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Global existence and uniqueness for viscoelastic equations with nonstandard growth conditions

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Abstract. This paper is devoted to the study of generalized viscoelastic nonlinear equations with Dirichlet-Neumann boundary conditions. We establish the local and uniqueness of weak solutions results in Sobolev spaces with variable exponents. Solutions are constructed as a limit of approximate solutions by a method independent of a compactness argument. We also discuss the global existence of solutions in the energy space.

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1. Introduction

In this paper, we study the global existence and uniqueness of weak solutions for the nonlinear viscoelastic equation with the $m(x)$-Laplacian operator

$$\begin{cases} u_{tt} - \Delta_{m(x)} u + w_1 \Delta^2 u(t) - w_2 \Delta u_t(t) + \alpha(t) \displaystyle\int_0^t \beta(t-s) \Delta u(s)\, \mathrm{d}s \\ \qquad\qquad + |u|^{p(x)-2} u(t) + \lambda g(u_t(t)) = bf(u(t)) \text{ in } \Omega \times \mathbb{R}^+, \\ \qquad\qquad u = \partial_\eta u = 0 \text{ on } \Gamma \times [0, +\infty[\,, \\ \qquad\qquad u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $\Delta_{m(x)} u = div\left(|\nabla u|^{m(x)-2} \nabla u\right)$ is called the $m(x)$-Laplacian operator, $m(x)$ and $p(x)$ are two continuous functions on Ω, Ω is a bounded open subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$, β is a memory kernel that decays exponentially,

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$g(u_t)$ is a nonlinear damping term, $f(u)$ is a nonlinear generalized source term, u_0 and u_1 are given functions, and ∂_η denotes the normal derivative directed outside of Ω and $Q = \Omega\times[0,T]$. Problem (1.1), with its general memory term $\alpha(t)\int_0^t \beta(t-s)\,\Delta u(s)ds$, can be regarded as a fourth order viscoelastic plate equation with a lower–order perturbation of the usual m-Laplacian type ($m(x) = const \geq 2$). It can also be regarded as an elastoplastic flow equation with some kind of memory effect. We note that for viscoelastic plate equations, it is usual to consider a memory of the general form $\alpha(t)\int_0^t \beta(t-s)\,\Delta^2 u(s)ds$. However, because the main dissipation of the system (1.1) is given by strong damping $-\Delta u_t(t)$, here we consider a weaker memory, acting only on $\Delta u(t)$. There is a large body of literature about the stability and global existence of viscoelasticity. We refer the reader to, [9, 10, 8, 18, 19, 4, 2, 3, 1]. Our objective in the present work is to extend the results established in the study of the differential equation about global existence with standard m-growth in the study of generalized problem (1.1) with nonstandard $m(x)$-growth. Equations with nonstandard growth occur in the mathematical modeling of various physical phenomena, for example, the flows of electrorheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, processes of filtration through porous media and image processing.

2. Literature overview and new contributions

The semilinear case with the classical Laplace operator (when $m(x) = m = const$) and when ($p(x) = p = const$), was studied by many authors. Other related works include:

1. The asymptotic behavior of solutions of the equations of linear viscoelasticity at large times was considered first by Dafermos [9] in 1970, where the general decay was discussed.
$$u_{tt} - \Delta^2 u(t) - \Delta u_t(t) + \int_0^t \beta(t-s)\,\Delta u(s)\,\mathrm{d}s = 0.$$
From a physical point of view, this type of problem usually arises in viscoelasticity.
2. With the usual m–Laplacian operator $m(x) = p(p = \text{const} \geq 2)$, a more general problem concerning the energy decay for a class of plate equations with memory and lower order perturbation of the p–Laplacian type
$$u_{tt} - div\left(|\nabla u|^{p-2}\nabla u\right) + \Delta^2 u(t) - \Delta u_t(t) + \int_0^t \beta(t-s)\,\Delta u(s)\,\mathrm{d}s + f(u(t)) = 0$$
has been extensively studied in [5].
3. Problem (1.1) without the viscoelastic term, with the usual m-Laplacian operator ($m(x) = m-1$), ($p = const \geq 2$) has been extensively studied by Yang et al [6, 7] concerning existence, nonexistence and long-term dynamics,
$$u_{tt} - div\left(|\nabla u|^{m-1}\nabla u\right) + \Delta^2 u(t) - \Delta u_t(t) + g(u_t(t)) + h(u(t)) = f(x,t)$$

4. The following problem:

$$u_{tt} - \Delta u(t) + \int_0^t \beta(t-s)\,\Delta(u(s,x))\mathrm{d}s + |u|^{p-2}u + \sigma(x)\,u_t = 0$$

for $\sigma : \Omega \to \mathbb{R}^+$, a function, which may be null on a part of the domain Ω, has been considered and studied by many authors [8].

By assuming $\sigma(x) > \sigma_0$ on the subdomain $\overline{\varpi} \subset \Omega$, the authors obtained an exponential rate of decay, provided that the kernel β satisfies:

$$\begin{cases} -\zeta_1\beta(t) \le \beta'(t) \le -\zeta_2\beta(t), \ t \ge 0, \\ \|\beta\|_{L^\infty(0,+\infty)} \text{ is small enough.} \end{cases}$$

Motivated by previous works, the goal of this paper is to establish the local and uniqueness of weak solution results in Sobolev spaces with variable exponents. We also discuss the global existence of solutions in the energy space. We pay specific properties caused by the variable exponents $m(.)$ and $p(.)$.

3. Problem statement

In this section we list and recall some well-known results and facts from the theory of Sobolev spaces with variable exponents. (For the details see [11, 12, 13, 14, 15]). Throughout the rest of the paper we assume that Ω is a bounded domain of $\mathbb{R}^n$, $n \ge 2$ with smooth boundary Γ and assume that $p(x)$ and $m(x)$ satisfy:

$$\begin{cases} 2 < p_- \le p(x) \le p_+ < p_*(x) < \infty, \\ 2 < m_- \le m(x) \le m_+ < m_*(x) < \infty \end{cases} \tag{3.1}$$

where

$$\varphi_+ = ess\ \sup_{x\in\Omega}\varphi(x), \quad \varphi_- = ess\ \inf_{x\in\Omega}\varphi(x)$$

and

$$\varphi_*(x) \le \begin{cases} \dfrac{n\varphi(x)}{(n-\varphi(x))_+}, \text{ if } \varphi_+ < n \\ +\infty, \text{ if } \varphi_+ \ge n. \end{cases} \tag{3.2}$$

We also assume that

$$|m(x) - m(y)| \le \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2}, \tag{3.3}$$

with $M > 0$ and

$$m_* > ess\ \sup_{\{x\in\Omega\}} m(x) \tag{3.4}$$

Let $p : \Omega \to [1,\infty]$ be a measurable function. We denote by $L^{p(.)}(\Omega)$ the set of measurable functions u on Ω such that

$$A_{p(.)}(u) = \int_{\{x\in\Omega | p(x)<\infty\}} |u(x)|^{p(x)}\,\mathrm{d}x + \mathop{ess\ \sup}_{\{x\in\Omega | p(x)=\infty\}} |u(x)| < \infty$$

The set $L^{p(.)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{p(.)}=\|u\|_{L^{p(.)}(\Omega)}=\inf\left\{\mu>0,\ A_{p(.)}\left(\frac{u}{\mu}\right)\leq 1\right\},$$

is a Banach space with

$$\min\left(\|u\|_{p(.)}^{p_-},\|u\|_{p(.)}^{p_+}\right)\leq A_{p(.)}(u)\leq\max\left(\|u\|_{p(.)}^{p_-},\|u\|_{p(.)}^{p_+}\right)$$

and the generalized Hölder's inequality holds.

Let p satisfy the following Zhikov–Fan uniform local continuity condition :

$$|p(x)-p(y)|\leq\frac{M}{|\log|x-y||},\text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y|<\frac{1}{2},\ M>0, \tag{3.5}$$

with $\underset{\{x\in\Omega\}}{ess\ \inf}\left(p^*(x)-p(x)\right)>0$.

- If condition (3.5) is fulfilled, Ω has a finite measure and p, q are variable exponents so that $p(x)\leq q(x)$ almost everywhere in Ω, then the embedding $L^{q(.)}(\Omega)\hookrightarrow L^{p(.)}(\Omega)$ is continuous.
- If $p:\Omega\to[1,+\infty)$ is a measurable function and $p_*>\underset{\{x\in\Omega\}}{ess\ \sup}\,p(x)$ with $p_*\leq\frac{2n}{n-2}\,(n>2)$, $\left(p_*\leq\frac{2n}{n-4}\,(n>4)\right)$, then the embeddings $H_0^1(\Omega)\hookrightarrow L^{p(.)}(\Omega)$, and $\left(H_0^2(\Omega)\hookrightarrow L^{p(.)}(\Omega)\right)$ are continuous and compact respectively.

Let us state the precise hypotheses on g, f, α and β :

α is a measurable nonincreasing differentiable bounded function on $\mathbb{R}^+$ and

$$\alpha_+\geq\alpha(0)\geq\alpha(t)>0,\ t\geq 0. \tag{3.6}$$

Let g be increasing C^1–function such that:

$$\left\{\begin{aligned} xg(x)&\geq d_0|x|^{\sigma(x)},\ x\in\mathbb{R},\\ |g(x)|&\leq d_1|x|+d_2|x|^{\sigma(x)-1},\ x\in\mathbb{R},\ d_i\geq 0,\\ 2<\sigma_-\leq\sigma(x)\leq\sigma_+&\leq p(x)\leq p_+\leq\frac{2n}{n-2}<\infty,\ n\geq 3.\end{aligned}\right. \tag{3.7}$$

Let $f(x,s)\in C^1(\Omega\times\mathbb{R})$ satisfy:

$$sf(x,s)+k_1(x)|s|\geq p(x)\widehat{f}(x,s), \tag{3.8}$$

and the growth conditions

$$\left\{\begin{aligned} |f(x,s)|&\leq l_1\left(|s|^{\theta}+k_2(x)\right);\\ |f_s(x,s)|\leq l_1\left(|s|^{\theta-1}+k_3(x)\right)&\text{ in }\Omega\times\mathbb{R},\text{ and } 1<\theta\leq\frac{p_-}{2},\end{aligned}\right. \tag{3.9}$$

where $\widehat{f}(x,s)=\widehat{f}(s)=\int_0^s f(x,\zeta)\,\mathrm{d}\zeta$, with some l_0, $l_1>0$ and the nonnegative functions $k_1(x),k_2(x),k_3(x)\in L^{\infty}(\Omega)$, a.e. $x\in\Omega$.

The memory kernel $\beta : [0,+\infty[\to [0,+\infty[$ is a differentiable bounded function such that

$$\begin{cases} \beta(0)=\beta_0>0, \ \ \infty > \displaystyle\int_0^\infty \beta(t)\mathrm{d}t=\beta_1; \\ \hfill w_1\lambda_1-\alpha(0)\beta_1>0; \\ \alpha(t)\beta(t)+\alpha'(t)\displaystyle\int_0^t \beta(s)\,\mathrm{d}s \geq 0 \ \ t\in\mathbb{R}^+. \end{cases} \tag{3.10}$$

there exists $K>0$ such that

$$\beta'(t) \leq -K\beta(t) \quad \forall t \geq 0. \tag{3.11}$$

where $\lambda_1>0$ is determined by the imbedding inequality

$$\lambda_1 \left|\nabla u(t)\right|^2 \leq \left|\Delta u\right|^2. \tag{3.12}$$

Remark 3.1. Typical examples of functions satisfying (3.10) and (3.11), are

$$\beta(t)=\beta_0 e^{-at}, \quad a \geq \max\left(\frac{\beta_0\alpha(0)}{w_1\lambda_1}, K\right);$$

$$\alpha(t)=\alpha(0)\,e^{-\frac{\alpha(0)}{w_1\lambda_1}\int_0^t\beta(s)\mathrm{d}s}.$$

Remark 3.2. We remark from the first identity in (3.10) and assumption (3.6) that

$$w_1\lambda_1-\alpha(t)\int_0^t\beta(s)\mathrm{d}s \geq w_1\lambda_1-\alpha(0)\beta_1>0, \quad \text{for all } t\in\mathbb{R}^+.$$

4. Main result

In this section we establish an existence result for problem (1.1).

4.1. Local existence

Theorem 4.1. *Assume that* (3.6)-(3.11) *hold, given any* $(u_0,u_1)\in H_0^2(\Omega)\cap L^{p(\cdot)}(\Omega)\times L^2(\Omega)$*. Then problem* (1.1) *admits a solution* $u(t)$ *satisfying:*

$$u\in L^\infty(0,T;V\cap L^{p(\cdot)}(\Omega)), \tag{4.1}$$

where

$$V=\left\{\varphi\in H^2(\Omega) : \varphi=0 \ \mathit{on}\ \Gamma\right\}.$$

Proof. Let w_j $(j=1,2,...)$ satisfy the spectral problem

$$(w_j,v)_{H_0^2}=\lambda_j(w_j,v), \quad \forall v\in H_0^2,$$

where $(.,.)_{H_0^2}$ represents the inner product in H_0^2. The family of functions $\{w_1,w_2,...,w_m\}$ yield a Galerkin basis for both H_0^2 and $L^2(\Omega)$.

For any $m\in\mathbb{N}$, let us put $V_m=Span\{w_1,w_2,...,w_m\}$. We define

$$u_m(t)=\sum_{i=1}^m K_{jm}(t)w_j, \tag{4.2}$$

where K_{jm} satisfies:

$$\begin{aligned} &(u_{ttm}(t), w_j) + w_1(\Delta u_m, \Delta w_j) + w_2(\nabla u_{mt}, \nabla w_j) \\ &+a(u_m(t), w_j) + \left(|u_m|^{p(x)-2} u_m, w_j\right) \\ -&\left(\alpha(t) \int_0^t \beta(t-s) \nabla u_m(s) \mathrm{d}s, \nabla w_j\right) + \lambda(g(u_{mt}), w_j) = b(f(u_m), w_j), \end{aligned} \tag{Pm}$$

$$\begin{cases} u_m(0) = u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_j, \ u_{mt}(0) = u_{1m} = \sum_{i=1}^{m} \beta_{im} w_j \ , \\ \qquad u_{0m} \to u_0 \text{ in } V_m, \quad u_{1m} \to u_1 \text{ in } L^2(\Omega). \end{cases} \tag{4.3}$$

for $1 \leq j \leq m$, and

$$a(\psi, \Psi) = \int_{\Omega} |\nabla \psi|^{m(x)-2} \nabla \psi \nabla \Psi \mathrm{d}x.$$

As the family $\{w_1, w_2, ..., w_m\}$ is linearly independent, the problem (Pm) admits at least one local solution u_m in the interval $[0, t_m]$ verifying $u_m(t) \in L^2(0, t_m; V_m)$ and $u_{mt}(t) \in L^2(0, t_m; V_m)$. The estimate below will allow t_m to be independent of m.

A priori Estimate 1

Let us define

$$(\beta o \nabla u)(t) = \int_0^t \beta(t-s) \int_{\Omega} |\nabla u(s) - \nabla u(t)|^2 \mathrm{d}x\mathrm{d}s,$$

it is easy, by differentiating the term $\alpha(t)(\beta o \nabla u)(t)$ with respect to t, to show that

$$\begin{aligned} &\alpha(t) \int_{\Omega} \int_0^t \beta(t-s) \nabla u(s) \nabla u_t(t) \mathrm{d}x\mathrm{d}s \\ &= -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \alpha(t)(\beta o \nabla u)(t) - \alpha(t) |\nabla u(t)|^2 \int_0^t \beta(s) \mathrm{d}s \right\} \\ &+ \frac{1}{2} \alpha(t)(\beta' o \nabla u)(t) - \frac{1}{2} \alpha(t) \beta(t) |\nabla u(t)|^2 \\ &+ \frac{1}{2} \alpha'(t)(\beta o \nabla u)(t) - \frac{1}{2} \alpha'(t) |\nabla u(t)|^2 \int_0^t \beta(s) \mathrm{d}s. \end{aligned} \tag{4.4}$$

Next, replacing w_j in (Pm) by $u_{mt}(t)$, yields

$$\begin{aligned} &(u_{ttm}(t), u_{mt}(t)) + a(u_m(t), u_{mt}(t)) + w_1(\Delta u_m(t), \Delta u_{mt}(t)) \\ &+ w_2(\nabla u_m(t), \nabla u_{mt}(t)) \\ &+ \left(|u_m|^{p(x)-2} u_m(t), u_{mt}(t)\right) - \alpha(t) \int_0^t \beta(t-s)(\nabla u_m(s), \nabla u_{mt}(t)) \mathrm{d}s \\ &+ \lambda(g(u_{mt}), u_{mt}(t)) = b(f(u_m(t)), u_{mt}(t)). \end{aligned} \tag{4.5}$$

Using Young's inequality and (4.4), it results

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{array}{r} \frac{1}{2}\left|u_{mt}(t)\right|^2+\int_\Omega \frac{1}{m(x)}\left|\nabla u_m(t)\right|^{m(x)}\mathrm{d}x+\frac{1}{2}w_1\left|\Delta u_m\right|^2 \\ -\frac{1}{2}\left(\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u_m(t)\right|^2 \\ +\frac{1}{2}\alpha(t)\left(\beta o\nabla u_m\right)(t)+\int_\Omega\frac{1}{p(x)}\left|u_m(t)\right|^{p(x)}\mathrm{d}x-b\int_\Omega \widehat{f}\left(u_m(t)\right)\mathrm{d}x \end{array}\right) \tag{4.6}$$

$$\begin{aligned} &+\lambda\left(g\left(u_{mt}\right),u_{mt}(t)\right)+w_2\left|\nabla u_{mt}(t)\right|^2 \\ &=\frac{1}{2}\alpha(t)\left(\beta' o\nabla u_m\right)(t)+\frac{1}{2}\alpha'(t)\left(\beta o\nabla u_m\right)(t) \\ &-\frac{1}{2}\left(\alpha(t)\beta(t)+\alpha'(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u_m(t)\right|^2. \end{aligned}$$

We denote by E_m the energy functional associated with problem (1.1):

$$\begin{aligned} E_m(t)=&\frac{1}{2}\left|u_{mt}(t)\right|^2+\frac{1}{2}w_1\left|\Delta u_m\right|^2-\frac{1}{2}\left(\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u_m(t)\right|^2 \\ &+\frac{1}{2}\alpha(t)\left(\beta o\nabla u_m\right)(t)+\int_\Omega\frac{1}{m(x)}\left|\nabla u_m(t)\right|^{m(x)}\mathrm{d}x \\ &+\int_\Omega\frac{1}{p(x)}\left|u_m(t)\right|^{p(x)}\mathrm{d}x-b\int_\Omega\widehat{f}\left(u_m(t)\right)\mathrm{d}x. \end{aligned} \tag{4.7}$$

Using the conditions (3.6), (3.10) and (3.11), we see that

$$\begin{aligned} E_m'(t)\le&\frac{1}{2}\alpha(t)\left(\beta' o\nabla u_m\right)(t)-\frac{1}{2}\left(\alpha(t)\beta(t)+\alpha'(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u_m(t))\right|^2 \\ &+\frac{1}{2}\alpha'(t)\left(\beta o\nabla u_m\right)(t)\le 0 \quad \forall t\ge 0. \end{aligned} \tag{4.8}$$

The Young's inequality and (3.8), gives

$$-b\int_\Omega\widehat{f}\left(u_m(t)\right)\mathrm{d}x\ge-\int_\Omega\frac{b}{p(x)}k_1(x)\left|u_m\right|\mathrm{d}x-\int_\Omega\frac{b}{p(x)}u_m f\left(x,u_m\right)\mathrm{d}x \tag{4.9}$$

$$\begin{aligned} &\ge-\varepsilon_+\frac{1}{p_-^2}\int_\Omega\left|u_m(t)\right|^{p(x)}\mathrm{d}x-C_{\varepsilon_+}\int_\Omega\left|k_1(x)\right|^{p'(x)}\mathrm{d}x \\ &-\int_\Omega\frac{b}{p(x)}u_m f\left(x,u_m\right)\mathrm{d}x. \end{aligned}$$

Next, using hypothesis (3.9) and Young's inequality, we obtain

$$\begin{aligned}
\int_\Omega \frac{b}{p(x)} u_m f(x,u_m)\,\mathrm{d}x &\le \int_\Omega \frac{b}{p(x)} |f(x,u_m)|\,|u_m|\,\mathrm{d}x \\
&\le \frac{l_1^2}{p_-}\varepsilon_+ \int_\Omega \left(|u_m|^{2\theta} + |k_2(x)|^2\right)\mathrm{d}x + \frac{c(\varepsilon_+,p_-)}{p_-^2}\int_\Omega |u_m|^2\,\mathrm{d}x \\
\le \frac{l_1^2}{p_-}\varepsilon_+ &\left(\int_\Omega \frac{p(x)-2\theta}{p(x)}\mathrm{d}x + 2\theta\int_\Omega \frac{1}{p(x)}|u_m(t)|^{p(x)}\,\mathrm{d}x\right) + \frac{l_1^2}{p_-}\varepsilon_+ \|k_2(x)\|_\infty^2 \\
&+C'(\varepsilon_+,p_-) + \frac{\varepsilon_+}{p_-^2}\int_\Omega |u_m(t)|^{p(x)}\,\mathrm{d}x \qquad (4.10)\\
&\le \frac{l_1^2}{p_-}\varepsilon_+\left(|\Omega|\frac{p_+-2\theta}{p_-} + \frac{2\theta}{p_-}\int_\Omega |u_m(t)|^{p(x)}\,\mathrm{d}x\right) \\
&+\frac{l_1^2}{p_-}\varepsilon_+\|k_2(x)\|_\infty^2 + C'(\varepsilon_+,p_-) + \frac{\varepsilon_+}{p_-^2}\int_\Omega |u_m(t)|^{p(x)}\,\mathrm{d}x.
\end{aligned}$$

Now replace (4.10) in (4.9) and let $0<\varepsilon_+\le \frac{p_-^2}{p_+\left(2+2\theta l_1^2\right)}$; by using (3.10), (3.12) and Remark 3.2 from (4.7), we obtain:

$$\begin{aligned}
E_m(t) &\ge \frac{1}{2}|u_{mt}(t)|^2 + \frac{1}{2\lambda_1}\left(w_1\lambda_1 - \alpha(0)\beta_1\right)|\Delta u_m(t)|^2 \qquad (4.11)\\
&+C_1\int_\Omega |\nabla u_m(t)|^{m(x)}\,\mathrm{d}x + C_2\int_\Omega |u_m(t)|^{p(x)}\,\mathrm{d}x - C_3(1+K_1+K_2),
\end{aligned}$$

or

$$\begin{aligned}
|u_{mt}(t)|^2 + |\Delta u_m(t)|^2 &+ \int_\Omega |u_m(t)|^{p(x)}\,\mathrm{d}x + \int_\Omega |\nabla u_m(t)|^{m(x)}\,\mathrm{d}x \\
&\le C_4\left(E_m(t) + K_1 + K_2 + 1\right), \qquad (4.12)
\end{aligned}$$

where

$$\begin{aligned}
C_1 &\ge \frac{1}{m_+},\ 0<C_2 = \frac{p_-^2 - p_+\left(2+2\theta l_1^2\right)\varepsilon_+}{p_-^2 p_+}, \\
C_3 &= \max\left(C_{\varepsilon_+}; \frac{l_1^2}{p}\varepsilon_+; C'(\varepsilon_+,p_-) + \frac{l_1^2}{p_-}\varepsilon_+\frac{p_- - 2\theta}{p_-}\right), \\
C_4 &= \max\left(\frac{1}{\min\left(\frac{1}{2\lambda_1}\left(w_1\lambda_1 - \alpha(0)\beta_1\right), C_1, C_2\right)}, C_3\right).
\end{aligned}$$

Thus, it follows from (4.6), (4.8) and (4.12) that

$$\begin{aligned}
|u_{mt}(t)|^2 &+ \int_\Omega |\nabla u_m(t)|^{m(x)}\,\mathrm{d}x + |\Delta u_m|^2 + \int_\Omega |u_m(t)|^{p(x)}\,\mathrm{d}x \\
&+w_2\int_0^t |\nabla u_{mt}(s)|^2\,\mathrm{d}s + \lambda\int_0^t \left(g(u_{mt}(s)), u_{mt}(s)\right)\mathrm{d}s \qquad (4.13)\\
&\le C_4\left(E_m(0) + K_1 + K_2 + 1\right) \quad \text{for every } t\ge 0
\end{aligned}$$

where $K_1 = \|k_1\|_\infty^2$, $K_2 = \|k_2\|_\infty^2$.

According to Hölder's inequality, using (3.8) and (3.9), we have

$$\left| b \int_\Omega \widehat{f}(u_m(0))\,\mathrm{d}x \right| \le \frac{b}{p_-} \int_\Omega |k_1(x)|\,|u_{0m}|\,\mathrm{d}x + \frac{b}{p_-} \int_\Omega |u_{0m}|\,|f(x,u_{0m})|\,\mathrm{d}x$$
$$\le C\left(|u_{0m}|^2 + \|k_1\|_\infty^2 + \int_\Omega |u_{0m}|^{p(x)}\,\mathrm{d}x + \|k_2\|_\infty^2 + |u_{0m}|^2 \right).$$

Therefore from (4.7) one has

$$E_m(0) = \frac{1}{2}|u_{1m}|^2 + \int_\Omega \frac{1}{m(x)} |\nabla u_{0m}|^{m(x)}\,\mathrm{d}x + \frac{1}{2}|\Delta u_{0m}|^2$$
$$+ \int_\Omega \frac{1}{p(x)} |u_{0m}|^{p(x)}\,\mathrm{d}x - b\int_\Omega \widehat{f}(u_{0m})\,\mathrm{d}x$$
$$\le C\left(|u_{1m}|^2 + \int_\Omega |\nabla u_{0m}|^{m(x)}\,\mathrm{d}x + |\Delta u_{0m}|^2 + \int_\Omega |u_{0m}|^{p(x)}\,\mathrm{d}x + |u_{0m}|^2 + K_1 + K_2 \right).$$

Then from (4.3) and (4.13), we obtain

$$|u_{mt}(t)|^2 + \int_\Omega |\nabla u_m(t)|^{m(x)}\,\mathrm{d}x + |\Delta u_m|^2 + \int_\Omega |u_m(t)|^{p(x)}\,\mathrm{d}x$$
$$+ w_2 \int_0^t |\nabla u_{mt}(s)|^2\,\mathrm{d}s + \lambda \int_0^t (g(u_{mt}(s)), u_{mt}(s))\,\mathrm{d}s \le C,$$

for some positive constant $C > 0$.

Gronwall's inequality and assumption (3.7) gives

$$\begin{cases} u_m \text{ is bounded in } L^\infty\left(0,T; H_0^2(\Omega) \cap L^{p(\cdot)}(\Omega)\right), \\ u_{mt} \text{ is bounded in } L^\infty\left(0,T; L^2(\Omega)\right), \\ g(u_{mt}).u_{mt} \text{ is bounded in } L^1(\Omega \times (0,T)), \\ u_{mt} \text{ is bounded in } L^2\left(0,T; L^{\sigma(\cdot)}(\Omega)\right), \\ \nabla u_{mt} \text{ is bounded in } L^2\left(0,T; L^2(\Omega)\right), \\ \nabla u_m \text{ is bounded in } L^\infty\left(0,T; L^{m(\cdot)}(\Omega)\right), \\ \Delta_{m(.)}(u_m) \text{ is bounded in } L^\infty\left(0,T; W^{-1,m'(\cdot)}(\Omega)\right). \end{cases} \tag{4.14}$$

Since $H_0^1 \hookrightarrow W_0^{1,p_+}(\Omega)$, we can use the standard projection arguments as in Lions [16]. Then from (Pm) and the estimates (4.14), we obtain

$$u_{ttm} \text{ is bounded in } L^2\left(0,T; H_0^{-1}(\Omega)\right). \tag{4.15}$$

To estimate the term $g(u_{mt}(t))$ we need the following lemma.

Lemma 4.2. *For all $m \in \mathbb{N}$ there exists $M > 0$ such that*

$$||g(u_{mt}(t))||_{L^{\frac{\sigma(x)}{\sigma(x)-1}}(Q)} \le M.$$

Proof. Thanks to Holder's, and Young's inequalities, from (3.7), we get

$$\int_\Omega |g(u_{mt})|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x = \int_\Omega |g(u_{mt})|\,|g(u_{mt})|^{\frac{1}{\sigma(x)-1}}\,\mathrm{d}x$$
$$\leq \int_\Omega |g(u_{mt}(t))|\left(d_1|u_{mt}(t)| + d_2|u_{mt}(t)|^{\sigma(x)-1}\right)^{\frac{1}{\sigma(x)-1}}\,\mathrm{d}x$$
$$\leq C\int_\Omega |g(u_{mt}(t))|\left(|u_{mt}(t)|^{\frac{1}{\sigma(x)-1}} + |u_{mt}(t)|\right)\mathrm{d}x$$
$$= C\int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|^{\frac{1}{\sigma(x)-1}}\,\mathrm{d}x + C\int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|\,\mathrm{d}x$$
$$\leq \frac{\sigma_+-1}{\sigma_+}\int_\Omega |g(u_{mt})|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x + C(\sigma_+,\sigma_-)\int_\Omega |u_{mt}(t)|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x$$
$$+C\int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|\,\mathrm{d}x,$$

therefore

$$\frac{1}{\sigma_+}\int_\Omega |g(u_{mt}(t))|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x \leq C(\sigma_+,\sigma_-)\int_\Omega |u_{mt}(t)|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x$$
$$+C\int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|\,\mathrm{d}x \leq C\,||u_{mt}(t)||_2^{\frac{\sigma(x)}{\sigma(x)-1}} + C\int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|\,\mathrm{d}x,$$

hence, estimates (4.14) gives

$$\int_0^T\int_\Omega |g(u_{mt}(t))|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x\mathrm{d}t \leq M. \tag{4.16}$$

□

By estimate (4.16)

$$g(u_{mt}(t)) \to g(u_t(t)) \ \ a.e. \text{ in } \Omega\times(0,T)$$

Therefore from Lions [16, Lemma 1.3] we infer that

$$g(u_{mt}) \to g(u_t) \ \text{ in } L^{\frac{\sigma(.)}{\sigma(.)-1}}(\Omega\times(0,T)) \ \text{ weak star.} \tag{4.17}$$

Passage to the limit

On the other hand, we have from (4.14)

$$\begin{cases} u_m \longrightarrow u \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\cap L^{p(\cdot)}(\Omega)\right), \\ \Delta^2 u_m \longrightarrow \Delta^2 u \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\cap L^{p(\cdot)}(\Omega)\right), \\ u_{mt} \longrightarrow u_t \text{ weak star in } L^2\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H_0^1(\Omega)\right), \\ g(u_{mt}) \longrightarrow g(u_t) \ \text{ weak star in } L^{\frac{\sigma(.)}{\sigma(.)-1}}(\Omega\times(0,T)), \\ \Delta u_{mt}(t) \to \Delta u_t(t) \ \text{ weak star in } L^2\left(0,T;H^{-1}(\Omega)\right), \\ \Delta_{m(.)}(u_m) \to \psi \text{ weak star in } L^\infty\left(0,T;W^{-1,m'(.)}(\Omega)\right) \end{cases} \tag{4.18}$$

By applying the Lions-Aubin compactness lemma, we obtain, for any $T > 0$,

$$u_m \longrightarrow u \text{ strongly in } L^2\left(0, T; H_0^1(\Omega)\right). \tag{4.19}$$

Using the compactness of $H_0^1(\Omega)$ in $L^2(\Omega)$, it is easy to verify

$$\int_0^T \int_\Omega |u_m|^{p(.)-2} u_m v \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega |u|^{p(.)-2} uv\mathrm{d}x\mathrm{d}t \text{ for all } v \in L^{\sigma(.)}\left(0, T; H_0^1(\Omega)\right),$$

as $m \to \infty$.

Using growth conditions (3.9) and (4.18), we see that $\int_0^T \int_\Omega |f(u_m)|^{\frac{\theta+1}{\theta}} \mathrm{d}x\mathrm{d}t$ is bounded and

$$f(u_m) \longrightarrow f(u) \text{ a.e.in } \Omega \times (0, T),$$

then

$$f(u_m) \longrightarrow f(u) \text{ weak star in } L^{\frac{\theta+1}{\theta}}\left(0, T; L^{\frac{\theta+1}{\theta}}\right),$$

as $m \to \infty$, which implies that

$$\int_0^T \int_\Omega f(u_m) v\mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega f(u) v\mathrm{d}x\mathrm{d}t \text{ for all } v \in L^{\theta+1}\left(0, T; H_0^1(\Omega)\right).$$

Passing to the limit in (Pm), we have

$$\begin{aligned}&(u_{tt}(t), v) - (\psi, v) + w_1\left(\Delta^2 u, v\right) - w_2(\Delta u_t, v) + \left(|u|^{p(.)-2} u, v\right) \\ &-\left(\alpha(t)\int_0^t \beta(t-s)\nabla u(s)\mathrm{d}s, \nabla v\right) + \lambda(g(u_t), v) = b(f(u), v) \quad \forall v \in W^{1,p(.)}(\Omega).\end{aligned} \tag{4.20}$$

Finally, by strong convergence, we can use a standard monotonicity argument as done in Lions [16] or Ma & Soriano [17] to show that $\psi = \Delta_{m(.)}(u)$. Then we infer that limit u satisfies (4.1) and

$$\begin{aligned}u_{tt} - \Delta_{m(.)}(u) + w_1\Delta^2 u - w_2\Delta u_t + \alpha(t)\int_0^t \beta(t-s)\Delta u(s))\mathrm{d}s + |u|^{p(.)-2} u \\ + \lambda g(u_t) = bf(u).\end{aligned}$$

From where the proof of theorem (4.1). □

4.2. Uniqueness

In this subsection, the uniqueness of the solution will be proven.

Theorem 4.3. *Let the assumptions of theorem 4.1 hold. Assume further that*

$$p_+ \leq \frac{2n-2}{n-2},\ n \neq 2\ \ (p_+ < \infty \textit{ if } n \leq 2) \tag{4.21}$$

$$m_+ \leq \frac{2n-2}{n-2},\ n \neq 2\ \ (m_+ < \infty \textit{ if } n \leq 2), \tag{4.22}$$

$$1 < \theta \leq \frac{p_-}{2}, \tag{4.23}$$

Then, there exists a unique solution u to problem 1.1 and it satisfies (4.1).

Proof. Let u, v be two weak solutions of problem 1.1, and set $\Psi = u - v$. Then, Ψ satisfies the equation

$$\begin{aligned}\Psi_{tt}(t) - \left(\Delta_{m(.)}u(t) - \Delta_{m(.)}v(t)\right) + w_1\Delta^2\Psi(t) - w_2\Delta\Psi'(t)\\ +\lambda\left(g\left(u_t(t)\right) - g\left(v_t(t)\right)\right) + \left(|u(t)|^{p(.)-2}u(t) - |v(t)|^{p(.)-2}v(t)\right)\\ +\alpha(t)\int_0^t \beta(t-s)\Delta\Psi(s)\mathrm{d}s = b\left(f\left(u(t)\right) - f\left(v(t)\right)\right)\end{aligned} \tag{4.24}$$

in $L^2\left(0,T;L^2(\Omega)\right)$, $T>0$, with boundary conditions and null initial data.
As $\Psi' \in L^2\left(0,T;H_0^1(\Omega)\right)$, multiplying above equation by $\Psi'(t)$, to get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_t(t)|^2 + w_1\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\Psi(t)|^2 + w_2|\nabla\Psi_t|^2 + \left(g(u_t) - g(v_t), u_t - v_t\right)\\ +\left(|\nabla u|^{m(.)-2}\nabla u - |\nabla v|^{m(.)-2}\nabla v, \nabla\Psi_t\right) = \int_\Omega\left(|v|^{p(.)-2}v - |u|^{p(.)-2}u\right)\Psi_t\mathrm{d}x\\ +\left(f(u) - f(v), \Psi_t\right) + \alpha(t)\int_\Omega\int_0^t \beta(t-s)\nabla\Psi(s)\nabla\Psi_t(t)\,\mathrm{d}s\mathrm{d}x.\end{aligned} \tag{4.25}$$

From (3.7) we have:

$$\left(g(u_t) - g(v_t), u_t - v_t\right) \geq 0.$$

Thanks to Hölder's inequality, we estimated the first term on the right hand side of (4.25) as follows:

$$\begin{aligned}\left|\int_\Omega(|v|^{p(x)-2}v - |u|^{p(x)-2}u)\Psi_t\mathrm{d}x\right| &\leq (p_+ - 1)\int_\Omega \sup\left(|u|^{p(x)-2}, |v|^{p(x)-2}\right)|\Psi||\Psi_t|\,\mathrm{d}x\\ &\leq (p_+ - 1)\int_\Omega\left(|u|^{p_+-2} + |v|^{p_+-2} + |u|^{p_--2} + |v|^{p_--2}\right)|\Psi||\Psi_t|\,\mathrm{d}x\\ &\leq C\begin{pmatrix}\|u\|_{L^{n(p_+-2)}(\Omega)}^{p_+-2} + \|v\|_{L^{n(p_+-2)}(\Omega)}^{p_+-2}\\ +\|u\|_{L^{n(p_--2)}(\Omega)}^{p_--2} + \|v\|_{L^{n(p_--2)}(\Omega)}^{p_--2}\end{pmatrix}||\Psi(t)||_{L^q(\Omega)}|\Psi_t(t)|,\end{aligned}$$

where $\frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1$, and from (4.21), $n(p_- - 2) \leq n(p_+ - 2) \leq \frac{2n}{n-2} = q$ which gives by estimate (4.1), Young's inequality and as $H_0^1(\Omega) \subset L^q(\Omega)$, that:

$$\begin{aligned}&\left|\int_\Omega(|v|^{p(x)-2}v - |u|^{p(x)-2}u)\Psi_t\mathrm{d}x\right|\\ &\leq C\begin{pmatrix}\|\nabla u\|_{L^2(\Omega)}^{p_+-2} + \|\nabla v\|_{L^2(\Omega)}^{p_+-2}\\ +\|\nabla u\|_{L^2(\Omega)}^{p_--2} + \|\nabla v\|_{L^2(\Omega)}^{p_--2}\end{pmatrix}||\nabla\Psi(t)||_{L^2(\Omega)}|\Psi_t(t)|\\ &\leq C\left(|\nabla\Psi(t)|^2 + |\Psi_t(t)|^2\right).\end{aligned}$$

By the same manner and by condition (4.21), we have

$$\left|\int_{\Omega}(|\nabla u|^{m(x)-2}\nabla u-|\nabla v|^{m(x)-2}\nabla v\nabla\Psi_t\mathrm{d}x\right|$$
$$\leq(m_+-1)\int_{\Omega}\sup\left(|\nabla u|^{m(x)-2},|\nabla v|^{m(x)-2}\right)|\nabla\Psi|\,|\nabla\Psi_t|\,\mathrm{d}x$$
$$\leq C\begin{pmatrix}\|u\|^{m_+-2}_{L^{n(m_+-2)}(\Omega)}+\|v\|^{m_+-2}_{L^{n(m_+-2)}(\Omega)}\\+\|u\|^{m_--2}_{L^{n(m_--2)}(\Omega)}+\|v\|^{m_--2}_{L^{n(m_--2)}(\Omega)}\end{pmatrix}||\Psi(t)||_{L^q(\Omega)}\,|\Psi'(t)|\,,$$
$$\leq C\begin{pmatrix}\|\nabla u\|^{m_+-2}_{L^2(\Omega)}+\|\nabla v\|^{m_+-2}_{L^2(\Omega)}\\+\|\nabla u\|^{p_--2}_{L^2(\Omega)}+\|\nabla v\|^{p_--2}_{L^2(\Omega)}\end{pmatrix}||\nabla\Psi(t)||_{L^2(\Omega)}\,|\Psi_t(t)|$$
$$\leq C\left(|\nabla\Psi(t)|^2+|\Psi_t(t)|^2\right).$$

Now setting $U_\zeta=\zeta u+(1-\zeta)v$, $0\leq\zeta\leq1$, from the growth condition it follows that

$$\left|\int_0^t\int_{\Omega}|f(u)-f(v)|\,|\Psi_t|\,\mathrm{d}x\mathrm{d}t\right|=\left|\int_0^t\int_{\Omega}\int_0^1\frac{\mathrm{d}}{\mathrm{d}\zeta}f(U_\zeta)\,\mathrm{d}\zeta\Psi_t\mathrm{d}x\mathrm{d}t\right|$$
$$\leq\int_0^t\int_{\Omega}\left|\int_0^1\frac{\mathrm{d}}{\mathrm{d}\zeta}f(U_\zeta)\,\mathrm{d}\varepsilon\right||\Psi_t|\,\mathrm{d}x\mathrm{d}s$$
$$\leq\int_0^t\int_{\Omega}\int_0^1\left|\frac{\mathrm{d}}{\mathrm{d}\zeta}f(U_\zeta)\,\mathrm{d}\zeta\right||\Psi_t|\,\mathrm{d}x\mathrm{d}s$$
$$\leq l_1\int_0^t\int_{\Omega}\int_0^1\left(|U_\zeta|^{\theta-1}+|k_3(x)|\right)|u-v|\,|\Psi_t|\,\mathrm{d}\zeta\mathrm{d}x\mathrm{d}s$$
$$\leq C\int_0^t\int_{\Omega}\left(|u|^{\theta-1}+|v|^{\theta-1}+|k_3(x)|\right)|\Psi(s)|\,|\Psi_t(s)|\,\mathrm{d}x\mathrm{d}s=I.$$

Using generalized Hölder's, Young's inequalities, estimates (4.1), and let λ satisfy:

$$1<\lambda+1\leq\min\left(\frac{n}{(n-2)(\theta-1)},\frac{n}{n-2}\right),\ n\neq2\ \ (\lambda<\infty\text{ if }n\leq2)\tag{4.26}$$

from (4.23), the following estimates hold,

$$I\leq C\int_0^t\left\|l_1\left(|u|^{\theta-1}+|v|^{\theta-1}+|k_3(x)|\right)\right\|^{\lambda}_{2(\lambda+1)}||\Psi||_{2(\lambda+1)}\,\|\Psi_t\|_2$$
$$\leq C\int_0^t\left(\left\||u|^{\theta-1}\right\|^{\lambda}_{2(\lambda+1)}+\left\||v|^{\theta-1}\right\|^{\lambda}_{2(\lambda+1)}+\|k_3(x)\|^{\lambda}_{2(\lambda+1)}\right)||\Psi||_{2(\lambda+1)}\,\|\Psi_t\|_2\,\mathrm{d}s$$
$$\leq C\int_0^t\left(\|\nabla u\|_2^{\lambda(\theta-1)}+\|\nabla v\|_2^{\lambda(\theta-1)}+\|k_3(x)\|^{\lambda}_{\infty}\right)||\nabla\Psi||_2\,\|\Psi_t\|_2\,\mathrm{d}s$$
$$\leq C\int_0^t\left(|\Psi_t(s)|^2+|\nabla\Psi(s)|^2\right)\mathrm{d}s.$$

because by (4.26) we have $||\Psi||_{2(\lambda+1)}\leq||\nabla\Psi||_2$.

Combining the above inequalities with identity (4.4), from (4.25), we derive

$$\begin{aligned}
&\frac{1}{2}\left|\Psi_t(t)\right|^2+\frac{1}{2}C\left(w_1\lambda_1-\alpha\left(t\right)\int_0^t\beta\left(s\right)\mathrm{d}s\right)\left|\nabla\Psi(t)\right|^2\\
&+C_2\int_0^t\left|\nabla\Psi_t\left(s\right)\right|^2\mathrm{d}s+\frac{1}{2}\alpha\left(t\right)\left(\beta o\nabla\Psi\right)\left(t\right)\\
&\leq C\int_0^t\left(\left|\Psi_t\left(s\right)\right|^2+\left|\nabla\Psi(s)\right|^2\right)\mathrm{d}s+\frac{1}{2}\int_0^t\alpha'\left(s\right)\left(\beta o\nabla\Psi\right)\left(s\right)\mathrm{d}s\\
&+\frac{1}{2}\int_0^t\alpha\left(s\right)\left(\beta' o\nabla\Psi\right)\left(s\right)\mathrm{d}s-\frac{1}{2}\int_0^t\left(\alpha\left(s\right)\beta\left(s\right)+\alpha'\left(s\right)\int\limits_0^s\beta\left(\zeta\right)\mathrm{d}\zeta\right)\left|\nabla\Psi\left(s\right)\right|^2\mathrm{d}s
\end{aligned}$$

Then, from remark (3.2), assumptions (3.10) gives

$$\left|\Psi_t(t)\right|^2+\left(w_1\lambda_1-\alpha\left(0\right)\beta_1\right)\left|\nabla\Psi(t)\right|^2\leq C\int_0^t\left(\left|\Psi_t\left(s\right)\right|^2+\left|\nabla\Psi(s)\right|^2\right)\mathrm{d}s.$$

and then by Gronwall's inequality we deduce that: $\Psi\left(t\right)=\Psi\left(0\right)=0$ in $H_0^2(\Omega)$. □

To study the global existence of the energy function, we define some functionals and establish several lemmas. Let the functions:

$$\begin{aligned}
I\left(t\right)=I\left(u\left(t\right)\right)=&\frac{p\left(x\right)}{4}\left(w_1\lambda_1-\alpha\left(t\right)\int_0^t\beta\left(s\right)\mathrm{d}s\right)\left|\nabla u\left(t\right)\right|^2\\
&-b\int_\Omega f\left(u\left(t\right)\right)u\left(t\right)\mathrm{d}x-b\int_\Omega k_1\left(x\right)\left|u\left(t\right)\right|\mathrm{d}x;
\end{aligned}\tag{4.27}$$

$$J\left(t\right)=J\left(u\left(t\right)\right)=\frac{1}{2}\left(\lambda_1w_1-\alpha\left(t\right)\int_0^t\beta\left(s\right)\mathrm{d}s\right)\left|\nabla u\left(t\right)\right|^2-b\int_\Omega\widehat{f}\left(x,u\right)\mathrm{d}x;\tag{4.28}$$

$$\begin{aligned}
E(t)=E(u(t),u_t(t))\geq J\left(u\left(t\right)\right)&+\frac{1}{2}\left|u_t(t)\right|^2+\int_\Omega\frac{1}{p\left(x\right)}\left|u(t)\right|^{p(x)}\mathrm{d}x\\
&+\int_\Omega\frac{1}{m\left(x\right)}\left|\nabla u(t)\right|^{m(x)}\mathrm{d}x+\frac{1}{2}\alpha\left(t\right)\left(\beta o\nabla u\right)\left(t\right).
\end{aligned}\tag{4.29}$$

And the set as

$$W=\left\{u:u\in H_0^2\left(\Omega\right),\,I\left(t\right)>0\right\}\cup\left\{0\right\}.\tag{4.30}$$

where

$$\begin{aligned}
E(t)=&\frac{1}{2}\left|u_t(t)\right|^2+\frac{1}{2}w_1\left|\Delta u\right|^2-\frac{1}{2}\left(\alpha\left(t\right)\int_0^t\beta\left(s\right)\mathrm{d}s\right)\left|\nabla u\left(t\right)\right|^2+\frac{1}{2}\alpha\left(t\right)\left(\beta o\nabla u\right)\left(t\right)\\
&+\int_\Omega\frac{1}{m\left(x\right)}\left|\nabla u(t)\right|^{m(x)}\mathrm{d}x+\int_\Omega\frac{1}{p\left(x\right)}\left|u(t)\right|^{p(x)}\mathrm{d}x-b\int_\Omega\widehat{f}\left(u(t)\right)\mathrm{d}x.
\end{aligned}\tag{4.31}$$

5. Global existence

In this section we show that the solution of problem 1.1 global in in infinite time under the assumption

$$E(0) < 4(w_1\lambda_1 - \alpha(0)\beta_1)\left(\frac{p_-(\lambda_1 w_1 - \alpha(0)\beta_1)}{4(l_1 + l_1\|k_2(x)\|_\infty + \|k_1(x)\|_\infty) bC_*^{\theta+1}}\right)^{\frac{2}{\theta-1}}.$$

and

$$p_+ \leq \frac{2n}{n-2},\ n \neq 2\ \ (p_+ < \infty \text{ if } n \leq 2).$$

The next lemma shows that our energy functional (4.29) is a nonincreasing function along the solution of (1.1).

Lemma 5.1. *$E(t)$ is a nonincreasing for $t \geq 0$ and*

$$E'(t) = -w_2|\nabla u_t|^2 - \lambda\int_\Omega u_t(t)\, g(u_t(t))\,\mathrm{d}x + \frac{1}{2}\alpha'(t)\int_\Omega(\beta o\nabla u)(t)\,\mathrm{d}x$$
$$+\frac{1}{2}\alpha(t)\int_\Omega(\beta' o\nabla u)(t)\,\mathrm{d}x - \frac{1}{2}\left(\alpha(t)\beta(t) + \alpha'(t)\int_0^t\beta(s)\,\mathrm{d}s\right)|\nabla u(t)|^2 \leq 0. \quad (5.1)$$

Proof. Multiplying the equation of (1.1) by u_t and integrating by parts over Ω, using (3.6), (3.7), (3.10) and remark 3.2, summing up the product results, obtains

$$E(t) - E(0) = -w_2\int_0^t|\nabla u_t(s)|^2\,\mathrm{d}s - \lambda\int_0^t\int_\Omega u_t(s)\, g(u_t(s))\,\mathrm{d}x\mathrm{d}s$$
$$+\frac{1}{2}\int_0^t\alpha'(t)\int_\Omega(\beta o\nabla u)(s)\,\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^t\alpha(s)\int_\Omega(\beta' o\nabla u)(t)\,\mathrm{d}x\mathrm{d}s$$
$$-\frac{1}{2}\int_0^t\left(\alpha(s)\beta(s) + \alpha'(s)\int_0^s\beta(\zeta)\,\mathrm{d}\zeta\right)|\nabla u(s)|^2\,\mathrm{d}s \leq 0 \ \text{ for } t \geq 0. \qquad \square$$

Lemma 5.2. *Let (3.6) and (3.8) hold, suppose $u_0 \in W$ and $u_1 \in H_0^1(\Omega)$ such that*

$$\gamma = bC_*^{\theta+1}\left(4\frac{E(0)}{w_1\lambda_1 - \alpha(0)\beta_1}\right)^{\frac{\theta-1}{2}}(l_1 + l_1\|k_2(x)\|_\infty + \|k_1(x)\|_\infty) \quad (5.2)$$
$$< \frac{p_-}{4}(\lambda_1 w_1 - \alpha(0)\beta_1).$$

then $u \in W$ for each $t \geq 0$, where C_* is the best Poincar's, Sovolev constant depending only on $p(x)$ and on Ω, which satisfy $2 < p(x) \leq p_+ \leq \frac{2n}{n-2}\,(n \geq 3)$ $(2 \leq p_+ < \infty$ if $n = 1, 2)$.

$$\|u(t)\|_{p(x)} \leq C_*\|\nabla u(t)\|_2 \quad \forall u \in H_0^1(\Omega).$$

Proof. Since $I(0) > 0$, by the continuity, there exists $0 < T_m < T$ such

$$I(t) \geq 0 \ \text{ in } [0, T_m],$$

this gives from (4.28) and (3.8):

$$\begin{aligned} E(t) \geq J(t) &= \frac{1}{p(x)} I(t) + \frac{1}{4}\left(\lambda_1 w_1 - \alpha(t)\int_0^t \beta(s)\,\mathrm{d}s\right)|\nabla u|^2 \\ &+ \frac{b}{p(x)}\left(\int_\Omega f(u)\,u\mathrm{d}x + \int_\Omega k_1(x)\,|u|\,\mathrm{d}x - p(x)\int_\Omega \widehat{f}(x)\,\mathrm{d}x\right) \\ &\geq \frac{1}{4}\left(\lambda_1 w_1 - \alpha(t)\int_0^t \beta(s)\,\mathrm{d}s\right)|\nabla u|^2 . \end{aligned} \tag{5.3}$$

since by (3.8) we have

$$\int_\Omega f(u)\,u\mathrm{d}x + \int_\Omega k_1(x)\,|u|\,\mathrm{d}x - p(x)\int_\Omega \widehat{f}(x)\,\mathrm{d}x \geq 0$$

Then by using (5.3), (4.29), (5.1) and remark 3.2, we obtain

$$\begin{aligned} |\nabla u|^2 &\leq 4\left(\lambda_1 w_1 - \alpha(t)\int_0^t \beta(s)\,\mathrm{d}s\right)^{-1} E(t) \\ &\leq 4\left(\lambda_1 w_1 - \alpha(t)\int_0^t \beta(s)\,\mathrm{d}s\right)^{-1} E(0). \end{aligned} \tag{5.4}$$

By recalling (3.9), Sobolev-Poincaré's embedding ($\theta + 1 \leq p$), condition (5.2), estimate (5.4) and Cauchy-Schwartz's inequality, we have the following estimates:

$$\begin{aligned} & b\int_\Omega f(u)\,u\mathrm{d}x + b\int_\Omega k_1(x)\,|u|\,\mathrm{d}x \leq b\int_\Omega |f(u)|\,|u|\,\mathrm{d}x + b\int_\Omega |k_1(x)|\,|u|\,\mathrm{d}x \\ &\leq bl_1\int_\Omega |u|^{\theta+1}\,\mathrm{d}x + bl_1\int_\Omega |k_2(x)|\,|u|\,\mathrm{d}x + b\int_\Omega |k_1(x)|\,|u|\,\mathrm{d}x \\ &\leq bl_1\,\|u(t)\|_{\theta+1}^{\theta+1} + b\left(l_1\,||k_2(x)||_\infty + ||k_1(x)||_\infty\right)\|u(t)\|_{\theta+1}^{\theta+1} \\ &\leq bl_1 C_*^{\theta+1}\,|\nabla u(t)|^{\theta+1} \\ &+ bC_*^{\theta+1}\left(l_1\,||k_2(x)||_\infty + ||k_1(x)||_\infty\right)|\nabla u(t)|^{\theta+1} \\ &= bl_1 C_*^{\theta+1}\,|\nabla u(t)|^{\theta-1}\,|\nabla u(t)|^2 \\ &+ bC_*^{\theta+1}\left(l_1\,||k_2(x)||_\infty + ||k_1(x)||_\infty\right)|\nabla u(t)|^{\theta-1}\,|\nabla u(t)|^2 \\ &\leq bC_*^{\theta+1}\left(4\left(\lambda_1 w_1 - \alpha(t)\int_0^t \beta(s)\,\mathrm{d}s\right)^{-1} E(0)\right)^{\frac{\theta-1}{2}} \\ &\quad\times\left(l_1 + l_1\,||k_2(x)||_\infty + ||k_1(x)||_\infty\right)|\nabla u|^2 \\ &\leq bC_*^{\theta+1}\left(4\frac{E(0)}{w_1\lambda_1 - \alpha(0)\,\beta_1}\right)^{\frac{\theta-1}{2}} \times\left(l_1 + l_1\,||k_2(x)||_\infty + ||k_1(x)||_\infty\right)|\nabla u|^2 \\ &< \frac{p_-}{4}\left(\lambda_1 w_1 - \alpha(0)\,\beta_1\right)|\nabla u|^2 \\ &\leq \frac{p(x)}{4}\left(\lambda_1 w_1 - \alpha(t)\int_0^t \beta(s)\,\mathrm{d}s\right)|\nabla u|^2 \text{ on } [0, T_m]. \end{aligned} \tag{5.5}$$

Therefore, from (4.27), we conclude that $I(t) > 0$ for all $t \in [0, T_m]$. By repeating this procedure, and using the fact that

$$\lim_{t\to T_m} bC_*^{\theta+1}\left(4\frac{E(t)}{w_1\lambda_1-\alpha(0)\beta_1}\right)^{\frac{\theta-1}{2}}\left(l_1+l_1\|k_2(x)\|_\infty+\|k_1(x)\|_\infty\right)\le D < \frac{p_-}{4}\left(\lambda_1 w_1-\alpha(0)\beta_1\right).$$

T_m is extended to T. □

Theorem 5.3. *Let the assumptions of theorem 4.1 hold. Let $u_0 \in W$ satisfying* (5.2)*. Then, the solution gotten in of theorem 4.1 is global.*

Proof. It sufficient independently to t to show that

$$|u_t|^2+|\nabla u|^2+\int_\Omega |\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega |u(t)|^{p(x)}\,\mathrm{d}x$$

is bounded.

For this aim, we use (4.27), (4.29), (3.8), (3.10) and Lemma 5.2 to obtain:

$$\begin{aligned}
E(0)\ge E(t)&\ge \frac{1}{2}\left(\lambda_1 w_1-\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)|\nabla u(t))|^2-b\int_\Omega \widehat{f}(x,u)\,\mathrm{d}x\\
&+\frac{1}{2}|u_t(t)|^2+\int_\Omega\frac{1}{m(x)}|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\,\mathrm{d}x+\frac{1}{2}\alpha(t)(\beta o\nabla u)(t)\\
&\ge\frac{1}{4}\left(\lambda_1 w_1-\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)|\nabla u(t))|^2+\frac{1}{p(x)}I(t)\\
&+\frac{b}{p(x)}\left(\int_\Omega f(u)u\mathrm{d}x+\int_\Omega k_1(x)|u|\,\mathrm{d}x-p(x)\int_\Omega\widehat{f}(x,u)\,\mathrm{d}x\right)\\
&+\frac{1}{2}|u_t(t)|^2+\int_\Omega\frac{1}{m(x)}|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\,\mathrm{d}x\\
&\ge\frac{1}{4}\left(\lambda_1 w_1-\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)|\nabla u(t)|^2\\
&+\frac{1}{2}|u_t(t)|^2+\int_\Omega\frac{1}{m(x)}|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\,\mathrm{d}x\\
&\ge\frac{1}{4}\left(\lambda_1 w_1-\alpha(0)\beta_1\right)|\nabla u(t))|^2+\frac{1}{2}|u_t(t)|^2\\
&+\int_\Omega\frac{1}{m(x)}|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\,\mathrm{d}x.
\end{aligned}$$

Therefore

$$\begin{aligned}
|u_t(t)|^2+|\nabla u(t)|^2+\int_\Omega|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega|u(t)|^{p(x)}\,\mathrm{d}x&\\
\le\max\left(p^+,m^+,4\left(\lambda_1 w_1-\alpha(0)\beta_1\right)^{-1}\right)E(0),&
\end{aligned}$$

These estimates ensure that the solution $u(t)$ exist globally in $[0,+\infty[$. □

Example 5.4. Consider the following functions:

$$f(x,u) = a(x)|u|^{\varpi-2}u - b(x)|u|^{\gamma-2}u$$

with appropriate functions $a(x)$ and $b(x)$, where $\varpi > \gamma \geq 1$.

$$\begin{aligned} g(u_t(t)) &= |u_t(t)|^{\sigma(x)-2}u_t(t); \quad \sigma(x) \text{ satisfies conditions in } (3.7); \\ \Delta_{m(x)}u &= div\left(|\nabla u|^{m-2}\nabla u\right); \quad m(x) = m > 2. \end{aligned}$$

Then, problem (1.1), is reduced to the following problem

$$\begin{cases} u_{tt} - div\left(|\nabla u|^{m-2}\nabla u\right) + w_1\Delta^2 u(t) - w_2\Delta u_t(t) + \alpha(t)\int_0^t \beta(t-s)\Delta u(s)\,\mathrm{d}s \\ \qquad +\lambda|u_t(t)|^{\sigma(x)-2}u_t(t) + |u|^{p(x)-2}u(t) = bf(u(t)) \text{ in } \Omega\times\mathbb{R}^+, \\ \qquad u = \partial_\eta u = 0 \text{ on } \Gamma\times[0,+\infty[, \\ \qquad u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \Omega, \end{cases} \tag{P}$$

Since f, g satisfies hypotheses (3.7)-(3.9). Then, Theorems (4.1), (4.3) and (5.3) are verified for problem (P), which gives importance to this general problem.

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