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Some classes of surfaces generated by Nielson and Marshall type operators on the triangle with one curved side

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. We construct some classes of surfaces which satisfy some given conditions, using some Hermite, Nielson and Marshall type interpolation operators defined on a triangle with one curved side.

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1. Introduction

In some recent papers, we have introduced and studied some interpolation operators for the functions defined on triangles with curved sides (see, e.g., [8]-[11], [13], [14], [16], [17]). They permit essential boundary conditions to be satisfied exactly and they come as an extension of the interpolation operators on triangles with all straight edges, introduced and studied for example in [1], [3]-[7], [12], [22]-[28].

We consider here a standard triangle, $\tilde{T}$, having the vertices $V_1 = (1, 0)$, $V_2 = (0, 1)$ and $V_3 = (0, 0)$, two straight sides Γ_1, Γ_2, along the coordinate axes, and the third side Γ_3 (opposite to the vertex V_3), which is defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = 1$. There is no restriction to consider this standard triangle $\tilde{T}$, since any triangle with one curved side can be obtained from this standard triangle by an affine transformation which preserves the form and order of accuracy of the interpolant [4].

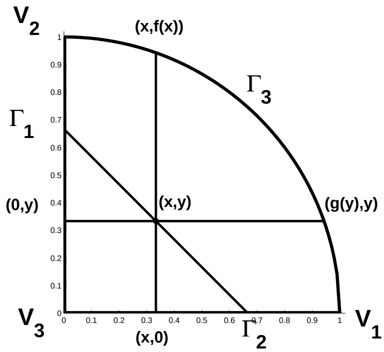

Figure 1: Triangle $\tilde{T}$.

The bending interpolants interpolate on an infinite set of points (segments, curves, etc.), so having such element as a boundary of an object, we may generate surfaces that contain the given boundary (see, e.g., [2], [18]-[21]). The aim of this paper is to construct some surfaces which satisfy some given conditions on the boundary of a domain that can be decomposed in triangles with one curved side. We construct some new surfaces using some Hermite, Nielson and Marshall type interpolation operators introduced in [13] and [14]. These operators come as extensions to triangle $\tilde{T}$, of some interpolation operators for triangles, given, for example, in [4], [5], [25].

2. Surfaces generation by Hermite, Nielson and Marshall type operators

Suppose that F is a real-valued function defined on $\tilde{T}$, and that it has all partial derivatives needed. We consider three types of interpolation operators defined on $\tilde{T}$:

- the Hermite interpolation operators H_1 and H_2 defined by [13]:

$$\begin{aligned}(H_1F)(x,y) =& \tfrac{[2x+g(y)][x-g(y)]^2}{g^3(y)} F(0,y) + \tfrac{x[x-g(y)]^2}{g^2(y)} F^{(1,0)}(0,y) \\ &+ \tfrac{x^2[-2x+3g(y)]}{g^3(y)} F(g(y),y) + \tfrac{x^2[x-g(y)]}{g^2(y)} F^{(1,0)}(g(y),y), \\ (H_2F)(x,y) =& \tfrac{[2y+f(x)][y-f(x)]^2}{f^3(x)} F(x,0) + \tfrac{y[y-f(x)]^2}{f^2(x)} F^{(0,1)}(x,0) \\ &+ \tfrac{y^2[-2y+3f(x)]}{f^3(x)} F(x,f(x)) + \tfrac{y^2[y-f(x)]}{f^2(x)} F^{(0,1)}(x,f(x)),\end{aligned} \tag{2.1}$$

- the Nielson type interpolation operators given by [14]:

$$\begin{aligned}(N_1F)(x,y) &= yF(x,f(x)) + (1-f(x))F(g(y),y), \\ (N_2F)(x,y) &= F(0,y) + F(x,0) - F(0,0),\end{aligned} \tag{2.2}$$

- the Marshall type operators defined by [14]:

$$\begin{aligned}
(Q_1F)(x,y) =& yF(0,1) + g(y)F(\tfrac{x}{g(y)},0), \qquad (2.3)\\
(Q_2F)(x,y) =& xF(1,0) + f(x)F(0,\tfrac{y}{f(x)}),\\
(Q_3F)(x,y) =& (f(x)-y)F(0,0) + (1-f(x)+y)F\left(\tfrac{x}{1-f(x)+y},\tfrac{y}{1-f(x)+y}\right).
\end{aligned}$$

For obtaining the first class of surfaces, we consider the boolean sum of the Nielson type operators N_1 and N_2, given in (2.2), namely,

$$\begin{aligned}
((N_1\oplus N_2)F)(x,y) =& y[F(x,f(x)) - F(0,f(x)) - F(x,0) + F(0,0)] \qquad (2.4)\\
&+ (1-f(x))[F(g(y),y) - F(0,y) - F(g(y),0)]\\
&+ F(0,y) + F(x,0) - f(x)F(0,0),
\end{aligned}$$

and we apply the condition that the roof stays on its support, i.e., $F|_{\Gamma_3} = 0$. We get

$$\begin{aligned}
S_N :=& -yF(0,f(x)) + (1-y)F(x,0) + [y-f(x)]F(0,0) \qquad (2.5)\\
&+ f(x)F(0,y) + [f(x)-1]F(g(y),0).
\end{aligned}$$

Theorem 2.1. *If $F|_{\Gamma_3} = 0$, then we have the following properties of the operator S_N :*

$$\begin{aligned}
(S_NF)(x,0) &= F(x,0).\\
(S_NF)(0,y) &= F(0,y).\\
(S_NF)(x,f(x)) &= 0.
\end{aligned}$$

Proof. The proof follows directly by the expression of S_N from (2.5). □

In the second level of approximation we use the Hermite interpolation operators, given in (2.1), taking into account the condition $F|_{\Gamma_3} = 0$, i.e.,

$$\begin{aligned}
(H_1^1F)(x,y) :=& \tfrac{[2x+g(y)][x-g(y)]^2}{g^3(y)}F(0,y) + \tfrac{x[x-g(y)]^2}{g^2(y)}F^{(1,0)}(0,y) \qquad (2.6)\\
&+ \tfrac{x^2[x-g(y)]}{g^2(y)}F^{(1,0)}(g(y),y),\\
(H_2^1F)(x,y) :=& \tfrac{[2y+f(x)][y-f(x)]^2}{f^3(x)}F(x,0) + \tfrac{y[y-f(x)]^2}{f^2(x)}F^{(0,1)}(x,0)\\
&+ \tfrac{y^2[y-f(x)]}{f^2(x)}F^{(0,1)}(x,f(x)).
\end{aligned}$$

We apply the following approximations:

$$\begin{aligned}
F(x,0) \approx (H_1^1F)(x,0) =& (2x+1)(x-1)^2F(0,0) + x(x-1)^2F^{(1,0)}(0,0) \qquad (2.7)\\
&+ x^2(x-1)F^{(1,0)}(1,0),
\end{aligned}$$

$$\begin{aligned}
F(0,y) \approx (H_2^1F)(0,y) =& (2y+1)(y-1)^2F(0,0) + y(y-1)^2F^{(0,1)}(1,0) \qquad (2.8)\\
&+ y^2(y-1)F^{(0,1)}(0,1),
\end{aligned}$$

and by (2.5), we obtain the following class of surfaces:

$$\begin{aligned}
(S_1F)(x,y) =& -y(H_2^1F)(0,f(x)) + (1-y)(H_1^1F)(x,0) + [y-f(x)]F(0,0)\\
&+ f(x)(H_2^1F)(0,y) + [f(x)-1](H_1^1F)(g(y),0),
\end{aligned}$$

i.e.,

$$\begin{aligned}(S_1F)(x,y) =& - y\{[2f(x)+1][f(x)-1]^2F(0,0) + f(x)[f(x)-1]^2F^{(0,1)}(1,0) \\ &+ f(x)^2[f(x)-1]F^{(0,1)}(0,1)\} + (1-y)[(2x+1)(x-1)^2F(0,0) \\ &+ x(x-1)^2F^{(1,0)}(0,0) + x^2(x-1)F^{(1,0)}(1,0)] + [y-f(x)]F(0,0) \\ &+ f(x)[(2y+1)(y-1)^2F(0,0) + y(y-1)^2F^{(0,1)}(1,0) \\ &+ y^2(y-1)F^{(0,1)}(0,1)] + [f(x)-1]\{[2g(y)+1][g(y)-1]^2F(0,0) \\ &+ g(y)[g(y)-1]^2F^{(1,0)}(0,0) + g(y)^2[g(y)-1]F^{(1,0)}(1,0)\}.\end{aligned} \tag{2.9}$$

For obtaining the second class of surfaces, we consider the boolean sum of the Marshall type operators Q_1, Q_2 and Q_3, given in (2.3):

$$\begin{aligned}((Q_1 \oplus Q_2 \oplus Q_3)F)(x,y) =& g(y)F(\tfrac{x}{g(y)},0) + f(x)F(0,\tfrac{y}{f(x)}) + (1-f(x)+y)\cdot \\ &\cdot F(\tfrac{x}{1-f(x)+y},\tfrac{y}{1-f(x)+y}) - yF(0,1) - g(y)f(\tfrac{x}{g(y)})F(0,0) \\ &- g(y)\left[1-f(\tfrac{x}{g(y)})\right]F\left(\tfrac{x}{g(y)}\Big/(1-f(\tfrac{x}{g(y)})),0\right),\end{aligned} \tag{2.10}$$

and supposing that the roof stays on its support we set the condition $F|_{\Gamma_3}=0$, hence we obtain

$$\begin{aligned}S_Q :=& g(y)F(\tfrac{x}{g(y)},0) + f(x)F(0,\tfrac{y}{f(x)}) \\ &- yF(0,1) - g(y)f(\tfrac{x}{g(y)})F(0,0) \\ &- g(y)\left[1-f(\tfrac{x}{g(y)})\right]F\left(\tfrac{x}{g(y)}\Big/(1-f(\tfrac{x}{g(y)})),0\right).\end{aligned} \tag{2.11}$$

Theorem 2.2. *If $F|_{\Gamma_3}=0$, then we have*

$$(S_QF)(x,f(x)) = 0.$$

Proof. The proof follows directly replacing in (2.11). □

In the second level we use the Hermite interpolation operators, given in (2.6), and the approximations (2.7) and (2.8), and we obtain the following class of surfaces:

$$\begin{aligned}(S_2F)(x,y) =& g(y)(H_1^1F)(\tfrac{x}{g(y)},0) + f(x)(H_2^1F)(0,\tfrac{y}{f(x)}) \\ &- yF(0,1) - g(y)f(\tfrac{x}{g(y)})F(0,0) \\ &- g(y)\left[1-f(\tfrac{x}{g(y)})\right](H_1^1F)\left(\tfrac{x}{g(y)}\Big/(1-f(\tfrac{x}{g(y)})),0\right),\end{aligned}$$

given below by

$$
\begin{aligned}
(S_2F)(x,y) =& \tfrac{[2x+g(y)][x-g(y)]^2}{g^2(y)} F(0,0) + \tfrac{x[x-g(y)]}{g^2(y)}^2 F^{(1,0)}(0,0) \qquad (2.12)\\
&+ \tfrac{x^2[x-g(y)]}{g^2(y)} F^{(1,0)}(1,0)\\
&+ \tfrac{[2y+f(x)][y-f(x)]^2}{f^2(x)} F(0,0) + \tfrac{y[y-f(x)]^2}{f^2(x)} F^{(0,1)}(1,0)\\
&+ \tfrac{y^2[y-f(x)]}{f^2(x)} F^{(0,1)}(0,1)\\
&- yF(0,1) - g(y)f(\tfrac{x}{g(y)})F(0,0)\\
&- g(y)\left[1 - f(\tfrac{x}{g(y)})\right] \cdot\\
&\cdot \{[2m(x,y)+1][m(x,y)-1]^2 F(0,0) \qquad (2.13)\\
&+ m(x,y)[m(x,y)-1]^2 F^{(1,0)}(0,0)\\
&+ m^2(x,y)[m(x,y)-1] F^{(1,0)}(1,0)\},
\end{aligned}
$$

where $m(x,y)$ denotes $\frac{x}{g(y)} \Big/ (1 - f(\frac{x}{g(y)}))$.

Other classes of surfaces may be obtained using the conditions

$$F|_{\Gamma_3} = F^{(0,1)}\Big|_{\Gamma_3} = F^{(1,0)}\Big|_{\Gamma_3} = 0. \qquad (2.14)$$

We consider the boolean sum of the Nielson type operators N_1 and $N_{2,}$ given in (2.4), taking into account the conditions (2.14), and we get the operator S_N given in (2.5).

In the second level we use the Hermite interpolation operators, given in (2.1), taking into account the conditions (2.14), so we have

$$
\begin{aligned}
(H_1^2F)(x,y) :=& \tfrac{[2x+g(y)][x-g(y)]^2}{g^3(y)} F(0,y) + \tfrac{x[x-g(y)]^2}{g^2(y)} F^{(1,0)}(0,y), \qquad (2.15)\\
(H_2^2F)(x,y) :=& \tfrac{[2y+f(x)][y-f(x)]^2}{f^3(x)} F(x,0) + \tfrac{y[y-f(x)]^2}{f^2(x)} F^{(0,1)}(x,0).
\end{aligned}
$$

Using the following approximations:

$$F(x,0) \approx (H_1^2F)(x,0) = (2x+1)(x-1)^2 F(0,0) + x(x-1)^2 F^{(1,0)}(0,0),$$

$$F(0,y) \approx (H_2^2F)(0,y) = (2y+1)(y-1)^2 F(0,0) + y(y-1)^2 F^{(0,1)}(1,0),$$

by (2.5), we obtain the following class of surfaces:

$$
\begin{aligned}
(S_3F)(x,y) =& - y(H_2^2F)(0,f(x)) + (1-y)(H_1^2F)(x,0) + [y - f(x)]F(0,0)\\
&+ f(x)(H_2^2F)(0,y) + [f(x)-1](H_1^2F)(g(y),0),
\end{aligned}
$$

further given as

$$
\begin{aligned}
(S_3F)(x,y) = &-y\{[2f(x)+1][f(x)-1]^2F(0,0)+f(x)[f(x)-1]^2F^{(0,1)}(1,0)\} \qquad (2.16)\\
&+(1-y)[(2x+1)(x-1)^2F(0,0)+x(x-1)^2F^{(1,0)}(0,0)]\\
&+[y-f(x)]F(0,0)+f(x)[(2y+1)(y-1)^2F(0,0)\\
&+y(y-1)^2F^{(0,1)}(1,0)]+[f(x)-1]\{[2g(y)+1][g(y)-1]^2F(0,0)\\
&+g(y)[g(y)-1]^2F^{(1,0)}(0,0)\}.
\end{aligned}
$$

Next we consider the boolean sum of the Marshall type operators, given in (2.10), taking into account the conditions (2.14), and we get S_Q given in (2.11).

Further, we apply the Hermite interpolation operators H_1^2 and H_2^2, given in (2.15), and we get the following class of surfaces:

$$
\begin{aligned}
(S_4F)(x,y) =&g(y)(H_1^2F)(\tfrac{x}{g(y)},0)+f(x)(H_2^2F)(0,\tfrac{y}{f(x)})\\
&-yF(0,1)-g(y)f(\tfrac{x}{g(y)})F(0,0)\\
&-g(y)\left[1-f(\tfrac{x}{g(y)})\right](H_1^2F)\left(\tfrac{x}{g(y)}\Big/(1-f(\tfrac{x}{g(y)})),0\right),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
(S_4F)(x,y) =&\tfrac{[2x+g(y)][x-g(y)]^2}{g^2(y)}F(0,0)+\tfrac{x[x-g(y)]^2}{g^2(y)}F^{(1,0)}(0,0) \qquad (2.17)\\
&+\tfrac{[2y+f(x)][y-f(x)]^2}{f^2(x)}F(0,0)+\tfrac{y[y-f(x)]^2}{f^2(x)}F^{(0,1)}(1,0)\\
&-yF(0,1)-g(y)f(\tfrac{x}{g(y)})F(0,0)\\
&-g(y)\left[1-f(\tfrac{x}{g(y)})\right]\cdot\\
&\cdot\{[2m(x,y)+1][m(x,y)-1]^2F(0,0)\\
&+m(x,y)[m(x,y)-1]^2F^{(1,0)}(0,0)\},
\end{aligned}
$$

where $m(x,y)$ denotes $\frac{x}{g(y)}\Big/\left(1-f\left(\frac{x}{g(y)}\right)\right)$.

3. Numerical examples

Example 3.1. Consider $F:\tilde{T}\to\mathbb{R}$,

$$
F(x,y)=\frac{(x^2+y^2-h^2)^2}{x^2+y^2+1} \quad \text{and} \quad f(x)=\sqrt{1-x^2}.
$$

In Figure 2 we plot the graphs of the surface S_1F, given in (2.9).

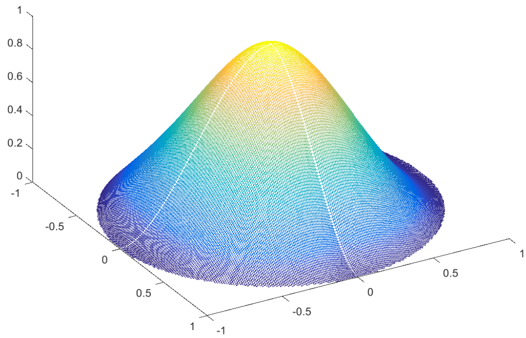

Figure 2: The surface S_1.

Example 3.2. Consider the function $f(x) = \sqrt{1-x^2}$ and $F : \tilde{T} \to \mathbb{R}$.
In Figure 3 we plot the graphs of surface S_2F, given in (2.12), assigning to the data $(F(0,0), F(0,1), F^{(1,0)}(0,0),\ F^{(1,0)}(1,0), F^{(0,1)}(1,0), F^{(0,1)}(0,1))$ the values $(-1/4, -1, 1, -1, 1)$.

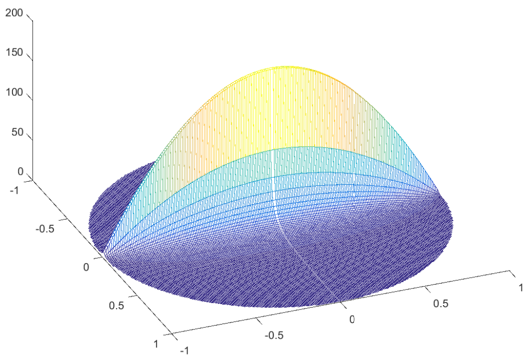

Figure 3: The surface S_2.

Example 3.3. Consider the data from Example 3.1. In Figure 4 we plot the graphs of the surface S_3F, given in (2.16).

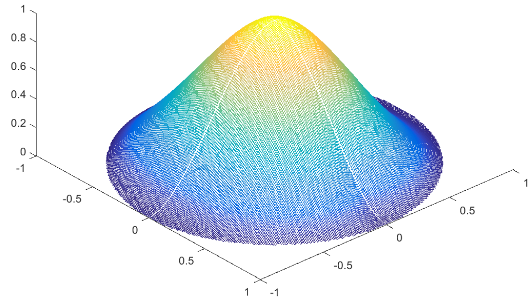

Figure 4: The surface S_3.

Example 3.4. Consider same data as in Example 3.2. In Figure 5 we plot the graphs of the surface S_4F, given in (2.17).

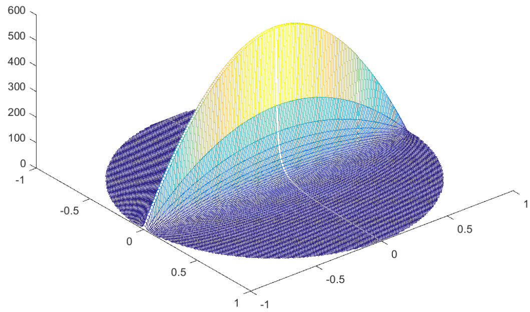

Figure 5: The surface S_4.

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