $\mathbf{0} \in \text{int}(\mathcal{C})$, the optimization problem (S) satisfies the properties of a convex program given in Section 28 of [21].

Using now so-called *primal-dual* information given by the *Karush-Kuhn-Tucker* conditions it is possible to prove the next result.

Lemma 5.2. *If $\boldsymbol{d}_0$ is an optimal solution of (S) with $\vartheta(S) < 0$ then the set $\partial\varphi_-(\boldsymbol{d}_0) - \vartheta(S)\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$ contains $\mathbf{0}$.*

Proof. If $\boldsymbol{d}_0$ is an optimal solution of (S) with $\varphi(\boldsymbol{d}_0) = \vartheta(S) < 0$ then $\lambda\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$ for every $\lambda > 0$. Also, by Lemma 5.1 we obtain that $\boldsymbol{d}_0 \in \text{bd}(\mathcal{C})$ implying that $\gamma_{\mathcal{C}}(\boldsymbol{d}_0) = 1$ and so $\gamma_{\mathcal{C}}(\lambda\boldsymbol{d}_0) = \lambda\gamma_{\mathcal{C}}(\boldsymbol{d}_0) < 1$ for every $0 < \lambda < 1$. Hence, by Corollary 28.2.1 of [21] a Karush-Kuhn-Tucker vector λ_1 of (S) exists and this yields by Theorem 28.3 of [21] that $\mathbf{0} \in \partial\varphi_-(\boldsymbol{d}_0) + \lambda_1\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$, $\lambda_1(\gamma_{\mathcal{C}}(\boldsymbol{d}_0) - 1) = 0$ and $\lambda_1 \geq 0$. If $\lambda_1 = 0$ it follows that $\mathbf{0} \in \partial\varphi_-(\boldsymbol{d}_0)$ and this yields $\varphi_-(\boldsymbol{d}) \geq \varphi_-(\boldsymbol{d}_0)$ for every $\boldsymbol{d} \in \mathbb{R}^n$. However, since φ_- is positively homogeneous and $\varphi_-(\boldsymbol{d}_0) < 0$ it follows that $\varphi_-(\lambda\boldsymbol{d}_0) = \lambda\varphi_-(\boldsymbol{d}_0) < \varphi_-(\boldsymbol{d}_0)$

for every $\lambda > 1$ contradicting $\mathbf{0} \in \partial\varphi_-(\boldsymbol{d}_0)$. Hence, $\lambda_1 > 0$ and to compute λ_1 we observe the following. It is well-known, [12, 21], that

$$\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0) = \{\boldsymbol{d}_0^\star \in \mathcal{C}^\circ : \langle \boldsymbol{d}_0^\star, \boldsymbol{d}_0 \rangle = \gamma_{\mathcal{C}}(\boldsymbol{d}_0)\}$$

with $\mathcal{C}^\circ$ the polar of $\mathcal{C}$ and so by the Karush-Kuhn-Tucker conditions and Lemma 5.1 there exists some $\boldsymbol{d}_0^\star \in \mathbb{R}^n$ with $-\boldsymbol{d}_0^\star \in \partial\varphi_-(\boldsymbol{d}_0)$, $\langle \boldsymbol{d}_0^\star, \boldsymbol{d}_0 \rangle = \lambda_1$ and $\langle \boldsymbol{d}_0^\star, \boldsymbol{d} \rangle \leq \lambda_1$ for every $\boldsymbol{d} \in \mathcal{C}$. Since $-\boldsymbol{d}_0^\star \in \partial\varphi_-(\boldsymbol{d}_0)$ it follows by Theorem 23.5 of [21] that

$$\varphi_-(\boldsymbol{d}_0) + \varphi_-^*(-\boldsymbol{d}_0^\star) = -\langle \boldsymbol{d}_0^\star, \boldsymbol{d}_0 \rangle$$

with φ_-^* the conjugate function of φ_-. Since φ_- is positively homogeneous and thus φ_-^* is either 0 or $+\infty$ we obtain by the above equality that

$$\varphi_-(\boldsymbol{d}_0) = -\langle \boldsymbol{d}_0^\star, \boldsymbol{d}_0 \rangle = -\lambda_1$$

and so the result is proven. □

The following result is an immediate consequence of the previous lemma.

Corollary 5.3. *If $\gamma_{\mathcal{C}}$ is differentiable in $\boldsymbol{d}_0$ then $-\nabla\gamma_{\mathcal{C}}(\boldsymbol{d}_0) \in \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$.*

Proof. The previous result shows for $\vartheta(S) < 0$ and $\boldsymbol{d}_0$ an optimal solution of (S) that the sets $\partial\varphi_-(\boldsymbol{d}_0)$ and $\vartheta(S)\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$ intersect. Hence, if $\gamma_{\mathcal{C}}$ is differentiable in $\boldsymbol{d}_0$ with gradient $\nabla\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$ then

$$\vartheta(S)\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0) = \{\vartheta(S)\nabla\gamma_{\mathcal{C}}(\boldsymbol{d}_0)\}$$

and so $\vartheta(S)\nabla\gamma_{\mathcal{C}}(\boldsymbol{d}_0) \in \partial\varphi_-(\boldsymbol{d}_0) \subseteq \partial\varphi_-(\mathbf{0})$. Now, by Lemma 3.3 it follows that $\partial\varphi_-(\mathbf{0}) \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ and since $\vartheta(S) < 0$ and $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ is a cone this leads to the stated result. □

In the next example we discuss the well know Lp norm.

Example 5.4. *Take $\mathcal{C} := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_p \leq 1\}$ with $\|\cdot\|_p$, $1 < p < \infty$, the ℓ_p-norm. Clearly, $\gamma_{\mathcal{C}}(\boldsymbol{x}) = \|\boldsymbol{x}\|_p$ and $\gamma_{\mathcal{C}}$ is differentiable everywhere except at $\mathbf{0}$. Moreover, for every $\boldsymbol{x} \neq \mathbf{0}$ it is easy to verify that*

$$\nabla\gamma_{\mathcal{C}}(\boldsymbol{x}) = \|\boldsymbol{x}\|_p^{1-p} \begin{bmatrix} \operatorname{sign}(x_1)|x_1|^{p-1} \\ \vdots \\ \operatorname{sign}(x_s)|x_s|^{p-1} \end{bmatrix}$$

with x_i the i^{th} component of $\boldsymbol{x}$ and sign(x) *the sign function.*

If $\vartheta(S) < 0$, $\boldsymbol{d}_0$ solves optimization problem (S) and $\gamma_{\mathcal{C}}$ is not differentiable in $\boldsymbol{d}_0$ while $\varphi(\boldsymbol{d})$ is differentiable in $\boldsymbol{d}_0$ then it is easy to show, due to $\boldsymbol{d}_0 \in \operatorname{int}(\mathcal{L}_\varphi^<(0))$ and the definition of φ_-, that $\nabla\varphi(\boldsymbol{d}_0) = \nabla\varphi_-(\boldsymbol{d}_0) \in \partial\varphi_-(\mathbf{0})$ with $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0; \boldsymbol{d})$. In this case, the optimization problem (S) is only used to identify an interior element of $\mathcal{L}_\varphi^<(0)$. This also shows that selecting some $\boldsymbol{d}_1 \in \operatorname{int}(\mathcal{L}_\varphi^<(0))$ with φ differentiable in $\boldsymbol{d}_1$ already yields an element of the normal cone $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$.

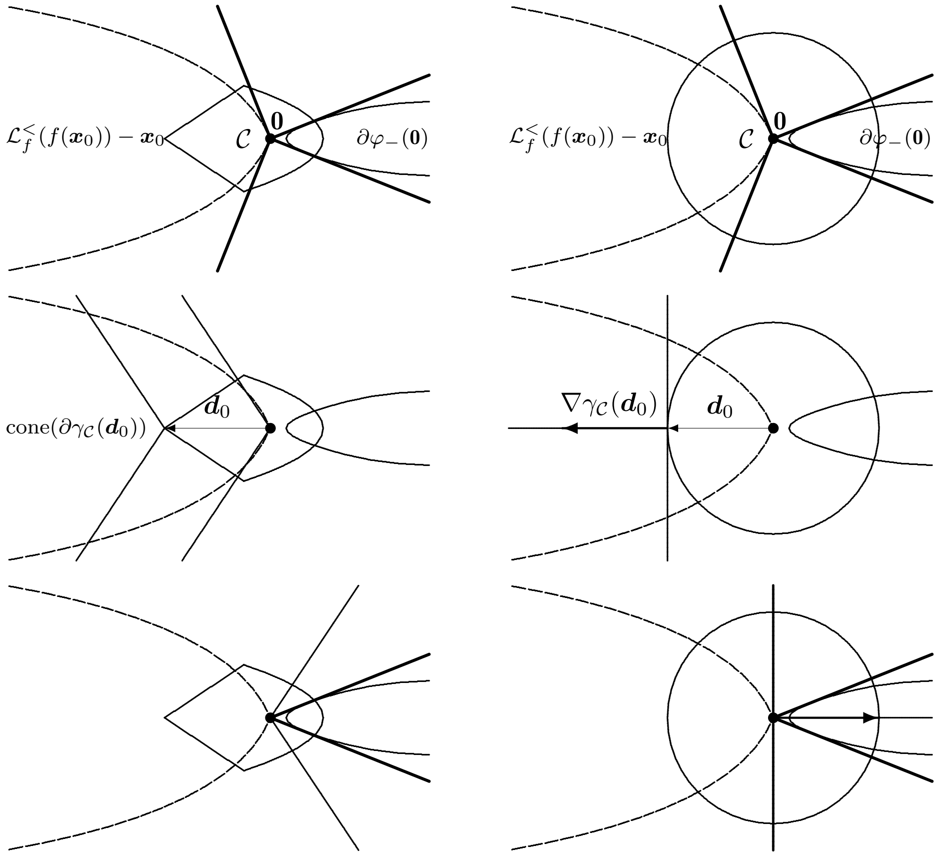

FIGURE 5. Geometric interpretation of the separation oracle

Finally, we provide in Figure 5 a geometrical interpretation of Lemma 5.2. The first picture shows a set $\mathcal{C}$ with a kink at $\boldsymbol{d}_0$ and for which cone$(\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0))$ is a cone (shifted in the picture to the vertex of $\mathcal{C}$ for clarity) whose symmetric cone intersects $\partial\varphi_-(\boldsymbol{d}_0)$ (by Lemma 5.2) but includes elements which do not belong to cone$(\partial\varphi_-(\boldsymbol{d}_0)) = \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$. On the other hand, the second picture corresponds to a smooth $\mathcal{C}$. Hence $\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$ is a singleton and so the symmetric of its conical hull (a half line) intersects (by Lemma 5.2 again) cone$(\partial\varphi_-(\boldsymbol{d}_0))$ and it must be included in $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$.

We triplicate each picture for clarity. The top picture shows the conical hull of the strict lower level set, the corresponding normal cone and the compact convex set $\mathcal{C}$ corresponding to the feasible region of optimization problem (S). The middle picture shows the solution of problem (S), direction $\boldsymbol{d}_0$, and the conical hull of $\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$. Finally, the bottom picture shows the intersection of the symmetric of this conical hull and the normal cone.

We consider in the next section several quasiconvex functions for which we do not have to solve the optimization problem (S).

6. Examples

This section illustrates classes of functions for which optimization problem (S) can be replaced by easier membership problems.

6.1. Regular functions

In this subsection we discuss a separation oracle for the following subclass of quasiconvex functions.

Definition 6.1. Let $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ be a quasiconvex function with $f(\boldsymbol{x}_0)$ finite. Then f is called regular at $\boldsymbol{x}_0$ if φ is a lower semicontinuous sublinear function with $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0; \boldsymbol{d})$.

Following Pshenichnyi in [20] these functions are sometimes called *quasidifferentiable*. However, we prefer to follow Clarke, [1], and call them *regular* since the term quasidifferentiable has nowadays a broader meaning, see [5].

As the next lemma shows the above class of functions is closed under the finite max operator.

Lemma 6.2. *Let $f_i : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$, $i = 1, \ldots, n$, be quasiconvex functions with $f_i(\boldsymbol{x}_0)$ finite for every $1 \le i \le n$. If each function f_i is regular at $\boldsymbol{x}_0$ then the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ given by $f(\boldsymbol{x}) := \max_{1 \le i \le n} f_i(\boldsymbol{x})$ is also regular at $\boldsymbol{x}_0$.*

Proof. It is easy to verify that f is quasiconvex and $f(\boldsymbol{x}_0)$ is finite. Moreover, by Lemma 2.5.3 of [11] we obtain that

$$f'_+(\boldsymbol{x}_0; \boldsymbol{d}) = \max_{i \in I(\boldsymbol{x}_0)} f'_{i+}(\boldsymbol{x}_0; \boldsymbol{d}) \tag{6.1}$$

with $I(\boldsymbol{x}_0) := \{1 \le i \le n : f(\boldsymbol{x}_0) = f_i(\boldsymbol{x}_0)\}$ the set of active indices of f at $\boldsymbol{x}_0$. Since by assumption it follows that $\boldsymbol{d} \longmapsto f'_{i+}(\boldsymbol{x}_0; \boldsymbol{d})$ is a lower semicontinuous sublinear function for every $1 \le i \le n$ the desired result follows by (6.1). □

An important class of regular functions is given by the next lemma. These functions are extremely important in location analysis, see [11].

Lemma 6.3. *Let $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a finite nondecreasing quasiconvex function and $\boldsymbol{v} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a finite-valued convex vector function, i.e. $\boldsymbol{v}(\boldsymbol{x}) := (v_1(\boldsymbol{x}), \ldots, v_m(\boldsymbol{x}))$ with $v_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, \ldots, m$, finite-valued convex functions. If the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is given by $f(\boldsymbol{x}) = g(\boldsymbol{v}(\boldsymbol{x}))$ and g is regular and locally Lipschitz at $\boldsymbol{v}(\boldsymbol{x}_0)$ then f is a quasiconvex function regular at $\boldsymbol{x}_0$. Moreover, $f'_+(\boldsymbol{x}_0; \boldsymbol{d}) = g'_+(\boldsymbol{v}(\boldsymbol{x}_0); \boldsymbol{v}'(\boldsymbol{x}_0; \boldsymbol{d}))$ with $\boldsymbol{v}'(\boldsymbol{x}_0; \boldsymbol{d}) = (v'_1(\boldsymbol{x}_0; \boldsymbol{d}), \ldots, v'_m(\boldsymbol{x}_0; \boldsymbol{d}))$ and the function $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0; \boldsymbol{d})$ is Lipschitz continuous.*

Proof. Since g is a nondecreasing quasiconvex function and $\boldsymbol{v}$ a convex vector function it is easy to verify that f is quasiconvex. Also, by Lemma 2.5.2 of [11] it follows that

$f'_+(\boldsymbol{x}_0;\boldsymbol{d}) = g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}))$. Moreover, since g is regular at $\boldsymbol{v}(\boldsymbol{x}_0)$ and nondecreasing we obtain for every $0<\lambda<1$ and $\boldsymbol{d}_1,\boldsymbol{d}_2\in\mathbb{R}^n$ that

$$\begin{aligned} & f'_+(\boldsymbol{x}_0;\lambda\boldsymbol{d}_1+(1-\lambda)\boldsymbol{d}_2) \\ &= g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{v}'(\boldsymbol{x}_0;\lambda\boldsymbol{d}_1+(1-\lambda)\boldsymbol{d}_2)) \\ &\leq g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\lambda\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}_1)+(1-\lambda)\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}_2)) \\ &\leq \lambda g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}_1))+(1-\lambda)g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}_2)) \\ &= \lambda f'_+(\boldsymbol{x}_0;\boldsymbol{d}_1)+(1-\lambda)f'_+(\boldsymbol{x}_0;\boldsymbol{d}_2) \end{aligned}$$

and by the Lipschitz continuity of $\boldsymbol{d}\longmapsto\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d})$ and $\boldsymbol{d}\longmapsto g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{d})$, the Lipschitz continuity of $\boldsymbol{d}\longmapsto f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ follows. □

For the class of functions given in Definition 6.1 it is now easy, using only classical results of convex analysis, to prove the next result.

Lemma 6.4. *If $f:\mathbb{R}^n\longrightarrow[-\infty,+\infty]$ is a quasiconvex function regular at $\boldsymbol{x}_0$ and $\mathcal{L}^<_\varphi(0)$ is nonempty then*

$$\mathrm{cl}(\mathrm{cone}(\partial\varphi(\mathbf{0}))) = \mathcal{N}_{\mathcal{L}^<_f}(\boldsymbol{x}_0)$$

with $\partial\varphi(\mathbf{0})$ the subgradient set of the convex function $\varphi(\boldsymbol{d}):=f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ at $\mathbf{0}$. Moreover, if $\mathcal{L}^<_f(f(\boldsymbol{x}_0))$ is nonempty and $\mathcal{L}^<_\varphi(0)$ is empty then

$$\mathrm{cl}(\mathrm{cone}(\partial\varphi(\mathbf{0}))) \subseteq \mathcal{N}_{\mathcal{L}^<_f}(\boldsymbol{x}_0).$$

Proof. To prove the first result we observe by relation (4.6) that

$$\mathcal{N}_{\mathcal{L}^<_f}(\boldsymbol{x}_0) = (\mathrm{cl}(\mathcal{L}^<_\varphi(0)))^\circ.$$

Since f is a quasiconvex function regular at $\boldsymbol{x}_0$ it follows by Proposition VI.1.3.3 of [12] that $\mathrm{cl}(\mathcal{L}^<_\varphi(0)) = \mathcal{L}^\leq_\varphi(0)$. Moreover, by Theorem V.3.1.1 of [12] we obtain that $\mathcal{L}^\leq_\varphi(0)$ equals $\{\boldsymbol{d}:\langle\boldsymbol{x}^\star,\boldsymbol{d}\rangle\leq 0$ for every $\boldsymbol{x}^\star\in\partial\varphi(\mathbf{0})\}$. Clearly this set also equals $(\mathrm{cone}(\partial\varphi(\mathbf{0})))^\circ$ and hence by (4.6) and Proposition III.4.2.7 of [12] we obtain that

$$\mathcal{N}_{\mathcal{L}^<_f}(\boldsymbol{x}_0) = (\mathrm{cone}(\partial\varphi(\mathbf{0})))^{\circ\circ} = \mathrm{cl}(\mathrm{cone}(\partial\varphi(\mathbf{0}))).$$

The second result can be proved in a similar way and this completes the proof. □

Finally we can show the main result of this subsection. Recall that the set of "bad" points Γ_f is defined in (4.4).

Lemma 6.5. *Let $g_i:\mathbb{R}^m\longrightarrow\mathbb{R}$, $i=1,\ldots,n$, be quasiconvex and continuously differentiable functions and suppose $v_i:\mathbb{R}^n\longrightarrow\mathbb{R}$, $1\leq i\leq m$, are finite-valued convex functions. Then it follows for $f(\boldsymbol{x}):=\max_{1\leq i\leq n}f_i(\boldsymbol{x})$ with $f_i(\boldsymbol{x}):=g_i(\boldsymbol{v}(\boldsymbol{x}))$ that $\boldsymbol{x}_0$ belongs to Γ_f if and only if*

$$\mathbf{0}\in\mathrm{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)}\sum_{j=1}^m \tfrac{\partial g_i}{\partial z_j}(\boldsymbol{v}(\boldsymbol{x}_0))\partial v_j(\boldsymbol{x}_0)\right)$$

where $I(\boldsymbol{x}_0) := \{1 \leq i \leq n : f(\boldsymbol{x}_0) = f_i(\boldsymbol{x}_0)\}$*. Moreover, if* $\mathcal{L}^<_\varphi(0)$ *is nonempty, i.e.* $\boldsymbol{x}_0 \notin \Gamma_f$*, then*

$$\operatorname{cone}\left(\operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \sum_{j=1}^m \tfrac{\partial g_i}{\partial z_j}(\boldsymbol{v}(\boldsymbol{x}_0))\partial v_j(\boldsymbol{x}_0)\right)\right) = \mathcal{N}_{\mathcal{L}^<_f}(\boldsymbol{x}_0)$$

Proof. Clearly every function $g_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ is regular and locally Lipschitz at $\boldsymbol{x}_0$. Applying now Lemma 6.3 yields

$$f_i'(\boldsymbol{x}_0;\boldsymbol{d}) = \sum_{j=1}^m \tfrac{\partial g_i}{\partial z_j}(\boldsymbol{v}(\boldsymbol{x}_0))v_j'(\boldsymbol{x}_0;\boldsymbol{d})$$

and so by Theorem V.3.1.1 of [12] we obtain for $1 \leq i \leq n$ that

$$f_i'(\boldsymbol{x}_0;\boldsymbol{d}) = \max\{\langle \boldsymbol{d}, \boldsymbol{x}^\star\rangle : \boldsymbol{x}^\star \in \partial\varphi_i(\mathbf{0})\}$$

with

$$\partial\varphi_i(\mathbf{0}) := \sum_{j=1}^m \frac{\partial g_i}{\partial z_j}(\boldsymbol{v}(\boldsymbol{x}_0))\partial v_j(\boldsymbol{x}_0).$$

By Lemma 2.5.3 of [11] this implies

$$\begin{aligned} f'(\boldsymbol{x}_0;\boldsymbol{d}) &= \max_{i\in I(\boldsymbol{x}_0)} f_i'(\boldsymbol{x}_0;\boldsymbol{d}) \\ &= \max_{i\in I(\boldsymbol{x}_0)} \max\{\langle \boldsymbol{d}, \boldsymbol{x}^\star\rangle : \boldsymbol{x}^\star \in \partial\varphi_i(\mathbf{0})\} \\ &= \max\left\{\langle \boldsymbol{d}, \boldsymbol{x}^\star\rangle : \boldsymbol{x}^\star \in \operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \partial\varphi_i(\mathbf{0})\right)\right\}. \end{aligned}$$

Using the above relation it follows that $f'(\boldsymbol{x}_0;\boldsymbol{d}) \geq 0$ for every $\boldsymbol{d} \in \mathbb{R}^n$ if and only if $\mathbf{0}$ belongs to $\operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \partial\varphi_i(\mathbf{0})\right)$. This proves the first part. To prove the second part we observe that $\operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \partial\varphi_i(\mathbf{0})\right)$ is the subgradient set of the finite valued convex function φ at $\mathbf{0}$ and since $\partial\varphi_i(\mathbf{0})$ is compact for each i and $\mathbf{0}$ does not belong to $\operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \partial\varphi_i(\mathbf{0})\right)$ the second result follows by Lemma 6.4 together with Lemma III.1.4.7 and Theorem III.1.4.3 of [12]. □

In the next subsection we consider another class of quasiconvex functions for which the separation problem is easy.

6.2. Another class of easy functions

Let $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable and convex function and $\gamma_i \in \mathbb{R}$, $1 \leq i \leq m$, and introduce the functions $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$f_i(\boldsymbol{x}) := \min\{g_i(\boldsymbol{x}), \gamma_i\}.$$

Clearly, the functions f_i are quasiconvex and so is the function

$$f(\boldsymbol{x}) := \max_{1\leq i\leq m} f_i(\boldsymbol{x}).$$

A representation of such a function for the case of affine g_i is given in Figure 6.

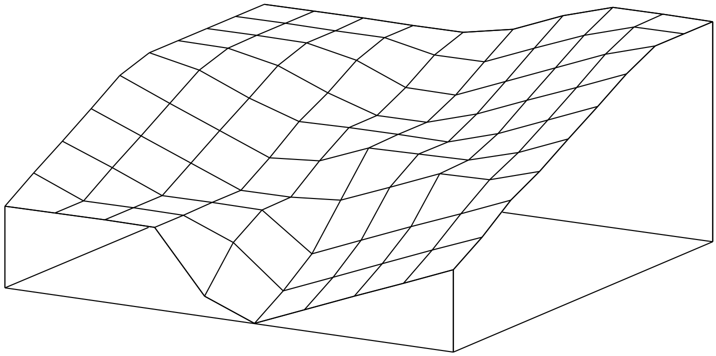

FIGURE 6. A bivariate quasiconvex function with horizontal regions

Moreover, if $I(\boldsymbol{x}_0) := \{1 \leq i \leq m : f(\boldsymbol{x}_0) = f_i(\boldsymbol{x}_0)\}$ it follows that

$$f'(\boldsymbol{x}_0; \boldsymbol{d}) = \max_{i \in I(\boldsymbol{x}_0)} f_i'(\boldsymbol{x}_0; \boldsymbol{d}) \tag{6.2}$$

and

$$f_i'(\boldsymbol{x}_0; \boldsymbol{d}) = \begin{cases} \min\{\langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle, 0\} & \text{if } g_i(\boldsymbol{x}_0) = \gamma_i \\ \langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle & \text{if } g_i(\boldsymbol{x}_0) < \gamma_i \\ 0 & \text{if } g_i(\boldsymbol{x}_0) > \gamma_i \end{cases}. \tag{6.3}$$

By (6.2) and (6.3) it is clear that $f'(\boldsymbol{x}_0; \boldsymbol{d}) \geq 0$ for every $\boldsymbol{d} \in \mathbb{R}^n$ if there exists some $i \in I(\boldsymbol{x}_0)$ satisfying $g_i(\boldsymbol{x}_0) > \gamma_i$ and this implies that $\vartheta(S) = \min_{\boldsymbol{d} \in \mathcal{C}} f'(\boldsymbol{x}_0; \boldsymbol{d}) \geq 0$. Therefore assume for every $i \in I(\boldsymbol{x}_0)$ that $g_i(\boldsymbol{x}_0) \leq \gamma_i$. If this holds the following result is easy to prove.

Lemma 6.6. *If for every $i \in I(\boldsymbol{x}_0)$ it follows that $g_i(\boldsymbol{x}_0) \leq \gamma_i$ then $\vartheta(S) < 0$ if and only if $\mathbf{0} \notin \operatorname{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\})$. Moreover, if this holds then*

$$\operatorname{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\}) \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0).$$

Proof. Clearly, by the assumption $g_i(\boldsymbol{x}_0) \leq \gamma_i$ for every $i \in I(\boldsymbol{x}_0)$, (6.2) and (6.3) we obtain that $\min_{\boldsymbol{d} \in \mathcal{C}} f'(\boldsymbol{x}; \boldsymbol{d})$ is equivalent to the optimization problem

$$\begin{array}{llll} \min & t & & \\ \text{st}: & t & \geq & \min\{\langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle, 0\} \quad \text{for every } i \in J(\boldsymbol{x}_0) \\ & t & \geq & \langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle \qquad\qquad\quad \text{for every } \ i \in I(\boldsymbol{x}_0) \setminus J(\boldsymbol{x}_0) \\ & \boldsymbol{d} & \in & \mathcal{C} \end{array}$$

with $J(\boldsymbol{x}_0) := \{i \in I(\boldsymbol{x}_0) : g_i(\boldsymbol{x}_0) = \gamma_i\}$. This implies that $\vartheta(S) < 0$ if and only if the optimization problem

$$\begin{array}{llll} \min & t & & \\ \text{st}: & t & \geq & \langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle \quad \text{for every } \ i \in I(\boldsymbol{x}_0) \\ & \boldsymbol{d} & \in & \mathcal{C} \end{array}$$

has a negative objective value. This problem in turn is equivalent to

$$\min_{\boldsymbol{d} \in \mathcal{C}} \max_{i \in I(\boldsymbol{x}_0)} \langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle = \min_{\boldsymbol{d} \in \mathcal{C}} \varphi(\boldsymbol{d})$$

with
$$\varphi(\boldsymbol{d}) := \max\{\langle \boldsymbol{d}, \boldsymbol{y}\rangle : \boldsymbol{y} \in \mathrm{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\})\}.$$
We finally obtain that $\vartheta(S) < 0$ if and only if there exists some $\boldsymbol{d} \in \mathcal{C}$ with $\varphi(\boldsymbol{d}) < 0$ or equivalently $\mathbf{0} \notin \mathrm{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\})$. Observe that for $s = 2$ this decision can be carried out by means of the linear time algorithm presented in [10]. By the definition of φ_- and the representation of $f'(\boldsymbol{x}_0; \boldsymbol{d})$ it follows that
$$\varphi_-(\boldsymbol{d}) = \begin{cases} \varphi(\boldsymbol{d}) & \text{if } \boldsymbol{d} \in \mathrm{cl}(\mathcal{L}^<_\varphi(0)) \\ +\infty & \text{otherwise} \end{cases}$$
and so any $\nabla g_i(\boldsymbol{x}_0)$, $i \in I(\boldsymbol{x}_0)$, belongs to $\partial\varphi_-(\mathbf{0})$. This implies
$$\mathrm{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\}) \subseteq \partial\varphi_-(\mathbf{0})$$
and by Lemma 3.3 the desired result follows. □

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