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On some cluster sets problems

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. We generalize some cluster sets theorems of Tsuji and Iversen from plane holomorphic mappings to the class of ring mappings.

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1. Introduction

A classical problem in complex analysis is the study of boundary behaviour of analytic mappings and a special case is the study of cluster sets (see the book of Noshiro [26]).

We shall extend some theorems of Iversen and Tsuji concerning cluster sets of holomorphic plane mappings in the class of so called ring mappings or mappings satisfying modular inequalities. Such mappings were intensively studied in the last 20 years (see the book of Martio, Ryazanov, Srebro, Yakubov [21], or some papers of Cristea [2-9], Golberg, Salimov, Sevost'yanov, Ryazanov [11-14], [16], [19-21], [27-33]).

The class of ring mappings preserves many geometric properties of the well-known quasiregular mappings, the last class being itself the best known extension of the plane analytic mappings.

Given a domain $D \subset \mathbb{R}^n$, we denote by $A(D)$ the set of all path families from D and if $\Gamma \in A(D)$ we set $F(\Gamma) = \{\rho : \mathbb{R}^n \to [0,\infty]$ Borel maps $\mid \int_\gamma \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$. If $E, F \subset \overline{D}$, we set $\Delta(E,F,D) = \{\gamma : [0,1] \to \overline{D}$ path such that $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma((0,1)) \subset D\}$ and if $x \in \mathbb{R}^n$ and $0 < a < b$ we set $\Gamma_{x,a,b} = \Delta(\overline{B}(x,a) \cap D, S(x,b) \cap D, C_{x,a,b} \cap D)$. Here $C_{x,a,b} = B(x,b) \setminus \overline{B}(x,a)$.

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We define for $p > 1$ and $\omega : D \to [0, \infty]$ measurable and finite a.e. the p-modulus of weight ω

$$M_\omega^p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \omega(x)\rho^p(x)dx \ \ for \ \ \Gamma \in A(D).$$

For $\omega = 1$ we have the classical p-modulus

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \rho^p(x)dx \ \ for \ \ \Gamma \in A(D).$$

We see that $M_\omega^p(\Gamma_1) \le M_\omega^p(\Gamma_2)$ if $\Gamma_1 > \Gamma_2$ and

$$M_\omega^p\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) \le \sum_{i=1}^{\infty} M_\omega^p(\Gamma_i)$$

for $\Gamma_1, \Gamma_2, ..., \Gamma_i, ... \in A(D)$. Here, if $\Gamma_1, \Gamma_2 \in A(D)$, we say that $\Gamma_1 > \Gamma_2$ if every path $\gamma_1 \in \Gamma_1$ has a subpath $\gamma_2 \in \Gamma_2$.

We say that a mapping $f : D \to \mathbb{R}^n$ is open if carries open sets to open sets and we say that f is discrete if either $f^{-1}(y) = \phi$ or $f^{-1}(y)$ is a discrete subset of D for every $y \in \mathbb{R}^n$. If $f : D \to \mathbb{R}^n$ is continuous, open and discrete, then for every path $p : [0, 1] \to \mathbb{R}^n$ and every $x \in D$ such that $f(x) = p(0)$ there exists $0 < a \le 1$, a path $q : [0, a) \to D$ such that $q(0) = x$ and $f \circ q = p|[0, a)$ and such a path is maximal with this property (we say that f has the property of path lifting and $q : [0, a) \to D$ is a maximal lifting of the path $p : [0, 1] \to \mathbb{R}^n$ from the point $x \in D$ such that $f(x) = p(0)$). We say that D is locally connected at $x \in \partial D$ if there exists $(U_m)_{m \in \mathbb{N}}$ a fundamental system of neighbourhoods of x such that $U_m \cap D$ is connected for every $m \in \mathbb{N}$. Let $p > 1$. We say that $E = (A, C)$ is a condenser if $C \subset A \subset \mathbb{R}^n$, A is open and C is compact and we define

$$cap_\omega^p(E) = \inf \int_{\mathbb{R}^n} \omega(x)\rho^p(x)dx,$$

where $u \in C_0^\infty(A)$ and $u \ge 1$ on C. For $\omega = 1$ we have the classical p-capacity. If $E = (A, C)$ is a condenser and $\Gamma_E = \Delta(C, \partial A, A)$, then $M_p(\Gamma_E) = cap_p(E)$. If $C \subset \mathbb{R}^n$ is compact, we say that $cap_p(C) = 0$ if $cap_p((A, C)) = 0$ for some bounded open set $A \subset \mathbb{R}^n$ and if $C \subset \mathbb{R}^n$ is arbitrary, we say that $cap_p(C) = 0$ if $cap_p(K) = 0$ for every compact $K \subset C$.

If $K \subset \mathbb{R}^n$, we say that $M_\omega^p(K) = 0$ if $M_\omega^p(\Gamma) = 0$, where $\Gamma = \{\gamma : [0, 1) \to \mathbb{R}^n$ path $|\gamma$ has at least a limit point in $K\}$. Here, for an open path $\gamma : [0, 1) \to \mathbb{R}^n$ we say that a point $x \in \overline{\mathbb{R}^n}$ is a limit point of γ if there exist $t_m \to 1$ such that $\gamma(t_m) \to x$.

The following capacity inequality is proved in [17]:

$$cap_p((A, C)) \ge C_1\left(\frac{d(C)^p}{\mu_n(A)^{1-n+p}}\right)^{\frac{1}{n-1}} \text{ if } n - 1 < p \le n. \tag{1.1}$$

Here $d(C)$ is the diameter of C, μ_n is the Lebesgue measure in $\mathbb{R}^n$ and C_1 is a constant which does not depends on n. We set V_n the volume of the unit ball from

$\mathbb{R}^n$. We also use a modular inequality from [1]:

$$M_p(\Delta(E_1, E_2, C_{x,a,b})) \geq \frac{C(n,p)}{n-p}(b^{n-p} - a^{n-p}) \text{ if } n-1 < p < n \tag{1.2}$$

$$M_p(\Delta(E_1, E_2, C_{x,a,b}) \geq C(n) \ln\left(\frac{b}{a}\right) \text{ if } p = n$$

where $E_1 \cap S(x,r) \neq \phi$, $E_2 \cap S(x,r) \neq \phi$ for every $a < r < b$ and $x \in \mathbb{R}^n$.

Here $C(n,p)$ is a constant depending only on n and p. Throughout this paper $C(n,p)$ means a constant depending only on n and p.

It is easy to see that the condition $M_\omega^p(E) = 0$ holds for instance if

$$\int_{\overline{D}} \omega(z)^{\frac{m}{m-p}} dz < \infty$$

and $cap_m(E) = 0$ and $p < m \leq n$ (see [8] page 4 or [12]).

Let us speak about the objects of cluster sets theory that will be used in this paper. Let $D \subset \mathbb{R}^n$ be open, $f : D \to \mathbb{R}^n$ and $x \in E \subset \partial D$. We set $C(f,x) = \{z \in \overline{\mathbb{R}^n} |$ there exists $x_m \in D$, $x_m \to x$ such that $f(x_m) \to z\}$ and we set $C(f,E) = \bigcup_{x \in E} C(f,x)$. Let $F : \overline{D} \to \mathcal{P}(\mathbb{R}^n)$ be defined by $F(x) = f(x)$ if $x \in D$, $F(x) = C(f,x)$ if $x \in \partial D$ and if $(U_m)_{m \in \mathbb{N}}$ is a fundamental system of neighbourhoods of x such that $U_{m+1} \subset U_m$ for every $m \in \mathbb{N}$, we set $C(f,x,E) = \bigcap_{m=1}^{\infty} F(U_m \cap (E \setminus \{x\}))$. We set for $x \in \overline{D}$ the range of values $R_D(f,x) = \bigcap_{r>0} f(B(x,r) \cap D)$. If $x \in \partial D$, we set $A(f,x) = \{z \in \mathbb{R}^n |$ there exists $\gamma : [0,1) \to D$ path such that $\gamma(0) \in D$, $\lim_{t \to 1} \gamma(t) = x$ and $\lim_{t \to 1} f(\gamma(t)) = z\}$. If $\gamma : [0,1] \to D$ is a path, we set $|\gamma| = \{z \in \mathbb{R}^n |$ there exists $t \in [0,1]$ such that $z = \gamma(t)\}$.

The theory of cluster sets was also studied for quasiregular mappings by many mathematicians like Näkki, Vuorinen, Martio, Zoric [22-25], [34-36], [18], [37].

In some recent papers [8-9] we studied classes of continuous, open, discrete mappings $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ for which the following modular inequality holds:

$$M_p(f(\Gamma)) \leq \gamma(M_\omega^q(\Gamma)) \text{ for every } \Gamma \in A(D), \tag{1.3}$$

where $n \geq 2$, $q > 1$, $n-1 < p \leq n$, $\omega : D \to [0,\infty]$ is measurable and finite a.e. and $\gamma : (0,\infty) \to (0,\infty)$ is increasing with $\lim_{t \to 0} \gamma(t) = 0$. We briefly say that such mappings satisfy condition (1.3). It must be mentioned that if $f : D \to \mathbb{R}^n$ is quasiregular, then the Poletski modular inequality holds, namely $M_n(f(\Gamma)) \leq K M_n(\Gamma)$ for every $\Gamma \in A(D)$ and a fixed constant $K \geq 1$.

It is surprisingly that for the mappings satisfying relation (1.3) with $p \neq n$, $n-1 < p < n$ we can give analogous of Liouville, Montel, Picard type theorems, boundary extension results and estimates of the modulus of continuity. Interesting examples of such mappings satisfying relation (1.3) which are not quasiregular may be found in [12]. Also, the case $q = n$, $p = n$ was extensively studied in [20].

We continue our researches and related researches on these mappings from [3], [8-9] and [12].

In this paper we study the cluster sets and the boundary cluster sets for such mappings. Tsuji proved the following theorem for analytic mappings see Theorem 3.5 in [26]).

Theorem A. *Let $D \subset \mathbb{C}$ be a domain, $E \subset \partial D$ compact with $cap_2(E) = 0$, $x \in E \cap \overline{(\partial D \setminus E)}$ and let $f \in H(D)$. Then if the open set $Q = C(f,x) \setminus C(f,x,\partial D \setminus E)$ is nonempty, it results that $cap_2(Q \setminus R_D(f,x)) = 0$.*

We prove the following extension of Tsuji's theorem for mappings satisfying relation (1.3):

Theorem 1.1. *Let $E \subset \partial D$, $x \in E \cap \overline{\partial D \setminus E}$ such that D is locally connected at x, $\omega \in L^1_{loc}(D)$ such that $M^q_\omega(E) = 0$ and let $f : D \to \mathbb{R}^n$ satisfying condition (1.3). Then, if the open set $Q = C(f,x) \setminus C(f,x,\partial D \setminus E)$ is nonempty, it results that $cap_p(Q \setminus R_D(f,x)) = 0$.*

Theorem 1.2. *Let $E \subset \partial D$, $x \in E \cap \overline{(\partial D \setminus E)}$ such that D is locally connected at x, $\omega \in L^1_{loc}(D)$ such that $M^q_\omega(E) = 0$ and let $f : D \to \mathbb{R}^n$ be bounded satisfying condition (1.3) such that there exists $K \subset \mathbb{R}^n$ compact such that $C(f,x,\partial D \setminus E) \subset K$ and $\mathbb{R}^n \setminus K$ is connected. Then $C(f,x) \subset K$.*

The following theorem is due to Iversen and Tsuji (see Theorem 3.2 in [26]).

Theorem B. *Let $D \subset \mathbb{C}$ be a domain, $E \subset \partial D$ compact, with $cap_2(E) = 0$, $x \in E \cap \overline{(\partial D \setminus E)}$ and let $f \in H(D)$ be bounded. Then $\limsup\limits_{y \to x} |f(y)| = \limsup\limits_{\substack{y \to x \\ y \in \partial D \setminus E}} (\limsup\limits_{z \to y} |f(z)|)$.*

We prove:

Theorem 1.3. *Let $E \subset \partial D$, $x \in E \cap \overline{(\partial D \setminus E)}$ such that D is locally connected at x, $\omega \in L^1_{loc}(D)$ such that $M^q_\omega(E) = 0$ and let $f : D \to \mathbb{R}^n$ be bounded satisfying condition (1.3). Then $\limsup\limits_{y \to x} |f(y)| = \limsup\limits_{\substack{y \to x \\ y \in \partial D \setminus E}} (\limsup\limits_{z \to y} |f(z)|$.*

Using the method from [3] and [10] we prove the following generalization of some theorems of Noshiro [26] and Martio and Rickman [20]:

Theorem 1.4. *Let $E \subset \partial D$ such that $\dim \partial D \geq 1$, $\dim E = 0$ and $M^q_\omega(E) = 0$, let $x \in (\partial D \setminus E)'$ and $z \in C(f,x) \setminus (C(f,x,\partial D \setminus E) \cup R_D(f,x))$ and $f : D \to \mathbb{R}^n$ satisfying condition (1.3). Then either $x \in E$ and $z \in A(f,x)$ or there exists $x_k \in E$ such that $x_k \in E$, $x_k \to x$ and $z \in A(f,x_k)$ for every $k \in \mathbb{N}$.*

We also see that our extensions given to the theorems of Tsuji and Iversen-Tsuji are effective even for plane analytic mappings, since we don't impose the exceptional set $E \subset \partial D$ to be compact.

2. Proofs of the results

Proof of Theorem 1.1. Let $y \in Q$, $\delta_y = d(y, \partial Q)$ and $0 < \delta < \frac{\delta_y}{3}$ and let

$$F_r = C(f, \overline{B}(x,r) \cap ((\partial D \setminus E) \setminus \{x\})) \text{ for } r > 0.$$

Since $C(f,x,\partial D \setminus E) = \bigcap\limits_{r>0} \overline{F}_r$, there exists $r_0 > 0$ such that $\overline{F}_r \cap B(y, \frac{5\delta}{2}) = \phi$ for every $0 < r < r_0$. Let $0 < r < r_0$ be fixed and let $\epsilon > 0$. We can find $0 < b < r$ such that $B(x,b) \cap D$ is connected and $M^q_\omega(\Gamma_{x,b,r}) < \epsilon$. Let x_m, $z_m \to x$ be such that $x_m, z_m \in B(x,b) \cap D$, $d(f(z_m), y) > \frac{5\delta}{2}$ and $d(f(x_m), y) < \frac{\delta}{2}$ for every $m \in \mathbb{N}$.

Since $B(x,b)\cap D$ is connected, there exists a path $q_m:[0,1]\to B(x,b)\cap D$ such that $q_m(0)=x_m$, $q_m(1)=z_m$ and $|q_m|\subset B(x,b)\cap D$ for every $m\in\mathbb{N}$.

Let us fix such $m\in\mathbb{N}$ and let $p_m=f\circ q_m$. Since $p_m(0)=f(q_m(o))=f(x_m)$ and $p_m(1)=f(q_m(1))=f(z_m)$, we see that $d(|p_m|)>2\delta$ and that there exists a subpath γ_m of p_m such that $f(x_m)\in|\gamma_m|$, $d(|\gamma_m|)>\frac{\delta}{2}$ and $|\gamma_m|\subset B(y,\delta)$.
Suppose that there exists a compact set $F\subset B(y,\delta)\setminus f(B(x,r))$ such that

$$cap_p(B(y,2\delta),F)=\delta_0>0.$$

Let $\Gamma'_m=\Delta(|\gamma_m|,F,B(y,2\delta))$ and let $\rho\in F(\Gamma'_m)$.
Let $\Delta_0=\Delta(F,S(y,2\delta),B(y,2\delta))$ and $\Delta_m=\Delta(|\gamma_m|,S(y,2\delta),B(y,2\delta))$.
If $3\rho\in F(\Delta_0)\cup F(\Delta_m)$, then

$$\int\limits_{\mathbb{R}^n}\rho^p(x)dx\geq\frac{M_p(\Delta_0)}{3^p}=\frac{\delta_0}{3^p}$$

and using relation (1.1) we have

$$\int\limits_{\mathbb{R}^n}\rho^p(x)dx\geq\frac{M_p(\Delta_m)}{3^p}\geq\frac{1}{3^p}\left(\frac{C_1d(|\gamma_m|)^p}{2^nV_n(2\delta)^{1-n+p}}\right)^{\frac{1}{n-1}}\geq C(n,p)\delta.$$

If $3\rho\notin F(\Delta_0)\cup F(\Delta_m)$, we can find paths $\lambda_1\in\Delta_0$, $\lambda_2\in\Delta_m$ such that

$$\int\limits_{\lambda_k}\rho ds<1\text{ for }k=1,2.$$

Let $\Gamma=\Delta(|\lambda_1|,|\lambda_2|,C_{y,\delta,2\delta})$. Using relation (1.2) we see that

$$M_p(\Gamma)\geq C(n,p)((2\delta)^{n-p}-\delta^{n-p})=C(n,p)\delta^{n-p}.$$

Let $\gamma_0\in\Gamma$. We can find subpaths α_1 of λ_1 and β_1 of λ_2 such that the path

$$\gamma=\alpha_1\vee\gamma_o\vee\beta_1\in\Gamma'_m$$

and since $\rho\in F(\Gamma'_m)$, we see that

$$1\leq\int\limits_{\gamma}\rho ds\leq\int\limits_{\alpha_1}\rho ds+\int\limits_{\gamma_0}\rho ds+\int\limits_{\beta_1}\rho ds\leq\frac{1}{3}+\int\limits_{\gamma_0}\rho ds+\frac{1}{3}$$

and hence

$$1\leq\int\limits_{\gamma_0}3\rho ds.$$

It results that if $3\rho\notin F(\Delta_0)\cup F(\Delta_m)$, then

$$\int\limits_{\mathbb{R}^n}\rho^n(x)dx\geq\frac{1}{3^p}M_p(\Gamma)\geq C(n,p)\delta^{n-p}.$$

We therefore proved that

$$M_p(\Gamma'_m)\geq\frac{1}{3^p}\min\{\delta_0,C(n,p)\delta,C(n,p)\delta^{n-p}\}=\rho>0$$

and the constant ρ does not depends on m.

Let Γ_m be the family of all maximal liftings of some path from Γ'_m starting from some point of $|q_m|$. Let $S_m = \Delta(|q_m|, E, B(x,r)\cap D)$ and $T_m = \Delta(|q_m|, \partial D\setminus E, B(x,r)\cap D)$. Let $\beta : [0,1] \to \mathbb{R}^n$, $\beta \in \Gamma'_m$ and let $\alpha : [0,a) \to B(x,r)\cap D$, $0 < a < 1$ be a maximal lifting of β with $\alpha(0) \in |q_m|$. Such a path α exists due to the openness and discreteness of the mapping f and since $\lim\limits_{t\to 1} \beta(t) \notin f(B(x,r)\cap D)$. Also, using the openness of the mapping f we see that any limit point of the open path $\alpha : [0,a) \to B(x,r)\cap D$ either belongs to ∂D, or intersects $S(x,r)$.
We see that $\Gamma_m \subset \Gamma_{x,b,r} \cup S_m \cup T_m$ and let us show that $T_m = \phi$. Indeed, if $\alpha : [0,a) \to B(x,r)\cap D$ is a maximal lifting of some path $\beta \in \Gamma'_m$ which has a limit point $z \in \partial D\setminus E$, let $t_k \nearrow a$. Then $f(\alpha(t_k)) = \beta(t_k) \in |\beta|$ and $\beta \in \Gamma'_m$ and hence $f(\alpha(t_k)) \in B(y, 2\delta)$ for $k \in \mathbb{N}$. On the other side, we see that $\beta(t_k) \in B(C(f,x,\overline{B}(x,r)\cap((\partial D\setminus E)|\{x\})),\delta)$ for k great enough and since $B(y,3\delta)\cap C(f,x,\overline{B}(x,r)\cap((\partial D\setminus E)\setminus\{x\})) = \phi$, we reached a contradiction.
We proved that $T_m = \phi$ and hence $\Gamma_m \subset \Gamma_{x,b,r} \cup S_m$ and we also see that $\Gamma'_m > f(\Gamma_m)$. We have

$$\begin{aligned} 0 < \rho < M_p(\Gamma'_m) &\le M_p(f(\Gamma_m)) \\ &\le M_p(f(\Gamma_{x,b,r} \cup S_m)) \\ &\le M_p f((\Gamma_{x,b,r})) + M_p(f(S_m)) \\ &\le \gamma(M^q_\omega(\Gamma_{x,b,r})) + \gamma(M^q_\omega(S_m)) \le \gamma(\epsilon). \end{aligned}$$

Letting now ϵ small enough such that $\gamma(\epsilon) < \rho$, we reached a contradiction.

We proved that $cap_p(B(y,2\delta),F) = 0$ for every $0 < \delta < \frac{\delta_y}{3}$ and every set $F \subset B(y,\delta)\setminus f(B(x,r)\cap D)$ and this implies that $cap_p(B(y,\frac{1}{3}\delta_y)\setminus f(B(x,r)\cap D)) = 0$. Since this holds for every $r > 0$, it results that $cap_p(B(y,\frac{1}{3}\delta_y)\setminus R_D(f,x)) = 0$. Let $W = \{y \in Q | cap_p(B(y,\frac{1}{3}\delta_y)\setminus R_D(f,x)) = 0\}$. We see that W is an open subset of Q and using the preceeding arguments we show that W is also a closed subset of Q and since Q is connected, we see that $Q = W$ and the theorem is now proved.

Proof of Theorem 1.2. Suppose that $C(f,x)\cap(\mathbb{R}^n\setminus K) \ne \phi$ and let $Q \supset \mathbb{R}^n\setminus K$ be the unbounded component of $C(f,x)\setminus C(f,x,\partial D\setminus E)$. We see that $Q \ne \phi$ and using Theorem 1 we find a set $F \subset \mathbb{R}^n$ with $cap_p(F) = 0$ and $f(D) \supset Q\setminus F \supset \mathbb{R}^n\setminus(K\cup F)$. This implies that f is unbounded and we reached a contradiction. We therefore proved that $C(f,x) \subset K$.

Proof of Theorem 1.3. Let $M = \limsup\limits_{\substack{y\to x\\ y\in\partial D\setminus E}} (\limsup\limits_{z\to y} |f(z)|)$.

Since f is bounded, we see that $M < \infty$ and since $n \ge 2$ it results that $\mathbb{R}^n\setminus\overline{B}(0,M)$ is connected. Now $C(f,x,\partial D\setminus E) \subset \overline{B}(0,M)$ and from Theorem 2 we find that $C(f,x) \subset \overline{B}(0,M)$. Then $\limsup\limits_{y\to x} |f(y)| \le M$ and the theorem is now proved.

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