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Well-posedness for set-valued equilibrium problems

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. In this paper we extend a concept of well-posedness for vector equilibrium problems to the more general framework of set-valued equilibrium problems in topological vector spaces using an appropriate reformulation of the concept of minimality for sets. Sufficient conditions for well-posedness are given in the generalized convex settings and we are able to single out classes of well-posed set-valued equilibrium problems.
On the other hand, in order to relax some conditions, we introduce a concept of minimizing sequences for a set-valued problem, in the set criterion sense, and further we will have a concept of well-posedness for the set-valued equilibrium problem we are interested in. Sufficient results are also given for this well-posedness concept.

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1. Introduction

In the last few years, set-valued optimization problems have received much attention by many authors due to their extensive applications in many fields such as optimal control, economics, game theory, multiobjective optimization and so on(see, e.g., [1], [2], [8] and the references therein). For some motivating examples one may refer also to the book by Khan et al.[11].
Approaches in set-valued optimization can be made using two types of criteria of solutions: the vector criterion and the set optimization criterion. The first criterion is equivalent to finding efficient solutions of the image set but this criterion is not always suitable for all types of set-valued optimization problems.

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Kuroiwa [14] introduced an alternative criterion of solutions for set-valued optimization problems, called the set optimization criterion, which is based on a comparison among the values of the objective set-valued map.
On the other hand, well-posedness plays a crucial role in the stability theory for optimization problems. The classical notion of well-posednesss for a scalar optimization problem was first introduced by Tykhonov [19] and is known as Tykhonov well-posedness. In the literature, various notions of well-posedness for vector optimization problems have been introduced and studied(see, e.g., [4], [5], [9], [12], [16] and the references therein).
Apart from its theoretical interest, important problems arising from economics, mechanics, electricity, chemistry and other practical sciences motivate the study of equilibrium problems. Recently, equilibrium problems for vector mappings have been considered by many authors. For a nice survey, we refer to the research monograph devoted to the analysis of equilibrium problems in pure and applied nonlinear analysis and mathematical economics by Kassay et al.[10].
Some concepts of well-posedness for the strong vector equilibrium problem in topological vector spaces were introduced and studied by Bianchi et al.[3]. Also, they gave sufficient conditions, in concave settings, in order to guarantee the well-posedness.
Inspired by the work of Bianchi et al.[3], in this paper we study the well-posedness of a set-valued equilibrium problem in topological vector spaces. We consider and study two notions of well-posedness; the first one generalizes the concept of well-posedness of strong vector equilibrium problem introduced by Bianchi et al.[3] and the second one is linked to the behaviour of a suitable set-valued problem.
The first concept of well-posedness for our set-valued equilibrium problem is also named M-well-posedness like in vectorial case and we are able to give sufficient conditions for M-well-posedness in generalized convex settings assuming alternative conditions only on a suitable set-valued map.
In order to drop some assumptions, we consider a concept of well-posedness for a suitable set-valued map with respect to a quasi-order relation, strongly related to a concept of well-posedness of our set-valued equilibrium problem. Some sufficient conditions concerning this kind of well-posedness are also established.
The paper in four sections is organized as follows. Section 2 presents the preliminaries required throughout the paper. Section 3 generalizes the concept of well-posedness of the strong vector equilibrium problem to a set-valued equilibrium problem and establishes some sufficient conditions for well-posedness in finite and infinite dimensional settings pointing out classes of well-posed set-valued equilibrium problems. Section 4 introduces a new concept of well-posedness for our set-valued equilibrium problem under weaker assumptions than those in Section 3. Some sufficient results for well-posedness are also obtained in infinite dimensional settings. For a clear understanding of the concepts and to illustrate our results, we give also some examples.

2. Preliminaries

Let X and Y be topological vector spaces with countable local bases. Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y and K be a proper nonempty closed convex

pointed cone in the real topological vector space Y. For $A \in \mathcal{P}(Y)$ we denote the topological interior, the topological closure, the topological boundary and complement of A by $intA$, clA, ∂A and A^c, respectively.
We consider also a preference relation on $\mathcal{P}(Y)$ introduced by Kuroiwa [14]: the lower set less quasi-order relation induced by the cone K. Also, we denote by $K_0 = K \setminus \{0\}$. For $A, B \in \mathcal{P}(Y)$

$$A \preceq_K B \Leftrightarrow B \subseteq A + K.$$

We now consider S a nonempty proper subset of Y. A preference relation based on the solution concept equipped with the set S was proposed by Flores-Bazán et al.[6]. For $a, b \in S$,

$$a \preceq_S b \Longleftrightarrow a - b \in S.$$

Khushboo et al.[13] reformulate a notion of minimality for a set $A \in \mathcal{P}(Y)$ considered for vector optimization problems by Flores-Bazan et al.[6]. An element $\overline{a} \in A$ is said to be an S-minimal point of A if

$$a \not\preceq_S \overline{a}, \text{ for all } a \in A \setminus \ \{\overline{a}\},$$

or, equivalently,

$$A \setminus \{\overline{a}\} \subseteq \overline{a} + S^c.$$

We denote the set of S-minimal points of A by $E_S(A)$.
It is obvious that if $0 \in S^c$ then

$$\overline{a} \in E_S(A) \Longleftrightarrow \overline{a} \in A \text{ and } A \subseteq \overline{a} + S^c. \tag{2.1}$$

It is well-known that vector equilibrium problems are natural extensions of several problems of practical interest like vector optimization and vector variational inequality problems. In the literature, there are some kinds of extensions of scalar equilibrium problems to the vector equilibrium problems. Further, vector equilibrium problems are extended to set-valued equilibrium problems in several manners.
In this paper we consider the set-valued equilibrium problem (SEP) which consists in finding $\overline{x} \in D$ such that

$$f(\overline{x}, y) \subseteq (-K_0)^c \text{ for all } y \in D,$$

where $D \subseteq X, f : D \times D \rightrightarrows Y$. This problem generalizes, in a certain sense, the strong vector equilibrium problem considered by Bianchi et al.[3]
We denote by S_0 the solution set of the problem (SEP) and we will suppose in the sequel that S_0 is nonempty.
Our purpose is to try to assign reasonable definitions of well-posedness for (SEP) that recover some previous existing concepts in vector criterion, see Bianchi et al.[3]. In order to start our approach, we introduce the set-valued map $\varphi : D \rightrightarrows Y$ given by

$$\varphi(x) = E_{-K_0}(f(x, D)).$$

The map φ generalizes the definition of the function ϕ in Bianchi et al.[3]; indeed, taking into account (2.1) we have that

$$z \in \varphi(x) \Leftrightarrow z \in f(x, D) \text{ and } f(x, D) \subseteq z + (-K_0)^c$$
$$\Leftrightarrow z \in f(x, D) \text{ and } (f(x, D) - z) \cap (-K) = \{0\}.$$

Throughout the paper is assumed that $\varphi(x) \neq \emptyset$ for every $x \in D$. The domain of φ, denoted by domφ, is defined as $\text{dom}\varphi := \{x \in D : \varphi(x) \neq \emptyset\}$ and therefore $\text{dom}\varphi = D$.
In the sequel, we shall denote by $\mathcal{V}_X(x_0)$ a neighbourhood base of x_0 in the topological space X. The same notation will be used for other spaces.
We now recall some notions of continuity for set-valued maps. Let $\varphi : D \rightrightarrows Y$ be a set-valued map.

Definition 2.1. [11] The map φ is said to be

(i) upper semicontinuous at $x_0 \in D$ if for every $W \subseteq Y, W$ open, $\varphi(x_0) \subseteq W$, there exists a neighbourhood $U \in \mathcal{V}_X(x_0)$ such that $\varphi(x) \subseteq W$ for every $x \in U \cap D$.

(ii) lower semicontinuous at $x_0 \in D$ if for every $W \subseteq Y, W$ open, $\varphi(x_0) \cap W \neq \emptyset$, there exists a neighbourhood $U \in \mathcal{V}_X(x_0)$ such that $\varphi(x) \cap W \neq \emptyset$ for every $x \in U \cap D$.

Definition 2.2. [7] The map φ is said to be upper Hausdorff continuous at $x_0 \in D$ if for every $W \in \mathcal{V}_Y(0)$, there exists a neighbourhood $U \in \mathcal{V}_X(x_0)$ such that $\varphi(x) \subseteq \varphi(x_0) + W$ for every $x \in U \cap D$.

The graph of φ, denoted by graph φ, is defined as $\text{graph}\varphi := \{(x, y) \in D \times Y : y \in \varphi(x)\}$.

Definition 2.3. [7] The map φ is said to be compact at $x_0 \in D$ if for every sequence $((x_n, y_n))_{n\in\mathbb{N}} \subseteq \text{graph}\varphi$ with $x_n \to x_0$ there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k} \to y_0 \in \varphi(x_0)$. Also φ is said to be compact on D if φ is compact at every $x_0 \in D$.

In metric spaces, Crespi et al.[4] pointed out, the following results obtained by Göpfert et al.[7] regarding the compactness of a set-valued map. These results also hold when we deal with topological vector spaces with countable local bases.

Theorem 2.4. [7],[4] *The following statements are equivalent*

(i) *φ is compact at $x_0 \in D$;*

(ii) *φ is upper semicontinuous at x_0 and $\varphi(x_0)$ is compact;*

(iii) *φ is upper Hausdorff continuous at x_0 and $\varphi(x_0)$ is compact.*

In order to obtain our main results we need the following characterization of upper and lower semicontinuity for set-valued maps.

Theorem 2.5. *Let $\varphi : D \rightrightarrows Y$ be a set-valued map.*

(i) *If $x_0 \in D$ and $\varphi(x_0)$ is compact, then φ is upper semicontinuous at x_0 if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ with $x_n \to x_0$ and for any $y_n \in \varphi(x_n)$, $n \in \mathbb{N}$, there exist $y_0 \in \varphi(x_0)$ and a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k} \to y_0$(see [7]).*

(ii) *φ is lower semicontinuous at $x_0 \in D$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ with $x_n \to x_0$ and for any $y_0 \in \varphi(x_0)$, there exists $y_n \in \varphi(x_n)$, $n \in \mathbb{N}$, such that $y_n \to y_0$(see [1]).*

In particular, we focus on l-type K-convex set-valued maps, a concept of generalized convexity introduced by Kuroiwa [14], see also Seto et al.[18].

Definition 2.6. Let $D \subseteq X$ be a nonempty convex subset of X. A set-valued map $\varphi : D \rightrightarrows Y$ is said to be l-type K-convex if for any $x_0, x_1 \in$domφ and $\lambda \in (0,1)$,

$$\varphi((1-\lambda)x_0 + \lambda x_1) \preceq_K (1-\lambda)\varphi(x_0) + \lambda\varphi(x_1).$$

3. M-well-posed set-valued equilibrium problems

In this section we keep the assumption that $0 \in f(x, D)$ for all $x \in D$ (see Bianchi et al.[3]) and investigate the properties of the set-valued map φ. Also, the concept of maximizing sequence for the set-valued map φ and a concept of well-posedness for the problem (SEP) are provided, similarly with those considered in Bianchi et al.[3](see also [15]). Further, sufficient conditions for the problem (SEP) to be well-posed are given and we discuss the role of l-type K-convexity of the set-valued map φ in order to single out classes of well-posed set-valued equilibrium problems in finite and infinite dimensional spaces.

Proposition 3.1. *For the map φ the following assertions hold:*

(i) $\varphi(x) \cap K_0 = \emptyset$ *for all* $x \in D$;
(ii) $\overline{x} \in S_0 \Longleftrightarrow 0 \in \varphi(\overline{x})$;
(iii) $\overline{x} \in S_0 \Longleftrightarrow \varphi(\overline{x}) \cap K \neq \emptyset$.

Proof. (i) Assume that for some $x_0 \in D, \varphi(x_0) \cap K_0 \neq \emptyset$. Therefore, there exists $z \in K_0, z \neq 0$, such that $z \in E_{-K_0}(f(x_0, D))$. Hence $z \in f(x_0, D)$ and $f(x_0, D) \subseteq z + (-K_0)^c$. Since $0 \in f(x, D)$ for all $x \in D$, we obtain that $0 \in z + (-K_0)^c$, i.e., $-z \in (-K_0)^c$ which contradicts the fact that $z \in K_0$.
(ii) Since $0 \in f(x, D)$ for each $x \in D$,

$$\overline{x} \in S_0 \Longleftrightarrow f(\overline{x}, y) \subseteq (-K_0)^c \text{ for every } y \in D \Longleftrightarrow$$

$$\Longleftrightarrow f(\overline{x}, D) \subseteq (-K_0)^c \Longleftrightarrow 0 \in f(\overline{x}, D), f(\overline{x}, D) \subseteq 0 + (-K_0)^c,$$

i.e., $0 \in E_{-K_0}(f(\overline{x}, D)) = \varphi(\overline{x})$.
(iii) Trivial, by (i) and (ii). □

Let us recall the following notion of upper Hausdorff convergence of a sequence of points to a set (see, e.g., Miglierina et al.[17]).

Definition 3.2. The sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be upper Hausdorff convergent to the set $A \subseteq X$ ($x_n \rightharpoonup A$) if for every neighbourhood $W \in \mathcal{V}_X(0)$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in A + W$, for every $n \geq n_0$.

It is well-known that the well-posedness concepts are formulate in terms of convergence of suitable minimizing sequences. Bianchi et al.[3] introduced the following concept for a sequence and proved that is related to some concept for sequences introduced by Miglierina et al.[17].

Definition 3.3. [3] A sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ is said to be a maximizing sequence for φ if for every $V \in \mathcal{V}_Y(0)$ there exists $n_0 \in \mathbb{N}$ such that

$$\varphi(x_n) \cap V \neq \emptyset,\ \forall n \geq n_0.$$

Clearly, every sequence $(x_n)_{n\in\mathbb{N}} \subseteq S_0$ is a maximizing sequence.

The following definition reproduces, in set-valued settings, the classical notion of Tykhonov well-posedness given in metric spaces, see also [3].

Definition 3.4. [3] We say that the set-valued equilibrium problem (SEP) is M-well-posed if every maximizing sequence is upper Hausdorff convergent to S_0.

Next theorem gives sufficient conditions for the set-valued equilibrium problem (SEP) to be M-well-posed. It is given in finite dimensional spaces and is a variant of Theorem 1 in Bianchi et al.[3] where the hypotheses are given with respect to the maps φ and f. In our version, the hypotheses are imposed only on the map φ which makes our result to be much easier to verify.
Similarly with Bianchi et al.[3], we suppose that the topological vector space Y is regular, i.e., every nonempty closed set and every singleton disjoint from it can be separated by open sets.

Theorem 3.5. *Let X be a finite dimensional vector space and $D \subseteq X$ be a closed convex set such that:*

(i) *$S_0 \subseteq D$ is bounded;*
(ii) *φ compact on $D \setminus S_0$;*
(iii) *φ is l-type $(-K)$-convex on D.*

Then the problem (SEP) is M-well-posed.

Proof. Suppose by contradiction that there exists a maximizing sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ which is not upper Hausdorff convergent to the set S_0. Therefore, there exists a neighbourhood $V \in \mathcal{V}_X(0)$ such that

$$x_n \notin S_0 + V, \text{ for infinitely many } n. \tag{3.1}$$

Since S_0 is bounded, the set $S_0 + V$ is bounded, $V \in \mathcal{V}_X(0)$, and therefore the set $cl(S_0+V)$ is compact. Consider the compact set $bd(S_0+V) = cl(S_0+V)\setminus int(S_0+V)$. Fix an arbitrary $\overline{x} \in S_0$. We can always find $\lambda_n \in (0,1)$ such that

$$\overline{x}_n = \lambda_n \overline{x} + (1-\lambda_n)x_n \in bd(S_0+V).$$

The set $bd(S_0+V)$ being compact, we can extract from the sequence

$$(\lambda_n \overline{x} + (1-\lambda_n)x_n)_{n\in\mathbb{N}}$$

a subsequence $(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k})_{k\in\mathbb{N}}$ converging to $x^* \in bd(S_0+V)$. By the l-type $(-K)$-convexity of φ, we have for every $k \in \mathbb{N}$,

$$\lambda_{n_k}\varphi(\overline{x}) + (1-\lambda_{n_k})\varphi(x_{n_k}) \subseteq \varphi(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k}) - K.$$

Therefore, since $0 \in \varphi(\overline{x})$ and $(x_{n_k})_{k\in\mathbb{N}} \subseteq D$ is a maximizing sequence, there exists $u_{n_k} \in \varphi(x_{n_k})$ such that $u_{n_k} \to 0$; hence we have

$$\lambda_{n_k}0 + (1-\lambda_{n_k})u_{n_k} \in \varphi(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k}) - K.$$

Thus, there exists $v_{n_k} \in \varphi(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k})$ such that

$$\lambda_{n_k}0 + (1-\lambda_{n_k})u_{n_k} - v_{n_k} \in -K.$$

Since φ is compact at $x^* \in bd(S_0+V)$, from Theorem 2.4(ii) and Theorem 2.5(i), it follows that there exist a subsequence $(v_{n_{k_l}})_{l\in\mathbb{N}}$ of $(v_{n_k})_{k\in\mathbb{N}}$ and $v^* \in \varphi(x^*)$ such that $v_{n_{k_l}} \to v^*$. Hence, we have

$$\lambda_{n_{k_l}}0 + (1-\lambda_{n_{k_l}})u_{n_{k_l}} - v_{n_{k_l}} \in -K. \tag{3.2}$$

Since $\lambda_{n_{k_l}} \in (0,1)$, there exist a subsequence of $\lambda_{n_{k_l}}$ (denoted also $\lambda_{n_{k_l}}$) and $\lambda_0 \in [0,1]$ such that $\lambda_{n_{k_l}} \to \lambda_0$. Now taking in (3.2) the limit as $l \to \infty$, from the closedness of K, we obtain that $0 \in v^* - K$ and then $0 \in \varphi(x^*) - K$, a contradiction.
Indeed, if not, $0 \in \varphi(x^*) - K$. Hence there exist $\overline{z} \in \varphi(x^*), \overline{z} \neq 0$ and $k \in K$ such that $\overline{z} - k = 0$. Therefore $\overline{z} = k$ and it follows that $\varphi(x^*) \cap K \neq \emptyset$, a contradiction since $x^* \notin S_0$.
The proof is complete. □

We now provide sufficient conditions for M-well-posedness in infinite dimensional settings assuming that the set $D \setminus S_0$ is compact.

Theorem 3.6. *If the following conditions hold:*

(i) *the set D is convex and $D \setminus S_0$ is compact;*
(ii) *φ compact on $D \setminus S_0$;*
(iii) *φ is l-type $(-K)$-convex on D,*

then the problem (SEP) is M-well-posed.

Proof. Suppose by contradiction that there exist a maximizing sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ and a neighbourhood $V \in \mathcal{V}_X(0)$ such that

$$x_n \notin S_0 + V, \text{ for infinitely many } n.$$

Since $(x_n)_{n\in\mathbb{N}}$ is a maximizing sequence, one can choose a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in \varphi(x_n)$ such that $u_n \to 0$. Let now $\overline{x} \in S_0$. Therefore, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0,1)$ such that $\lambda_n\overline{x} + (1-\lambda_n)x_n \in D \setminus S_0$. Since $D \setminus S_0$ is compact there exists a subsequence $(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k})_{k\in\mathbb{N}} \subseteq D \setminus S_0$ such that $\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k} \to x^* \in D \setminus S_0$ when $k \to \infty$.
From the l-type $(-K)$-convexity of φ on D we obtain that

$$\lambda_{n_k}\varphi(\overline{x}) + (1-\lambda_{n_k})\varphi(x_{n_k}) \subseteq \varphi(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k}) - K. \tag{3.3}$$

Since $0 \in \varphi(\overline{x})$ and $u_{n_k} \in \varphi(x_{n_k})$, $u_{n_k} \to 0$, from (3.3) we have

$$\lambda_{n_k}0 + (1-\lambda_{n_k})u_{n_k} \in \lambda_{n_k}\varphi(\overline{x}) + (1-\lambda_{n_k})\varphi(x_{n_k}) \subseteq \varphi(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k}) - K.$$

Therefore, there exists $v_{n_k} \in \varphi(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k})$, $k \in \mathbb{N}$, such that

$$\lambda_{n_k}0 + (1-\lambda_{n_k})u_{n_k} - v_{n_k} \in -K. \tag{3.4}$$

The map φ is compact at $x^* \in D \setminus S_0$; taking into account Theorem 2.4(ii) and Theorem2.5(i), it follows that there exist a subsequence $(v_{n_{k_l}})_{l\in\mathbb{N}}$ of $(v_{n_k})_{k\in\mathbb{N}}$ and $v^* \in \varphi(x^*)$ such that $v_{n_{k_l}} \to v^*$. By (3.4) we have that

$$\lambda_{n_{k_l}}0 + (1-\lambda_{n_{k_l}})u_{n_{k_l}} - v_{n_{k_l}} \in -K.$$

Similarly with Theorem 3.5 we obtain that $0 \in \varphi(x^*) - K$, which is a contradiction because $x^* \notin S_0$.
The proof is complete. □

4. Well-posedness with respect to set criterion

In the previous section, the assumption that $0 \in f(x,D)$ for every $x \in D$, gave us the possibility to characterize the solutions of the problem (SEP) via the set-valued map φ.
Since there exist set-valued equilibrium problems which are well-posed without fulfilling the condition above, in this section we want to drop the assumption that $0 \in f(x,D)$ for every $x \in D$. To this point, we consider the following set-valued problem (SP) which consists in finding $\overline{x} \in D$ such that

$$\varphi(\overline{x}) \subseteq (-K_0)^c,$$

where $D \subseteq X$ and $\varphi : D \rightrightarrows Y$, $\varphi(x) = E_{-K_0}(f(x,D))$. We will denote by $\overline{S}_0$ the solution set of this problem. From the definition of the map φ it follows that $S_0 \subseteq \overline{S}_0$. Since we supposed that $S_0 \neq \emptyset$ we have also that $\overline{S}_0 \neq \emptyset$.
Further we will introduce a well-posedness concept for the set-valued problem (SP) which will lead to some concept of well-posedness for (SEP). While dealing with set-valued problems it is more relevant to consider solution concepts based on comparison among the sets corresponding to each value of the objective map.
For $0 \neq e \in Y$ and $A \in \mathcal{P}(Y)$, Khushboo et al.[13] considered the scalarization function $\phi_{e,A} : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined as $\phi_{e,A}(y) = \inf\{t \in \mathbb{R} : y \in te + A - S\}$, where S is a nonempty proper subset of Y. When $S = -K$ we obtain that $\phi_{e,A}(y) = \inf\{t \in \mathbb{R} : y \in te + A + K\}$.
We now consider the following generalized Gerstewitz function introduced by Khushboo et al.[13].

Definition 4.1. Let $H_e : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$ be defined as

$$H_e(A,B) = sup_{b \in B}\phi_{e,A}(b). \tag{4.1}$$

Further, under the assumptions that $S = -K$ and $e \in -K$, since the set $-K$ is closed and $\text{cl}(-K) + \mathbb{R}_{++}e \subseteq -K$ holds, the following two lemmas are particular cases of Theorem 4.1(ii) and Lemma 4.2 in Khushboo et al.[13], respectively.

Lemma 4.2. *If $r \in \mathbb{R}$ and $A \in \mathcal{P}(Y)$ then*

$$\{y \in Y : \phi_{e,A}(y) \leq r\} = re + A + K.$$

Lemma 4.3. *If $r \in \mathbb{R}$ and $A, B \in \mathcal{P}(Y)$ then*

$$H_e(A,B) \leq r \Longleftrightarrow B \subseteq A + re + K. \tag{4.2}$$

In the sequel, we suppose that $S = -K$ and $e \in -K$. Inspired by the two lemmas above, we introduce the following notion for a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ to be a minimizing sequence for the set-valued problem φ.

Definition 4.4. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ is said to be a minimizing sequence for φ if there exist a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \overline{S}_0$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}, \varepsilon_n > 0, \varepsilon_n \to 0$ such that

$$H_e(\varphi(y_n), \varphi(x_n)) \leq \varepsilon_n.$$

Remark 4.5. In our settings, it is obvious that a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ is a minimizing sequence for φ if and only if there exist a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \overline{S}_0$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}, \varepsilon_n > 0, \varepsilon_n \to 0$ such that

$$\varphi(x_n) \subseteq \varphi(y_n) + e\varepsilon_n + K.$$

We observe that every sequence from $\overline{S}_0$ is a minimizing sequence.

The following example shows that the maximizing sequence and the minimizing sequence concepts introduced before for φ, are different.

Example 4.6. Let $e = (-1, 0)$ and $f : D \times D \rightrightarrows Y$ where $D = [-1, 1]$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, be defined as

$$f(x, y) = \{(x, - \mid y \mid)\}.$$

Is is easy to check that $\varphi : D \rightrightarrows Y, \varphi(x) = E_{-K_0}(f(x, D))$, is defined by

$$\varphi(x) = \{(x, -1)\},$$

and $S_0 = \overline{S}_0 = (0, 1]$. Let $x_n = (-\frac{1}{n})_{n\in\mathbb{N}^*}$. Since there exist the sequence $(y_n)_{n\in\mathbb{N}} \subseteq \overline{S}_0, y_n = \frac{1}{n}, n \in \mathbb{N}^*$, and the sequence $(\varepsilon_n)_{n\in\mathbb{N}}, \varepsilon_n = \frac{2}{n}, n \in \mathbb{N}^*$, such that

$$\varphi(x_n) \subseteq \varphi(y_n) + e\varepsilon_n + K,$$

it follows that $(x_n)_{n\in\mathbb{N}^*}$ is a minimizing sequence for the map φ. We can notice that for each $V \in \mathcal{V}_Y(0)$, $\varphi(x_n) \cap V = \emptyset$; therefore $(x_n)_{n\in\mathbb{N}}$ is not a maximizing sequence. Also, $0 \in f(x, D)$ does not hold for every $x \in D$.

Definition 4.7. We say that the set-valued problem (SP) is M_1-well-posed if every minimizing sequence is upper Hausdorff convergent to the set $\overline{S}_0$.

Now we provide sufficient conditions for the problem (SP) to be M_1-well-posed in infinite dimensional spaces.

Theorem 4.8. *If D is compact, $\overline{S}_0$ is closed, φ is lower semicontinuous on $D \setminus \overline{S}_0$ and compact on $\overline{S}_0$, then the problem (SP) is M_1-well-posed.*

Proof. Suppose by contradiction that there exist a minimizing sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ and $V \in \mathcal{V}_X(0)$ such that

$$x_n \notin \overline{S}_0 + V, \tag{4.3}$$

for infinitely many n. Since the set D is compact it follows that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \to x_0$, $x_0 \in D$. From (4.3), obviously $x_0 \notin \overline{S}_0$.
On the other hand, $(x_n)_{n\in\mathbb{N}} \subseteq D$ is a minimizing sequence, therefore there exist a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \overline{S}_0$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}, \varepsilon_n > 0, \varepsilon_n \to 0$, such that

$$H_e(\varphi(y_n), \varphi(x_n)) \leq \varepsilon_n. \tag{4.4}$$

By Lemma 4.3, we have that

$$\varphi(x_n) \subseteq \varphi(y_n) + \varepsilon_n e + K, \ \varepsilon_n \to 0.$$

The set $\overline{S}_0 \subseteq D$ is closed, D is compact and therefore $\overline{S}_0$ is compact. Thus, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k} \to y_0$ for some $y_0 \in \overline{S}_0$.
Let $v_0 \in \varphi(x_0)$. The map φ is lower semicontinuous on $D \setminus \overline{S}_0$ and therefore at

$x_0 \in D \setminus \overline{S}_0$. From Theorem 2.5$(ii)$, it follows that there exists $v_{n_k} \in \varphi(x_{n_k})$ such that $v_{n_k} \to v_0$.
The inclusion

$$\varphi(x_{n_k}) \subseteq \varphi(y_{n_k}) + \varepsilon_{n_k} e + K,\ \varepsilon_{n_k} \to 0,$$

implies that there exists $u_{n_k} \in \varphi(y_{n_k})$ such that

$$v_{n_k} \in u_{n_k} + \varepsilon_{n_k} e + K,\ \text{for every } k \in \mathbb{N}.$$

By Theorem 2.4, the map φ is upper semicontinuous at $y_0 \in \overline{S}_0$ and $\varphi(y_0)$ is compact; thus, taking into account Theorem 2.5(ii), there exist a subsequence $(u_{n_{k_l}})_{l\in\mathbb{N}}$ of $(u_{n_k})_{k\in\mathbb{N}}$ and $u_0 \in \varphi(y_0)$ such that $u_{n_{k_l}} \to u_0$. We have that

$$v_{n_{k_l}} \in u_{n_{k_l}} + \varepsilon_{n_{k_l}} e + K,\ \text{for every } l \in \mathbb{N}.$$

When $l \to \infty$, it follows from the closedness of K that $v_0 \in u_0 + K \subseteq \varphi(y_0) + K$ and therefore $\varphi(x_0) \subseteq \varphi(y_0) + K$, which is a contradiction since $\varphi(y_0) \subseteq (-K_0)^c$ and $x_0 \notin \overline{S}_0$. □

In the next example all the assumptions of the theorem above are fulfilled and the problem (SP) is well-posed. Also, $0 \in f(x, D)$ does not hold for every $x \in D$.

Example 4.9. Let $f : D \times D \rightrightarrows Y$ where $D = [-1, 1]$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, be defined as

$$f(x, y) = \{(x, \mid y \mid)\}, x \in [-1, 1],\ y \in [-1, 1].$$

The map $\varphi : D \rightrightarrows Y$ is defined by

$$\varphi(x) = \{(x, 0)\}, x \in [-1, 1].$$

The solution set for (SP) is $\overline{S}_0 = S_0 = [0, 1]$.

Remark 4.10. Obviously, if $(x_n)_{n\in\mathbb{N}}$ is a minimizing sequence of φ we have that there exist $(y_n)_{n\in\mathbb{N}} \subseteq \overline{S}_0$ and $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}, \varepsilon_n > 0$, $\varepsilon_n \to 0$, such that

$$\varphi(x_n) \subseteq \varphi(y_n) + \varepsilon_n e + K \subseteq f(y_n, D) + \varepsilon_n e + K.$$

Now we introduce the concept of M_1-well-posedness for the problem (SEP), strongly related to the concept of M_1-well-posedness of (SP).

Definition 4.11. The set-valued equilibrium problem (SEP) is said to be M_1-well-posed if every minimizing sequence $(x_n)_{n\in\mathbb{N}}$ is upper Hausdorff convergent to the set S_0.

The following theorem makes the connection between the special set-valued problem (SP) and the set-valued equilibrium problem (SEP) we are interested in. Also, it provides sufficient conditions for M_1-well-posedness of the set-valued equilibrium problem (SEP).

Theorem 4.12. *If the problem (SP) is M_1-well-posed and for every $V \in \mathcal{V}_X(0)$, $\overline{S}_0 \subseteq S_0 + V$, then the set-valued equilibrium problem (SEP) is M_1-well-posed.*

Proof. Let $V \in \mathcal{V}_X(0)$ be a neighbourhood of 0. Hence there exists $W \in \mathcal{V}_X(0)$ such that $W + W \subseteq V$.
Let $(x_n)_{n\in\mathbb{N}} \subseteq D$ be a minimizing sequence for φ. Since the problem (SP) is M_1-well-posed it follows that for W there exists $n_0 \in \mathbb{N}$ such that $x_n \in \overline{S}_0 + W$ for every $n \geq n_0$. Also, from the hypothesis $\overline{S}_0 \subseteq S_0 + W$. Therefore,

$$x_n \in \overline{S}_0 + W \subseteq S_0 + W + W \subseteq S_0 + V,$$

for every $n \geq n_0$.
Hence for every $V \in \mathcal{V}_X(0)$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in S_0 + V$ for every $n \geq n_0$.
The proof is complete. □

The following example illustrates Theorem 4.12.

Example 4.13. Let $e = (0, -1)$ and $f : D \times D \rightrightarrows Y$, where

$$D = [-1, 1],\ Y = \mathbb{R}^2,\ K = \mathbb{R}^2_+,$$

be defined as

$$f(x,y) = \begin{cases} \{((1-\mid y\mid)x, \mid y \mid x)\}, x \in [0,1], x \neq \frac{1}{n}, n \geq 2; y \in [-1,1]; \\ \{((1-\mid y\mid)(-x), (1-\mid y\mid)(\pm x))\}, x = \frac{1}{n}, n \geq 2, y \neq 0; \\ \{(-x, x)\}, x = \frac{1}{n}, n \geq 2, y = 0; \\ \{(x, \mid y \mid)\}, x \in [-1, 0), y \in [-1, 1]. \end{cases}$$

The map $\varphi : D \rightrightarrows Y$ is defined by

$$\varphi(x) = \begin{cases} [(0, x); (x, 0)], x \in [0,1], x \neq \frac{1}{n}, n \geq 2; \\ \{(-x, x)\}, x = \frac{1}{n}, n \geq 2; \\ \{(x, 0)\}, x \in [-1, 0). \end{cases}$$

The solution set for (SP) is $\overline{S}_0 = [0, 1]$ and the solution set of the set-valued equilibrium problem (SEP) is $S_0 = [0, 1] \setminus \{\frac{1}{n}, n \geq 2\}$. It is easy to observe that $\overline{S}_0 \subseteq S_0 + V$ for every $V \in \mathcal{V}_X(0)$. Also, every minimizing sequence of the set-valued problem (SP) is upper Hausdorff convergent to the set $\overline{S_0}$ and therefore the problem (SP) is M_1-well-posed. Finally, we observe that the problem (SEP) is also M_1-well-posed.

5. Conclusions

In this paper, we introduce some concepts of well-posedness for a set-valued equilibrium problem; the first of them generalizes a concept of well-posedness of the strong vector equilibrium problem studied by Bianchi et al.[3] in topological vector spaces. First, we focus on several properties of a suitable set-valued map φ and we obtain some sufficient results for well-posedness for our set-valued equilibrium problem in the presence of l-type K-convexity of the set-valued map φ in finite and infinite settings. The quasi-order relation induced by the nonempty closed convex pointed cone K in the topological vector space Y and the nice properties of the Gerstewitz map considered by Khushboo et al.[13], conducted us to another well-posedness concept for the set-valued equilibrium problem we are interested in. Some sufficient conditions for this well-posedness concept have been obtained via an appropriate set-valued problem.

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