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Generalization of Jack's lemma for functions with fixed initial coefficient and its applications

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Abstract. In this paper, by using the theory of differential subordination, we will generalize Jack's lemma for functions with fixed initial coefficient. Then extensions of the well-known open-door lemma for analytic and meromorphic functions with fixed initial coefficient are given. Also we consider some applications of the extension of Jack's lemma.

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1. Introduction and preliminaries

Let $\mathcal{H}$ denote the set of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We define

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\},$$

where n is a positive integer number and $a \in \mathbb{C}$. Suppose $n \in \mathbb{N}$, we introduce the subclass $\mathcal{A}_n$ of $\mathcal{H}$ as follows:

$$\mathcal{A}_n = \{f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots\}.$$

In addition to, in particular, we set $\mathcal{A}_1 = \mathcal{A}$. Also we define the subclass $\mathcal{S}$ of $\mathcal{A}$ consisting of univalent functions in the open unit disk $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order $0 \leq \gamma < 1$, written $f \in S^*(\gamma)$, if it satisfies

$$\mathfrak{Re} \frac{zf'(z)}{f(z)} > \gamma \qquad (z \in \mathbb{U}).$$

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Especially we set $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. Now for analytic functions in $\mathbb{U}$ with fixed initial coefficient, we define the class $\mathcal{H}_\beta[a,n]$ as follows:

$$\mathcal{H}_\beta[a,n] = \{f \in \mathcal{H} : f(z) = a + \beta z^n + a_{n+1}z^{n+1} + \dots\},$$

where n is a positive integer number, $a \in \mathbb{C}$ and $\beta \in \mathbb{C}$ is a fixed number. Moreover we assume

$$\mathcal{A}_{n,b} = \{f \in \mathcal{H} : f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \dots\},$$

where n is a positive integer number and $b \in \mathbb{C}$ is a fixed number. Also we set $\mathcal{A}_b = \mathcal{A}_{1,b}$. Let f and g be in $\mathcal{H}$. We say that the function f is subordinate to g, denoted by $f \prec g$, if there exists an analytic function in $\mathbb{U}$ as ω, with $\omega(0) = 0$ and $|\omega(z)| \leq |z| < 1$, such that $f(z) = g(\omega(z))$. Moreover if g is an univalent function in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

It is important to note that coefficients of analytic functions play important role in geometric functions theory. For example, the bound on the second coefficient of an univalent function leads to well-known results such as growth, distortion and covering theorems (see [8]). Recently the subject of second order differential subordination for analytic functions with fixed initial coefficient was considered by Ali et al.[2]. Then in the papers [7, 6, 9] the authors by applying first order differential subordination for functions with fixed initial coefficient related to univalent functions, obtained some good results.

Furthermore in [1], the problem of radius of starlikeness for analytic functions with fixed second coefficient is discussed. Also, Amani et al., [3, 4] have obtained some results for functions with fixed initial coefficient.

Motivated by [3] and [4], in this paper we extend the famous Jake's Lemma for analytic functions with fixed second coefficient.

We organize the contents as follows. In Section 2, we will bring extension of Jack's Lemma and open-door lemma for analytic and meromophic functions with fixed initial coefficient and then we include some corollaries from them. In Section 3, we apply the results in the sections 2, for obtaining some sufficient conditions for starlikeness and carathedory functions.

In the continuation of work, for proving main results, we require to express a definition and a basic lemma.

Definition 1.1. (see [8]) Let Q denote the set of functions q that are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$, where

$$E(q) := \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$.

Lemma 1.2. *(see [2]) Let $q \in Q$ with $q(0) = a$ and $p \in \mathcal{H}_c[a, n]$ with $p(z) \not\equiv a$. If there exist a point $z_0 \in \mathbb{U}$ such that $p(z_0) \in q(\partial\mathbb{U})$ and $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$ then*

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) \tag{1.1}$$

and

$$\mathfrak{Re}\left\{1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq m\mathfrak{Re}\left\{1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\} \tag{1.2}$$

where $q^{-1}(p(z_0)) = \zeta_0 = e^{i\theta_0}$ and

$$m \geq n + \frac{|q'(0)| - |c||z_0|^n}{|q'(0)| + |c||z_0|^n} \tag{1.3}$$

2. Main results

In the beginning, we prove extension of Jake's Lemma [5] as follows:

Theorem 2.1. *Let $c = re^{it}$ with $-\frac{\pi\alpha}{\alpha+\lambda} < t < \frac{\pi\lambda}{\alpha+\lambda}$, where $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$. Also let $0 \leq \beta \leq (\alpha + \lambda)|c|^{\frac{\alpha+\lambda}{2}} \cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})$ and $p \in \mathcal{H}_\beta[c^{\frac{\alpha+\lambda}{2}}, n]$ with $p(z) \neq 0$ in $\mathbb{U}$. If there exist elements $z_1 \in \mathbb{U}$ and $z_2 \in \mathbb{U}$ such that $|z_1| = |z_2| = r$ and for all $z \in \mathbb{U}_r = \{z \in \mathbb{C}, |z| < r\}$*

$$-\frac{\pi\alpha}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\lambda}{2}, \tag{2.1}$$

then we have

$$z_1 p'(z_1) = -i\frac{\lambda+\alpha}{2} m_1 p(z_1), \tag{2.2}$$

and

$$z_2 p'(z_2) = i\frac{\lambda+\alpha}{2} m_2 p(z_2), \tag{2.3}$$

where

$$m_1 > \left(n + \frac{|c|^{\frac{\alpha+\lambda}{2}} \cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}} \cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}\right) \frac{1 + \sin(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}, \tag{2.4}$$

and

$$m_2 > \left(n + \frac{|c|^{\frac{\alpha+\lambda}{2}} \cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}} \cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}\right) \frac{1 - \sin(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}. \tag{2.5}$$

Proof. Let us define

$$q(z) = \exp\left\{\frac{\pi i(\lambda-\alpha)}{4}\right\}\left(\frac{c_1 + \bar{c}_1 z}{1-z}\right)^{\frac{\lambda+\alpha}{2}}$$

with $c_1 = c\exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}$. It is easy to find that q is analytic in $\mathbb{U}$, $q(0) = c^{\frac{\lambda+\alpha}{2}}$ and

$$-\frac{\pi\alpha}{2} < \arg q(\mathbb{U}) < \frac{\pi\lambda}{2},$$

moreover $q \in Q$ and $E(q) = 1$. Upon assumption and the properties of the function q, we have $p(z_1) \in q(\partial\mathbb{U})$ and $p(z_2) \in q(\partial\mathbb{U})$, also $p(\{z : |z| < r\}) \subset q(\mathbb{U})$. Define

$$p_1(z) = \exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}\{p(z)\}^{\frac{2}{\lambda+\alpha}} \qquad (z \in \mathbb{U}),$$

and

$$q_1(z) = \frac{c_1 + \bar{c_1} z}{1-z} \qquad (z \in \mathbb{U}),$$

with $c_1 = c\exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}$. Then it can be readily considered that $q_1 \in Q$, $q_1(0) = p_1(0)$, $q_1(\mathbb{U}) = \{w \in \mathbb{C} : \mathfrak{Re} w > 0\}$(note that $\mathfrak{Re} c_1 > 0$) and $p_1(\{z : |z| < r\}) \subset q_1(\mathbb{U})$. Also $p_1(z_1) = -ix_1$ and $p_1(z_2) = ix_2$, with $x_1, x_2 > 0$. By means of calculating the inverse of q_1 and obtaining the derivative of q_1, we reach to

$$q_1^{-1}(z) = \frac{z-c_1}{z+\bar{c_1}} \quad \textit{and} \quad q_1'(z) = \frac{2\mathfrak{Re} c_1}{(1-z)^2}.$$

On the other hand, since $p \in \mathcal{H}_\beta[c^{\frac{\alpha+\lambda}{2}}, n]$, we have $p_1 \in \mathcal{H}_{c_2}[a, n]$, with

$$a = c\exp\left\{\frac{\pi i(\alpha-\lambda)}{2(\lambda+\alpha)}\right\} = c_1 \quad \textit{and} \quad c_2 = \frac{2c^{\frac{2-\alpha-\lambda}{2}}\beta}{\alpha+\lambda}\exp\left\{\frac{\pi i(\alpha-\lambda)}{2(\lambda+\alpha)}\right\}.$$

Hence by applying Lemma 1.1 we deduce that there exist complex numbers ζ_1 and ζ_2 in $\partial\mathbb{U}$ such that $p_1(z_1) = q_1(\zeta_1)$ and $p_1(z_2) = q_1(\zeta_2)$ and also

$$z_1 p_1'(z_1) = k_1\zeta_1 q_1'(\zeta_1) \quad \textit{and} \quad z_2 p_1'(z_2) = k_2\zeta_2 q_1'(\zeta_2),$$

where

$$k_1 \geq n + \frac{|q_1'(0)| - |c_2||z_1|^n}{|q_1'(0)| + |c_2||z_1|^n} \quad \textit{and} \quad k_2 \geq n + \frac{|q_1'(0)| - |c_2||z_2|^n}{|q_1'(0)| + |c_2||z_2|^n}.$$

Since $p_1(z_1) = -ix_1$ with $x_1 > 0$ and $\zeta_1 = q_1^{-1}(p_1(z_1)) = \frac{ix_1+c_1}{ix_1-\bar{c_1}}$, we have

$$\begin{aligned}
\frac{z_1 p'(z_1)}{p(z_1)} &= \frac{\lambda+\alpha}{2}\frac{z_1 p_1'(z_1)}{p_1(z_1)} \\
&= \frac{\lambda+\alpha}{2}\frac{k_1\zeta_1 q_1'\zeta_1)}{p_1(z_1)} \\
&= k_1\frac{\lambda+\alpha}{2}\frac{ix_1+c_1}{ix_1-\bar{c_1}} \times \frac{1}{-ix_1} \times \frac{2\mathfrak{Re} c_1}{(1-\frac{ix_1+c_1}{ix_1-\bar{c_1}})^2} \\
&= k_1\frac{\lambda+\alpha}{2}\frac{1}{ix_1} \times \frac{x_1^2 + 2x_1\mathfrak{Im} c_1 + |c_1|^2}{2\mathfrak{Re} c_1} \\
&= -ik_1\left(\frac{\lambda+\alpha}{2}\right)\frac{x_1^2 + 2|c|x_1\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}.
\end{aligned}$$

Set

$$f(x) = \frac{x^2 + 2|c|x\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})} \qquad (x > 0).$$

By computing, it can be easily observed that

$$\min_{x>0} f(x) = f(|c|) = \frac{1+\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}.$$

Now using $q_1'(0) = 2|c|\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})$ and $|c_2| = \frac{2\beta|c|^{\frac{2-\alpha-\lambda}{2}}}{\lambda+\alpha}$, we obtain

$$m_1 = k_1 f(x_1) > \left(n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}\right)\frac{1+\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}$$

Thus assertions (2.2) and (2.4) hold. Now similar to the procedure of the former case, since $p_1(z_2) = ix_2$, with $x_2 > 0$ and $\zeta_2 = q_1^{-1}(ix_2) = \frac{ix_2-c_1}{ix_2+\bar{c_1}}$ we can obtain

$$\begin{aligned}
\frac{z_2p'(z_2)}{p(z_2)} &= \frac{\lambda+\alpha}{2}\frac{z_2p_1'(z_2)}{p_1(z_2)} \\
&= \frac{\lambda+\alpha}{2}\frac{k_2\zeta_2q_1'\zeta_2)}{p_1(z_2)} \\
&= k_2\frac{\lambda+\alpha}{2}\frac{ix_2-c_1}{ix_2+\bar{c_1}}\times\frac{1}{ix_2}\times\frac{2\mathfrak{Re}c_1}{(1-\frac{ix_2-c_1}{ix_2+\bar{c_1}})^2} \\
&= k_2\frac{\lambda+\alpha}{2}\frac{1}{ix_2}\times\frac{-x_2^2+2x_2\mathfrak{Im}c_1-|c_1|^2}{2\mathfrak{Re}c_1} \\
&= ik_2\left(\frac{\lambda+\alpha}{2}\right)\frac{x_2^2-2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})+|c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}.
\end{aligned}$$

Set

$$g(x) = \frac{x^2-2|c|x\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})+|c|^2}{2|c|x\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})} \qquad (x>0).$$

By computing, we have

$$\min_{x>0} g(x) = g(|c|) = \frac{1-\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}.$$

Thus in view of $q_1'(0) = 2|c|\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})$ and $|c_2| = \frac{2\beta|c|^{\frac{2-\alpha-\lambda}{2}}}{\lambda+\alpha}$, as the former case, we can conclude assertions (2.3) and (2.5). $\square$

Remark 2.2. Note that the above theorem extends Theorem 2.1 obtained in [3].

By applying the same trend of Theorem 2.1 and putting $\alpha = \lambda$ in this theorem, we obtain

Corollary 2.3. *Let $c = re^{it}$ be a complex number with $\mathfrak{Re}c > 0$. Let $0 \le \beta \le 2\lambda|c|^\lambda\cos t$ and $p \in \mathcal{H}_\beta[c^\lambda, n]$ with $p(z) \neq 0$ in $\mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\lambda\pi}{2} \qquad \textit{for} \ \ |z| < |z_0|,$$

and $p(z_0)^{\frac{1}{\lambda}} = \pm ia$, *where* $a > 0$ *and* $0 < \lambda \leq 1$, *Then we have*

$$z_0 p'(z_0) = im\lambda p(z_0),$$

where

$$m > \frac{a^2 - 2a|c|sint + |c|^2}{2a|c|\cos t}\left(n + \frac{|c|^\lambda \cos t - \frac{\beta}{2\lambda}}{|c|^\lambda \cos t + \frac{\beta}{2\lambda}}\right) \quad when \quad \arg p(z_0) = \frac{\lambda\pi}{2},$$

and

$$m < -\frac{a^2 + 2a|c|sint + |c|^2}{2a|c|\cos t}\left(n + \frac{|c|^\lambda \cos t - \frac{\beta}{2\lambda}}{|c|^\lambda \cos t + \frac{\beta}{2\lambda}}\right) \quad when \quad \arg p(z_0) = \frac{-\lambda\pi}{2}.$$

By putting $\lambda = 1$ in Corollary 2.1, we have

Corollary 2.4. *Let* $c = re^{it}$ *be a complex number with* $\mathfrak{Re}c > 0$. *Let* $0 \leq \beta \leq 2\mathfrak{Re}c$ *and* $p \in \mathcal{H}_\beta[c, n]$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg p(z)| < \frac{\pi}{2} \quad for \ \ |z| < |z_0|,$$

and $p(z_0) = \pm ia$ *where* $a > 0$, *Then we have*

$$z_0 p'(z_0) = imp(z_0),$$

where

$$m > \frac{a^2 - 2a\mathfrak{Im}p(0) + |p(0)|^2}{2a\mathfrak{Re}p(0)}\left(n + \frac{2\mathfrak{Re}p(0) - \beta}{2\mathfrak{Re}p(0) + \beta}\right) \quad when \quad \arg p(z_0) = \frac{\pi}{2},$$

and

$$m < -\frac{a^2 + 2a\mathfrak{Im}p(0) + |p(0)|^2}{2a\mathfrak{Re}p(0)}\left(n + \frac{2\mathfrak{Re}p(0) - \beta}{2\mathfrak{Re}p(0) + \beta}\right) \quad when \quad \arg p(z_0) = -\frac{\pi}{2}.$$

Remark 2.5. Letting $p \in \mathcal{H}[c, 1]$ in corollary 2.2 and using the corrections needed in this Corollary, one can gain Theorem 2.1 in [11].

By setting $c = 1$ in Corollary 2.2, we attain

Corollary 2.6. *Let* $p \in \mathcal{H}_\beta[1, n]$ *and* $0 \leq \beta \leq 2$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg p(z)| < \frac{\pi}{2} \quad for \ \ |z| < |z_0|,$$

and $p(z_0) = \pm ia$ *where* $a > 0$, *Then we have*

$$z_0 p'(z_0) = imp(z_0),$$

where

$$m > \frac{1}{2}(a + a^{-1})\left(n + \frac{2 - \beta}{2 + \beta}\right) \quad when \quad \arg p(z_0) = \frac{\pi}{2},$$

and

$$m < -\frac{1}{2}(a + a^{-1})\left(n + \frac{2 - \beta}{2 + \beta}\right) \quad when \quad \arg p(z_0) = -\frac{\pi}{2}.$$

Remark 2.7. Letting $p \in \mathcal{H}[1, 1]$ in Corollary 2.3 and implying the alternations required in this corollary, we can obtain Theorem 1 in [10].

Theorem 2.8. *(extension of open door Lemma) Let* $c = re^{it}$ *with* $-\frac{\pi\alpha}{\alpha+\lambda} < t < \frac{\pi\lambda}{\alpha+\lambda}$, *where* $0 < \alpha \leq 1$ *and* $0 < \lambda \leq 1$. *Also let* $0 \leq \beta \leq (\alpha+\lambda)|c|^{\frac{\alpha+\lambda}{2}} \cos B$ *and* $p \in \mathcal{H}_\beta[c^{\frac{\alpha+\lambda}{2}}, n]$ *with* $p(z) \neq 0$ *in* $\mathbb{U}$. *If*

$$\gamma p(z)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{zp'(z)}{p(z)} \neq iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$ *where*

$$y > \frac{\sqrt{M}}{cosB}(\sqrt{M + 2|c|\cos B} - \sqrt{M}\sin B),$$

or

$$y < -\frac{\sqrt{M}}{cosB}(\sqrt{M + 2|c|\cos B} + \sqrt{M}\sin B),$$

then

$$-\frac{\alpha\pi}{2} < \arg p(z) < \frac{\lambda\pi}{2} \qquad (z \in \mathbb{U}), \tag{2.6}$$

where $\gamma = \exp\{-i\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}\}$, $B = t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}$ *and* $M = n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}$.

Proof. Let us set

$$p_1(z) = \exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}\{p(z)\}^{\frac{2}{\lambda+\alpha}} \qquad (z \in \mathbb{U}),$$

and

$$q_1(z) = \frac{c_1 + \bar{c_1}z}{1-z} \qquad (z \in \mathbb{U}),$$

where $c_1 = c\exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}$. We know that $p_1 \in \mathcal{H}_{c_2}[a,n]$, with

$$a = c\exp\left\{\frac{\pi i(\alpha-\lambda)}{2(\lambda+\alpha)}\right\} = c_1 \qquad and \qquad c_2 = \frac{2c^{\frac{2-\alpha-\lambda}{2}}\beta}{\alpha+\lambda}\exp\left\{\frac{\pi i(\alpha-\lambda)}{2(\lambda+\alpha)}\right\}.$$

and $p_1(0) = q_1(0)$. If $p(\mathbb{U})$ is not contained in the sector $\{w : -\frac{\pi\alpha}{2} < \arg w < \frac{\pi\lambda}{2}\}$, then $p_1\mathbb{U})$ is not contained in the right half plane $\mathfrak{Re}w > 0$. On the other hand $q_1(\mathbb{U}) = \{w : \mathfrak{Re}w > 0\}$, thus we follow that $p_1 \not\prec q_1$, then there exists a point $z_1 \in \mathbb{U}$ such that $p_1(\{z : |z| < |z_1|\}) \subset q_1(\mathbb{U})$ and $p_1(z_1) = -ix_1$ or $p_1(z_1) = ix_2$ with $x_1, x_2 > 0$. Let $p_1(z_1) = -ix_1$, with $x_1 > 0$. Similar to the argument of Theorem 2.1 we have

$$\frac{z_1p'(z_1)}{p(z_1)} = -ik_1\left(\frac{\lambda+\alpha}{2}\right)\frac{x_1^2 + 2|c|x_1\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})},$$

where $k_1 > M$. Then it yields

$$\begin{aligned}
&\mathfrak{Im}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{zp'(z_1)}{p(z_1)}\right\} \\
&= \mathfrak{Im}\left\{-ix_1 - ik_1\frac{x_1^2 + 2|c|x_1\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}\right\} \\
&= -(x_1 + k_1\frac{x_1^2 + 2|c|x_1\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}) \\
&< -(x_1 + M\frac{x_1^2 + 2|c|x_1\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})})
\end{aligned}$$

Suppose

$$f(x) = x + M\frac{x^2 + 2|c|x\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})} \qquad (x > 0).$$

By computing, we can readily find that

$$\min_{x>0} f(x) = f\left(\frac{|c|\sqrt{M}}{\sqrt{M+2|c|\cos B}}\right) = \frac{\sqrt{M}}{\cos B}\left(\sqrt{M+2|c|\cos B} + \sqrt{M}\sin B\right),$$

this implies that

$$\mathfrak{Im}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{z_1p'(z_1)}{p(z_1)}\right\} < -\frac{\sqrt{M}}{cosB}(\sqrt{M+2|c|\cos B} + \sqrt{M}\sin B),$$

where $\gamma = \exp\{-i\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}\}$, $B = t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}$ and $M = n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}$.
On the other hand we have

$$\mathfrak{Re}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{z_1p'(z_1)}{p(z_1)}\right\} = 0,$$

that this contradicts with the hypothesis. For the case $p_1(z_1) = ix_2$, Similar to the argument of Theorem 2.1 we have

$$\frac{z_1p'(z_1)}{p(z_1)} = ik_2\left(\frac{\lambda+\alpha}{2}\right)\frac{x_2^2 - 2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})},$$

where $k_2 > M$. Then it yields

$$\begin{aligned}
&\mathfrak{Im}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{zp'(z_1)}{p(z_1)}\right\} \\
&= \mathfrak{Im}\left\{ix_2 + ik_2\frac{x_2^2 - 2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}\right\} \\
&= x_2 + k_2\frac{x_2^2 - 2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})} \\
&> x_2 + M\frac{x_2^2 - 2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}
\end{aligned}$$

Suppose

$$g(x) = x + M\frac{x^2 - 2|c|x\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})} \qquad (x > 0).$$

By computing we can easily conclude that

$$\min_{x>0} g(x) = g\left(\frac{|c|\sqrt{M}}{\sqrt{M+2|c|\cos B}}\right) = \frac{\sqrt{M}}{\cos B}\left(\sqrt{M+2|c|\cos B} - \sqrt{M}\sin B\right),$$

thus we have

$$\mathfrak{Im}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{zp'(z_1)}{p(z_1)}\right\} > \frac{\sqrt{M}}{cosB}(\sqrt{M+2|c|\cos B} - \sqrt{M}\sin B),$$

where $\gamma = \exp\{-i\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}\}$, $B = t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}$ and $M = n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}$.
On the other hand we have

$$\mathfrak{Re}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{z_1p'(z_1)}{p(z_1)}\right\} = 0,$$

that this contradicts with the hypothesis. Hence the assertion (2.6) holds. □

Remark 2.9. we note that Theorem 2.2 extends Theorem 2.1 in [4]

Also we can write the other version of extension of open door Lemma as follows:

Corollary 2.10. *Let* $c = re^{it}$ *be a complex number with* $\mathfrak{Re}c > 0$. *Also Let* $0 < \lambda \le 1$, $0 \le \beta \le 2\lambda|c|^\lambda \cos t$ *and* $p \in \mathcal{H}_\beta[c^\lambda, n]$ *with* $p(z) \ne 0$ *in* $\mathbb{U}$. *If*

$$p(z)^{\frac{1}{\lambda}} + \frac{1}{\lambda}\frac{zp'(z)}{p(z)} \ne iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$, *where*

$$y > \frac{\sqrt{M}}{\cos t}\left(\sqrt{M+2|c|\cos t} - \sqrt{M}\sin t\right),$$

or

$$y < -\frac{\sqrt{M}}{\cos t}\left(\sqrt{M+2|c|\cos t} + \sqrt{M}\sin t\right),$$

then

$$-\frac{\lambda\pi}{2} < \arg p(z) < \frac{\lambda\pi}{2} \qquad (z \in \mathbb{U}),$$

where $M = n + \frac{|c|^\lambda \cos t - \frac{\beta}{2\lambda}}{|c|^\lambda \cos t + \frac{\beta}{2\lambda}}$.

Proof. The proof of this corollary is similar to that of Theorem 2.2 (put $\alpha = \lambda$), so we omit its details. □

Corollary 2.11. *Let $f \in \mathcal{A}_{n,b}$ with $f(z)f'(z) \neq 0$ in $\mathbb{U} - \{0\}$. Also let $\alpha + \lambda = \frac{2}{t_1}$ with $t_1 \geq 1$ and $0 \leq b \leq \frac{2}{n}\cos\{-\frac{\pi t_1(\lambda-\alpha)}{4}\}$. If*

$$(\gamma - 1)\frac{zf'(z)}{f(z)} + (1 + \frac{zf''(z)}{f'(z)}) \neq iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$ where

$$y > \frac{\sqrt{M}}{\cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}}\left(\sqrt{M + \cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}} - \sqrt{M}\sin\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}\right),$$

or

$$y < -\frac{\sqrt{M}}{\cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}}\left(\sqrt{M + \cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}} + \sqrt{M}\sin\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}\right),$$

then

$$-\frac{\pi}{2}\alpha t_1 < \arg\frac{zf'(z)}{f(z)} < \frac{\pi}{2}\lambda t_1 \qquad (z \in \mathbb{U}),$$

where $\gamma = \exp(-i\pi\frac{t_1(\lambda-\alpha)}{4})$ *and* $M = n + \frac{\cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\} - \frac{nb}{2}}{\cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\} + \frac{nb}{2}}$.

Proof. Let $p(z) = (\frac{zf'(z)}{f(z)})^{\frac{1}{t_1}}$, then we have $p \in \mathcal{H}_{\frac{nb}{t_1}}[1, n]$ with $p(z) \neq 0$ in $\mathbb{U}$. Then with applying Theorem 2.2 and with letting $c = 1$, $t = 0$, $\alpha + \lambda = \frac{2}{t_1}$ and $\beta = \frac{nb}{t_1}$ in this theorem, the proof is complete. □

Theorem 2.12. *Let $c = re^{it}$ with $-\frac{\pi\alpha}{\alpha+\lambda} < t < \frac{\pi\lambda}{\alpha+\lambda}$, where $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$. Also let $M > \frac{2|c|}{\cos B}$, $0 \leq \beta \leq (\alpha+\lambda)|c|^{\frac{\alpha+\lambda}{2}}\cos B$ and $p \in \mathcal{H}_\beta[c^{\frac{\alpha+\lambda}{2}}, n]$ with $p(z) \neq 0$ in $\mathbb{U}$. If*

$$\gamma p(z)^{\frac{2}{\alpha+\lambda}} - \frac{2}{\alpha+\lambda}\frac{zp'(z)}{p(z)} \neq iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$ where

$$y > \frac{\sqrt{M}}{cosB}(\sqrt{M - 2|c|\cos B} + \sqrt{M}\sin B),$$

or

$$y < -\frac{\sqrt{M}}{cosB}(\sqrt{M - 2|c|\cos B} - \sqrt{M}\sin B),$$

then

$$-\frac{\alpha\pi}{2} < \arg p(z) < \frac{\lambda\pi}{2} \qquad (z \in \mathbb{U}),$$

where $\gamma = \exp\{-i\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}\}$, $B = t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}$ *and* $M = n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})-\frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})+\frac{\beta}{\lambda+\alpha}}$.

Proof. The proof of this theorem is similar to Theorem 2.2, and we omit its details. □

Corollary 2.13. *Let* $f(z) = \frac{1}{z} + \beta z^n + \dots$ *be a meromorphic function with* $f'f \neq 0$ *in* $\mathbb{U} - \{0\}$*. Also let* $-\frac{2}{(n+1)} \leq \beta \leq 0$ *and* $M > 2$*. If*

$$-1 - \frac{zf''(z)}{f'(z)} \neq iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$ *where*

$$y > \sqrt{M}(\sqrt{M-2}),$$

or

$$y < -\sqrt{M}(\sqrt{M-2}),$$

then we have

$$-\frac{\pi}{2} < \arg\left\{-\frac{zf'(z)}{f(z)}\right\} < \frac{\pi}{2} \qquad (z \in \mathbb{U}),$$

where $M = (n+1) + \frac{2+(n+1)\beta}{2-(n+1)\beta}$.

Proof. Let $p(z) = -\frac{zf'(z)}{f(z)}$, then $p \in \mathcal{H}_{\beta_1}[1, n+1]$ with $\beta_1 = -(n+1)\beta > 0$. With a simple computation we obtain

$$p(z) - \frac{zp'(z)}{p(z)} = -1 - \frac{zf''(z)}{f'(z)} \qquad (z \in \mathbb{U}).$$

Then with using Theorem 2.3 and with letting $c = 1$, $t = 0$, $\alpha = \lambda = 1$ and also with substituting β by β_1 in this theorem, we obtain this result and the proof is complete. □

3. Further applications related to extension of Jake's Lemma

Corollary 3.1. *Let* $0 < \lambda \leq 1$, $c \in \mathbb{C}$ *and* β_1 *be a real number such that* $(c^\lambda - \beta_1)^{\frac{1}{\lambda}} = re^{it}$ *with* $\mathfrak{Re}(c^\lambda - \beta_1)^{\frac{1}{\lambda}} > 0$*. Suppose* $0 \leq \beta \leq 2\lambda|c^\lambda - \beta_1|\cos t$ *and* $p \in \mathcal{H}_\beta[c^\lambda, n]$ *with* $p(z) \neq \beta_1$ *in* $\mathbb{U}$*. If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg(p(z) - \beta_1)| < \frac{\lambda\pi}{2} \qquad for \ \ |z| < |z_0|,$$

and $(p(z_0) - \beta_1)^{\frac{1}{\lambda}} = \pm ia$*, where* $a > 0$*, Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0) - \beta_1} = im\lambda,$$

where for $\arg\{p(z_0) - \beta_1\} = \frac{\lambda\pi}{2}$

$$m > \frac{a^2 - 2a\mathfrak{Im}(c^\lambda - \beta_1)^{\frac{1}{\lambda}} + |c^\lambda - \beta_1|^{\frac{2}{\lambda}}}{2a\mathfrak{Re}(c^\lambda - \beta_1)^{\frac{1}{\lambda}}}\left(n + \frac{|c^\lambda - \beta_1|\cos t - \frac{\beta}{2\lambda}}{|c^\lambda - \beta_1|\cos t + \frac{\beta}{2\lambda}}\right),$$

and for $\arg\{p(z_0)-\beta_1\} = -\frac{\lambda\pi}{2}$

$$m < -\frac{a^2 + 2a\mathfrak{Im}(c^\lambda-\beta_1)^{\frac{1}{\lambda}} + |c^\lambda-\beta_1|^{\frac{2}{\lambda}}}{2a\mathfrak{Re}(c^\lambda-\beta_1)^{\frac{1}{\lambda}}}\left(n + \frac{|c^\lambda-\beta_1|\cos t - \frac{\beta}{2\lambda}}{|c^\lambda-\beta_1|\cos t + \frac{\beta}{2\lambda}}\right).$$

Proof. It is sufficient that we consider $q(z) = p(z) - \beta_1$. Then $q(z) \in \mathcal{H}_\beta[c_1^\lambda, n]$ with $c_1 = (c^\lambda - \beta_1)^{\frac{1}{\lambda}}$. Also from the hypothesis we have $\mathfrak{Re} c_1 > 0$ and there exists a point $z_0 \in \mathbb{U}$ such that $|\arg q(z)| < \frac{\lambda\pi}{2}$ for $|z| < |z_0|$ and $q(z_0)^{\frac{1}{\lambda}} = \pm ia$. Now using Corollary 2.1 for q, we get the result and the proof is complete. □

By using Corollary 3.1, we obtain

Corollary 3.2. *Let* $f \in \mathcal{A}_{n,b}$ *with* $\frac{f(z)}{z} \neq \beta$ *in* $\mathbb{U}$. *Suppose* $0 \le \beta < 1$ *and* $0 \le b \le 2(1-\beta)$. *If*

$$\frac{zf'(z) - f(z)}{f(z) - \beta z} \neq is \qquad (z \in \mathbb{U}),$$

for all $s \in \mathbb{R}$ *where* $|s| > n + \frac{2(1-\beta)-b}{2(1-\beta)+b}$, *then we have* $\mathfrak{Re}\frac{f(z)}{z} > \beta$.

Proof. Let us define $p(z) = \frac{f(z)}{z}$, then $p \in \mathcal{H}_b[1,n]$. Let there exists a point $z_0 \in \mathbb{U}$ such that $\mathfrak{Re} p(z) > \beta$ for $|z| < |z_0|$ and $\mathfrak{Re} p(z_0) = \beta$, so $|\arg(p(z) - \beta)| < \frac{\pi}{2}$ for $|z| < |z_0|$ and $p(z_0) = \beta \pm ia$, where $a > 0$. Now applying Corollary 3.1, we have

$$\frac{z_0 f'(z_0) - f(z_0)}{f(z_0) - \beta z_0} = \frac{z_0 p'(z_0)}{p(z_0) - \beta} = im \qquad (z \in \mathbb{U}),$$

where for $p(z_0) - \beta = ia$

$$m > \frac{a^2 - (1-\beta)^2}{2a(1-\beta)}\left(n + \frac{2(1-\beta)-b}{2(1-\beta)+b}\right) \ge \left(n + \frac{2(1-\beta)-b}{2(1-\beta)+b}\right),$$

and for $p(z_0) - \beta = -ia$

$$m < -\frac{a^2 - (1-\beta)^2}{2a(1-\beta)}\left(n + \frac{2(1-\beta)-b}{2(1-\beta)+b}\right) \le -\left(n + \frac{2(1-\beta)-b}{2(1-\beta)+b}\right),$$

which contradicts with the hypothesis. Hence the proof is complete. □

Also similar to Corollary 3.1, we can conclude

Corollary 3.3. *Let* $0 < \lambda \le 1$, $c \in \mathbb{C}$ *and* β_1 *be a real number such that* $(\beta_1 - c)^{\frac{1}{\lambda}} = re^{it}$ *with* $\mathfrak{Re}(\beta_1 - c)^{\frac{1}{\lambda}} > 0$. *Suppose* $-2\lambda|\beta_1 - c|\cos t \le \beta \le 0$ *and* $p \in \mathcal{H}_\beta[c,n]$ *with* $p(z) \neq \beta_1$ *in* $\mathbb{U}$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg(\beta_1 - p(z))| < \frac{\lambda\pi}{2} \quad \text{for } |z| < |z_0|,$$

and $(\beta_1 - p(z_0))^{\frac{1}{\lambda}} = \pm ia$, *where* $a > 0$, *Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0) - \beta_1} = im\lambda,$$

where for $\arg\{\beta_1 - p(z_0)\} = \frac{\lambda\pi}{2}$

$$m > \frac{a^2 - 2a\mathfrak{Im}(\beta_1 - c)^{\frac{1}{\lambda}} + |\beta_1 - c|^{\frac{2}{\lambda}}}{2a\mathfrak{Re}(\beta_1 - c)^{\frac{1}{\lambda}}} \left(n + \frac{|\beta_1 - c|\cos t + \frac{\beta}{2\lambda}}{|\beta_1 - c|\cos t - \frac{\beta}{2\lambda}} \right)$$

and for $\arg\{\beta_1 - p(z_0)\} = -\frac{\lambda\pi}{2}$

$$m < -\frac{a^2 + 2a\mathfrak{Im}(\beta_1 - c)^{\frac{1}{\lambda}} + |\beta_1 - c|^{\frac{2}{\lambda}}}{2a\mathfrak{Re}(\beta_1 - c)^{\frac{1}{\lambda}}} \left(n + \frac{|\beta_1 - c|\cos t + \frac{\beta}{2\lambda}}{|\beta_1 - c|\cos t - \frac{\beta}{2\lambda}} \right).$$

Proof. It is sufficient to consider $q(z) = \beta_1 - p(z)$. The rest of the proof is similar to the proof of Corollary 3.1. □

The same as Corollary 3.2 and by applying Corollary 3.3, we can obtain the following Corollary.

Corollary 3.4. *Let* $\beta > 1$ *and* $-2(\beta - 1) \leq b \leq 0$. *Suppose* $f \in \mathcal{A}_{n,b}$ *with* $\frac{f(z)}{z} \neq \beta$. *in* $\mathbb{U}$. *If*

$$\frac{zf'(z) - f(z)}{f(z) - \beta z} \neq is \qquad (z \in \mathbb{U}),$$

for all $s \in \mathbb{R}$ *where* $|s| > n + \frac{2(\beta-1)+b}{2(\beta-1)-b}$, *then we have*

$$\mathfrak{Re}\frac{f(z)}{z} < \beta.$$

Theorem 3.5. *Let* c, β_1 *and* γ *be real numbers with* $c^\alpha - \beta_1 > 0$. *Suppose* $\gamma > 0$, $0 < \alpha \leq 1$, $0 \leq \beta_1 < 1$ *and* $0 \leq \beta \leq 2\alpha(c^\alpha - \beta_1)$. *If* $p \in \mathcal{H}_\beta[c^\alpha, n]$ *with* $p(z) \neq \beta_1$ *in* $\mathbb{U}$ *and*

$$|\arg(p(z) - \beta_1 + \gamma zp'(z))| \leq \frac{\pi}{2}(\alpha + \frac{2}{\pi}\tan^{-1}(\alpha\gamma s)) \qquad (z \in \mathbb{U}),$$

then

$$|\arg(p(z) - \beta_1)| < \frac{\pi}{2}\alpha \qquad \in \mathbb{U},$$

where $s = n + \left(\frac{(c^\alpha - \beta_1) - \frac{\beta}{2\alpha}}{(c^\alpha - \beta_1) + \frac{\beta}{2\alpha}}\right)$.

Proof. If there exists a point $z_0 \in \mathbb{U}$ such that $|\arg(p(z) - \beta_1)| < \frac{\pi}{2}\alpha$ for $|z| < |z_0|$ and $|\arg(p(z_0) - \beta_1)| = \frac{\pi}{2}\alpha$, then from Corollary 3.1 we have

$$\frac{z_0 p'(z_0)}{p(z_0) - \beta_1} = i\alpha m,$$

where

$$|m| > \left(n + \frac{(c^\alpha - \beta_1) - \frac{\beta}{2\alpha}}{(c^\alpha - \beta_1) + \frac{\beta}{2\alpha}} \right) = s.$$

Thus for the case $\arg(p(z_0)-\beta_1)=\frac{\pi}{2}\alpha$ we have

$$\begin{aligned}\arg\{p(z_0)-\beta_1+\gamma z_0p'(z_0)\} &= \arg\left\{(p(z_0)-\beta_1)(1+\gamma\frac{z_0p'(z_0)}{p(z_0)-\beta_1})\right\}\\ &= \frac{\pi}{2}\alpha+\arg\{1+i\gamma\alpha m\}\\ &> \frac{\pi}{2}\alpha+\tan^{-1}(\gamma\alpha s)\end{aligned}$$

which contradicts with the hypothesis. Also for the case $\arg(p(z_0)-\beta_1)=-\frac{\pi}{2}\alpha$ we have

$$\begin{aligned}\arg\{p(z_0)-\beta_1+\gamma z_0p'(z_0)\} &= \arg\left\{(p(z_0)-\beta_1)(1+\gamma\frac{z_0p'(z_0)}{p(z_0)-\beta_1})\right\}\\ &= -\frac{\pi}{2}\alpha+\arg\{1+i\gamma\alpha m\}\\ &< -(\frac{\pi}{2}\alpha+\tan^{-1}(\gamma\alpha s))\end{aligned}$$

which contradicts with the hypothesis. Hence the proof is complete. □

By putting $c=\gamma=\alpha=n=1$ in Theorem 3.1 we have

Corollary 3.6. *Let $0\le\beta_1<1$ be a real number and $0\le\beta\le 2(1-\beta_1)$. If $p(z)=1+\beta z+\dots$ with $p(z)\neq\beta_1$ in $\mathbb{U}$ and*

$$|\arg(p(z)-\beta_1+zp'(z))|\le\frac{\pi}{2}+\tan^{-1}\left\{\frac{4-4\beta_1}{(2-2\beta_1)+\beta}\right\}\qquad (z\in\mathbb{U}),$$

then

$$\mathfrak{Re}\,p(z)>\beta_1\qquad z\in\mathbb{U}.$$

Remark 3.7. Letting $p\in\mathcal{H}[1,1]$ in the Corollary 3.5 and applying the reforms required in this corollary, we can obtain Theorem 3 in [13].

Theorem 3.8. *Let $-\lambda<b<\lambda$, $\lambda>0$ and $k>0$. Also let $p\in\mathcal{H}_\beta[1,n]$ with $p(z)\neq\frac{2\lambda}{b+\lambda}$ in $\mathbb{U}$ and $0\le\beta\le 1-\frac{b}{\lambda}$. If for all $z\in\mathbb{U}$*

$$\mathfrak{Re}\left\{p(z)+k\frac{zp'(z)}{p(z)}\right\}\le\begin{cases} Mk\frac{\lambda+b}{2(\lambda-b)} & if\quad -\lambda<b\le 0, M\ge\frac{2(\lambda-b)}{k(\lambda+b)}\\ \frac{Mk}{2}\frac{\lambda-b}{\lambda+b}+\frac{2\lambda}{\lambda+b} & if\quad \frac{\lambda}{1+kM}\le b<\lambda\\ \frac{Mk}{2}\frac{\lambda-b}{\lambda+b} & if\quad 0<b<\frac{\lambda}{1+kM}, M\ge\frac{2(\lambda+b)}{k(\lambda-b)},\end{cases}$$

then we have

$$\left|p(z)-\frac{\lambda}{b+\lambda}\right|<\frac{\lambda}{b+\lambda}\qquad z\in\mathbb{U},$$

where $M=n+\frac{\frac{\lambda-b}{\lambda}-\beta}{\frac{\lambda-b}{\lambda}+\beta}$.

Proof. Let us define

$$q(z)=\frac{\lambda(1-z)}{\lambda-bz}.$$

One can easily observe that $q \in Q$ with $q(0) = p(0) = 1$ and q maps the open unit disc $\mathbb{U}$ onto the disk with the center $\frac{\lambda}{\lambda+b}$ and the radius $\frac{\lambda}{\lambda+b}$. Moreover

$$q^{-1}(z) = \frac{\lambda(z-1)}{bz-\lambda} \qquad and \qquad q'(z) = \frac{\lambda(b-\lambda)}{(\lambda-bz)^2}.$$

We claim that $p \prec q$, otherwise if $p \not\prec q$, then there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that $p(z_0) = q(\zeta_0)$ and $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$. Therefore from lemma 1.1 we have

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0).$$

where

$$m \geq n + \frac{|q'(0)| - |\beta||z_0|^n}{|q'(0)| + |\beta||z_0|^n} > n + \frac{\lambda - b - \beta\lambda}{\lambda - b + \beta\lambda} = M.$$

Since

$$\zeta_0 = q^{-1}(p(z_0)) = \frac{\lambda(p(z_0)-1)}{bp(z_0)-\lambda},$$

we have

$$z_0 p'(z_0) = -m\frac{(1-p(z_0))(\lambda - bp(z_0))}{(\lambda-b)}.$$

Set

$$p(z_0) = \frac{\lambda}{\lambda+b} + \frac{\lambda}{\lambda+b}e^{it},$$

for a fix real t. Using the relations obtained at the above and with a simple computation we deduce that

$$\mathfrak{Re}\left\{p(z_0) + k\frac{z_0 p'(z_0)}{p(z_0)}\right\} = \left(\frac{\lambda(\lambda - b(1+km))}{(\lambda+b)(\lambda-b)}\right)(1+\cos t) + mk\frac{\lambda+b}{2(\lambda-b)}. \tag{3.1}$$

For completing our proof we consider three cases. If $-\lambda < b \leq 0$ then (3.1) implies that

$$\mathfrak{Re}\left\{p(z_0) + k\frac{z_0 p'(z_0)}{p(z_0)}\right\} > Mk\frac{\lambda+b}{2(\lambda-b)},$$

which contradicts with the assumption. Also for $0 < b < \lambda \leq b(1+kM)$, we put

$$f(x) = mk\frac{\lambda+b}{2(\lambda-b)} + (1+x)\frac{\lambda}{\lambda+b}\frac{\lambda - b(1+km)}{\lambda-b} \qquad (-1 \leq x \leq 1),$$

where $x = \cos t$. It is clear that

$$f'(x) = \frac{\lambda}{\lambda+b}\frac{\lambda-b(1+km)}{\lambda-b} < 0 \qquad (-1 \leq x \leq 1),$$

so

$$f(x) \geq f(1) = \frac{mk(\lambda-b)}{2(\lambda+b)} + \frac{2\lambda}{(\lambda+b)} > \frac{Mk(\lambda-b)}{2(\lambda+b)} + \frac{2\lambda}{(\lambda+b)} \qquad (-1 \leq x \leq 1),$$

which contradicts with the assumption. Ultimately, for the case $0 < b < \frac{\lambda}{1+kM}$ we set

$$g(x) = \frac{\lambda+b}{2} - \frac{\lambda b}{\lambda+b} - \frac{\lambda b}{\lambda+b}x \qquad (-1 \leq x \leq 1),$$

where $x = \cos t$. Now $g'(x) = -\frac{\lambda}{\lambda+b} < 0$, and so for all $-1 \le x \le 1$ we have

$$g(x) \ge g(1) = \frac{(\lambda - b)^2}{2(\lambda + b)} > 0.$$

Consequently,

$$\begin{aligned}\mathfrak{Re}\left\{p(z_0) + k\frac{z_0 p'(z_0)}{p(z_0)}\right\} &= \frac{\lambda}{\lambda+b}(1+x) + \left(\frac{mk}{\lambda-b}\right) g(x)\\ &> \frac{Mk}{(\lambda-b)}\frac{(\lambda-b)^2}{2(\lambda+b)} = Mk\frac{\lambda-b}{2(\lambda+b)},\end{aligned}$$

that contradicts with the assumption. Hence the proof is complete. □

Corollary 3.9. *Let $-\lambda < b < \lambda$, $\lambda > 0$ and $k > 0$. Also let $f \in \mathcal{A}_{n,b_1}$ with $\frac{zf'(z)}{f(z)} \neq \frac{2\lambda}{b+\lambda}$ in $\mathbb{U}$ and $0 \le b_1 \le \frac{\lambda-b}{n\lambda}$. If for all $z \in \mathbb{U}$*

$$\mathfrak{Re}\left\{(1-k)\frac{zf'(z)}{f(z)} + k\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}$$
$$\le \begin{cases} Mk\frac{\lambda+b}{2(\lambda-b)} & \text{if} \quad -\lambda < b \le 0, M \ge \frac{2(\lambda-b)}{k(\lambda+b)} \\ \frac{Mk}{2}\frac{\lambda-b}{\lambda+b} + \frac{2\lambda}{\lambda+b} & \text{if} \quad \frac{\lambda}{1+kM} \le b < \lambda \\ \frac{Mk}{2}\frac{\lambda-b}{\lambda+b} & \text{if} \quad 0 < b < \frac{\lambda}{1+kM}, M \ge \frac{2(\lambda+b)}{k(\lambda-b)}, \end{cases}$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{\lambda}{b+\lambda}\right| < \frac{\lambda}{b+\lambda} \qquad z \in \mathbb{U},$$

where $M = n + \frac{\frac{\lambda-b}{\lambda} - nb_1}{\frac{\lambda-b}{\lambda} + nb_1}$.

Proof. Let $p(z) = \frac{zf'(z)}{f(z)}$ then we have $p \in \mathcal{H}_{nb_1}[1, n]$. Therefore by applying Theorem 3.2, and replacing β by nb_1 in this theorem, we obtain the result. □

Remark 3.10. By putting $b_1 = 0$ in the Corollary 3.6, one can observe that this corollary improves and extends the result obtained in [12](see Theorem 3.1 in [12]).

By setting $k = 1$, $b = 1$, $b_1 = \frac{1}{3}$, $\lambda = 3$ and $n = 2$ in Corollary 3.6 we obtain

Corollary 3.11. *Let $f \in \mathcal{A}_{2,\frac{1}{3}}$ with $\frac{zf'(z)}{f(z)} \neq \frac{3}{2}$ in $\mathbb{U}$. If for all z in the open unit disc*

$$\mathfrak{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \le 2,$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{3}{4}\right| < \frac{3}{4} \qquad z \in \mathbb{U}.$$

By setting $k = 1$, $b = 1$, $b_1 = \frac{1}{9}$, $\lambda = 3$ and $n = 3$ in Corollary 3.6 we obtain

Corollary 3.12. *Let $f \in \mathcal{A}_{3,\frac{1}{9}}$ with $\frac{zf'(z)}{f(z)} \neq \frac{3}{2}$ in $\mathbb{U}$. If for all z in the open unit disc*

$$\mathfrak{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \leq \frac{11}{3},$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{3}{4}\right| < \frac{3}{4} \qquad z \in \mathbb{U}.$$

By putting $k = 1$, $b = 3$, $b_1 = \frac{2}{5}$, $\lambda = 5$ and $n = 1$ in Corollary 3.6 we obtain

Corollary 3.13. *Let $f \in \mathcal{A}_{1,\frac{2}{5}}$ with $\frac{zf'(z)}{f(z)} \neq \frac{5}{4}$ in $\mathbb{U}$. If for all z in the open unit disc*

$$\mathfrak{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \leq \frac{11}{8},$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{5}{8}\right| < \frac{5}{8} \qquad z \in \mathbb{U}.$$

By putting $k = 1$ and $b = 0$ in Corollary 3.6 we obtain

Corollary 3.14. *Let $n \geq 2$ and $0 \leq b_1 \leq \frac{1}{n}$. Also let $f \in \mathcal{A}_{n,b_1}$ with $\frac{zf'(z)}{f(z)} \neq 2$ in $\mathbb{U}$. If for all z in the open unit disc*

$$\mathfrak{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \leq \frac{M}{2},$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad z \in \mathbb{U},$$

where M is defined in the Corollary 3.6.

References

[1] Ali, R.M., Kumar, V., Ravichandran, V., *Radius of starlikeness for analytic functions with fixed second coefficient*, Kyungpook Math. J., **57**(2017), no. 3, 473-492.

[2] Ali, R.M., Nagpal, S., Ravichandran, V., *Second-order differential subordination for analytic functions with fixed initial coefficient*, Bull. Malays. Math. Sci. Soc., **34**(3)(2011), 611-629.

[3] Amani, M., Aghalary, R., Ebadian, E., *Extension of Nunokawa lemma for functions with fixed second coefficient and its applications*, Journal of Function Spaces, (2021), 1-11.

[4] Amani, M., Aghalary, R., Ebadian, E., *Open Door Lemma for functions with fixed second coefficient*, Bull. Malays. Math. Sci. Soc., **45**(2022), 513-536.

[5] Jack, I.S., *Functions starlike and convex of order α*, J. London Math. Soc., **3**(1971), 469-474.

[6] Kanika, S., Ravichandran, V., *Applications of theory of differential subordination of functions with fixed initial coefficient*, Journal of Classical Analysis, **8**(2016), no. 2, 113-121.

[7] Kwon, O.-S., *Some properties of analytic functions with fixed second coefficients*, Advances in Pure Math., **4**(2014), 194-202.

[8] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications*, Marcel Dekker Inc., New York, 2000.

[9] Nagpal, S., Ravichandran, V., *Applications of the theory of differential subordination for functions with fixed initial coefficient to univalent functions*, Ann. Polon. Math., **105**(3)(2012), 225-238.

[10] Nunokawa, M., *On properties of non-Carathéodory functions*, Proc. Japan Acad. Ser. A., **68**(6)(1992), 152-153.

[11] Nunokawa, M., Cho, N.E., Sokol, J., *On the Jack Lemma and its generalization*, Publications de L'Institut Mathematique, Nouvelle Serie, **107**(121)(2020), 63-65.

[12] Nunokawa, M., Sokol, J., *Remarks on some starlike functions*, J. Inequal. Appl., **593**(2013), 1-8.

[13] Nunokawa, M., Sokol, J., Cho, N.E., *Some applications of Nunokawa's Lemma*, Bull. Malays. Math. Sci. Soc., **40**(2)(2017), 1791-1800.

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