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Global smoothness preservation and simultaneous approximation by multivariate discrete operators

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. In this article we study the multivariate generalized discrete singular operators defined on $\mathbb{R}^N$, $N \geq 1$, regarding their simultaneus global smoothness preservation property with respect to L_p norm for $1 \leq p \leq \infty$, by using higher order moduli of smoothness. Furthermore, we study their simultaneous approximation properties.

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1. Background

In [1], Chapter 3, the author defined

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1,2,...,r, \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0, \end{cases} \tag{1.1}$$

for $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$ and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1,2,...,m \in \mathbb{N}. \tag{1.2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1. \tag{1.3}$$

Additionally,in [1], the author used

Definition 1.1. *Let $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$, $m \in \mathbb{N}$, the mth modulus of smoothness for $1 \leq p \leq \infty$, is given by*

$$\omega_m (f; h)_p := \sup_{\|t\|_2 \leq h} \|\Delta_t^m (f)\|_{p,x}, \tag{1.4}$$

$h > 0$, where

$$\Delta_t^m f(x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jt). \tag{1.5}$$

Denote

$$\omega_m (f; h)_\infty = \omega_m (f, h). \tag{1.6}$$

Above, $x, t \in \mathbb{R}^N$.

Additionally, in [4], the authors defined the following operators:

Let μ_{ξ_n} be a Borel measure on $\mathbb{R}^N$, $N \geq 1$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. Assume that $\nu := (\nu_1, ..., \nu_N)$, $x := (x_{1,...,}x_N) \in \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function.

i) When

$$\mu_{\xi_n}(\nu) = \frac{e^{-\frac{\sum_{i=1}^N |\nu_i|}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} ... \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^N |\nu_i|}{\xi_n}}}, \tag{1.7}$$

they defined generalized multiple discrete Picard operators as:

$$P_{r,n}^{*\,[m]}(f; x_1, ..., x_N) \tag{1.8}$$

$$= \frac{\sum_{\nu_1=-\infty}^{\infty} ... \sum_{\nu_N=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_{j,r}^{[m]} f(x_1 + j\nu_1, ..., x_N + j\nu_N) \right) e^{-\frac{\sum_{i=1}^N |\nu_i|}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} ... \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^N |\nu_i|}{\xi_n}}}.$$

ii) When

$$\mu_{\xi_n}(\nu) = \frac{e^{-\frac{\sum_{i=1}^N \nu_i^2}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} ... \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^N \nu_i^2}{\xi_n}}}, \tag{1.9}$$

they defined generalized multiple discrete Gauss-Weierstrass operators as:

$$W_{r,n}^{*\,[m]}\left(f;x_1,...,x_N\right) \tag{1.10}$$
$$= \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j,r}^{[m]}f\left(x_1+j\nu_1,...,x_N+j\nu_N\right)\right)e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}.$$

iii) Let $\hat{\alpha}\in\mathbb{N}$ and $\beta>\frac{1}{\hat{\alpha}}$. When

$$\mu_{\xi_n}\left(\nu\right)=\frac{\prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}, \tag{1.11}$$

they defined the generalized multiple discrete Poisson-Cauchy operators as:

$$Q_{r,n}^{*\,[m]}\left(f;x_1,...,x_N\right) \tag{1.12}$$
$$= \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j,r}^{[m]}f\left(x_1+j\nu_1,...,x_N+j\nu_N\right)\right)\prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}\right)}.$$

iv) When

$$\mu_{\xi_n}\left(\nu\right)=\frac{e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\left(1+2\xi_n e^{-\frac{1}{\xi_n}}\right)^N}, \tag{1.13}$$

they defined the generalized multiple discrete non-unitary Picard operators as:

$$P_{r,n}^{[m]}\left(f;x_1,...,x_N\right) \tag{1.14}$$
$$= \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j,r}^{[m]}f\left(x_1+j\nu_1,...,x_N+j\nu_N\right)\right)e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\left(1+2\xi_n e^{-\frac{1}{\xi_n}}\right)^N}.$$

v) When

$$\mu_{\xi_n}\left(\nu\right)=\frac{e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\left(\sqrt{\pi\xi_n}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi_n}}\right)\right)+1\right)^N}, \tag{1.15}$$

they defined the generalized multiple discrete non-unitary Gauss-Weierstrass operators as:

$$W_{r,n}^{[m]}\left(f;x_1,...,x_N\right) = \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j,r}^{[m]}f\left(x_1+j\nu_1,...,x_N+j\nu_N\right)\right)e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\left(\sqrt{\pi\xi_n}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi_n}}\right)\right)+1\right)^N}, \tag{1.16}$$

where $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-t^2}dt$ with $\operatorname{erf}(\infty)=1$.

Additionally, in [4], article they assumed that $0^0=1$.

In [4], for $\alpha_i\in\mathbb{N}$, the authors defined the sums

$$c_{\alpha,n,\tilde{j}} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\prod\limits_{i=1}^{N}\nu_i^{\alpha_i}\right)e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}, \tag{1.17}$$

$$p_{\alpha,n,\tilde{j}} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\prod\limits_{i=1}^{N}\nu_i^{\alpha_i}\right)e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}, \tag{1.18}$$

and for $\hat{\alpha}\in\mathbb{N}$ and $\beta>\frac{\alpha_i+r+1}{2\hat{\alpha}}$, they introduced

$$q_{\alpha,n,\tilde{j}} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\prod\limits_{i=1}^{N}\nu_i^{\alpha_i}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}\right)}{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}. \tag{1.19}$$

Furthermore, they proved that

$$c_{\alpha,n,\tilde{j}},\ p_{\alpha,n,\tilde{j}},\ q_{\alpha,n,\tilde{j}}<\infty,\ \forall\xi_n\in(0,1], \tag{1.20}$$

and for $\alpha_i\in\mathbb{N}$, as $\xi_n\to 0$ when $n\to\infty$, the authors showed that

$$c_{\alpha,n,\tilde{j}},\ p_{\alpha,n,\tilde{j}},\text{and } q_{\alpha,n,\tilde{j}}\to 0. \tag{1.21}$$

In [4], they also proved

$$m_{\xi_n,P}=\prod_{i=1}^{N}\left(\frac{\sum\limits_{\nu_i=-\infty}^{\infty}e^{-\frac{|\nu_i|}{\xi_n}}}{1+2\xi_n e^{-\frac{1}{\xi_n}}}\right)\to 1 \text{ as } \xi_n\to 0^+, \tag{1.22}$$

and

$$m_{\xi_n,W} = \prod_{i=1}^{N} \left(\frac{\sum\limits_{\nu_i=-\infty}^{\infty} e^{-\frac{\nu_i^2}{\xi}}}{1 + \sqrt{\pi \xi_n} \left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi_n}}\right)\right)} \right) \to 1 \text{ as } \xi_n \to 0^+. \tag{1.23}$$

Moreover, in [4], the authors defined the following error quantities:

$$E_{n,P}^{[0]}(f;x) := P_{r,n}^{[0]}(f;x) - f(x), \tag{1.24}$$

$$E_{n,W}^{[0]}(f;x) := W_{r,n}^{[0]}(f;x) - f(x).$$

Furthermore, they introduced the errors ($n \in \mathbb{N}$):

$$E_{n,P}^{[m]}(f;x) \tag{1.25}$$

$$: \quad = P_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{\tilde{c}_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\prod\limits_{i=1}^{N} \alpha_i!} \right),$$

and

$$E_{n,W}^{[m]}(f;x) \tag{1.26}$$

$$: \quad = W_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{\tilde{p}_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\prod\limits_{i=1}^{N} \alpha_i!} \right),$$

where

$$\tilde{c}_{\alpha,n,\widetilde{j}} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} \nu_i^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^{N} |\nu_i|}{\xi_n}}}{\left(1 + 2\xi_n e^{-\frac{1}{\xi_n}}\right)^N} \tag{1.27}$$

and

$$\tilde{p}_{\alpha,n,\widetilde{j}} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} \nu_i^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^{N} \nu_i^2}{\xi_n}}}{\left(\sqrt{\pi \xi_n} \left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi_n}}\right)\right) + 1\right)^N}. \tag{1.28}$$

In [4], the authors proved

Proposition 1.2. *Let* $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N \in \mathbb{N}$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = m \in \mathbb{N}$. *Then, there exist* K_1, K_2, $K_3 > 0$ *such that*

$$\begin{aligned} & u^*_{P,\xi_n} \\ = \quad & \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(\prod_{i=1}^{N} |\nu_i|^{\alpha_i} \right) \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r e^{-\frac{\sum_{i=1}^{N} |\nu_i|}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N} |\nu_i|}{\xi_n}}} \\ \leq \quad & K_1 < \infty, \end{aligned} \tag{1.29}$$

$$\begin{aligned} & u^*_{W,\xi_n} \\ = \quad & \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(\prod_{i=1}^{N} |\nu_i|^{\alpha_i} \right) \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r e^{-\frac{\sum_{i=1}^{N} \nu_i^2}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N} \nu_i^2}{\xi_n}}} \\ \leq \quad & K_2 < \infty, \end{aligned} \tag{1.30}$$

and

$$\begin{aligned} & u^*_{Q,\xi_n} \\ = \quad & \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(\prod_{i=1}^{N} |\nu_i|^{\alpha_i} \right) \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r \left(\prod_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta} \right)}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(\prod_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta} \right)} \\ \leq \quad & K_3 < \infty, \end{aligned} \tag{1.31}$$

for all $\xi_n \in (0, 1]$ *where* $\hat{\alpha}$, $n \in \mathbb{N}$, $\beta > \max\left\{\frac{1+r+\alpha_i}{2\hat{\alpha}}, \ \frac{r+2}{2\hat{\alpha}}\right\}$ *for all* $i = 1, ..., N$, *and* $\nu = (\nu_1, ..., \nu_N)$.

Additionally, in [4], the authors defined

$$\Phi^*_{P,\xi_n} := \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r e^{-\frac{\sum_{i=1}^{N} |\nu_i|}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N} |\nu_i|}{\xi_n}}}, \tag{1.32}$$

$$\Phi^*_{W,\xi_n} := \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r e^{-\frac{\sum_{i=1}^{N} \nu_i^2}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N} \nu_i^2}{\xi_n}}}, \tag{1.33}$$

and

$$\Phi^*_{Q,\xi_n} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(1+\frac{\|\nu\|_2}{\xi_n}\right)^r \prod\limits_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \prod\limits_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}. \tag{1.34}$$

They also showed Φ^*_{P,ξ_n}, Φ^*_{W,ξ_n}, and Φ^*_{Q,ξ_n} are uniformly bounded for all $\xi_n \in (0,1]$, where $\hat{\alpha} \in \mathbb{N}$, $\beta > \frac{r+2}{2\hat{\alpha}}$.

On the other hand, in [5], the authors proved

Proposition 1.3. *Let* $\nu := (\nu_1, ..., \nu_N)$, $\alpha := (\alpha_{1,...,}\alpha_N) \in \mathbb{R}^N$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N \in \mathbb{N}$, $|\alpha| := \sum\limits_{i=1}^{N} \alpha_i = m \in \mathbb{Z}^+$, *and* $p \geq 1$. *Then,*

$$S^{p,m}_{P^*,\xi_n} = \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} |\nu_i|^{\alpha_i}\right)^p \left(1+\frac{\|\nu\|_2}{\xi_n}\right)^{rp} e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}, \tag{1.35}$$

$$S^{p,m}_{W^*,\xi_n} = \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} |\nu_i|^{\alpha_i}\right)^p \left(1+\frac{\|\nu\|_2}{\xi_n}\right)^{rp} e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}, \tag{1.36}$$

and

$$S^{p,m}_{Q^*,\xi_n} = \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} |\nu_i|^{\alpha_i}\right)^p \left(1+\frac{\|\nu\|_2}{\xi_n}\right)^{rp} \left(\prod\limits_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}\right)}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}\right)}, \tag{1.37}$$

are uniformly bounded for all $\xi_n \in (0,1]$ *where* $\hat{\alpha}$, $n \in \mathbb{N}$,

$$\beta > \max\left\{\frac{1+\lceil \alpha_i p\rceil + \lceil rp\rceil}{2\hat{\alpha}}, \ \frac{2+\lceil rp\rceil}{2\hat{\alpha}}\right\}$$

for all $i = 1, ..., N$, *and* $\nu = (\nu_1, ..., \nu_N)$.

Finally, in [5], when $p \geq 1$, they obtained the following inequalities for the error quantities $E^{[0]}_{n,P}(f;x)$, $E^{[0]}_{n,P}(f;x)$, and the errors $E^{[m]}_{n,P}(f;x)$, $E^{[m]}_{n,P}(f;x)$:

$$\left\|E^{[0]}_{n,P}(f)\right\|_p \leq m_{\xi_n,P} \left\|P^{*[0]}_{r,n}(f) - f\right\|_p + \|f\|_p \left|m_{\xi_n,P} - 1\right|. \tag{1.38}$$

$$\left\| E_{n,W}^{[0]}(f) \right\|_p \leq m_{\xi_n,W} \left\| W_{r,n}^{*[0]}(f) - f \right\|_p + \|f\|_p \left| m_{\xi_n,W} - 1 \right|, \tag{1.39}$$

$$\begin{aligned} \left\| E_{n,P}^{[m]}(f;x) \right\|_p &\leq m_{\xi_n,P} \left\| P_{r,n}^{*\,[m]}(f) - f - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \right. \\ &\quad \times \left. \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \\ &\quad + \|f\|_p \left| m_{\xi_n,P} - 1 \right|, \end{aligned} \tag{1.40}$$

and

$$\begin{aligned} \left\| E_{n,W}^{[m]}(f) \right\|_p &\leq m_{\xi_n,W} \left\| W_{r,n}^{*\,[m]}(f) - f - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \right. \\ &\quad \times \left. \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{p_{\alpha,n,\widetilde{j}} f_\alpha}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \\ &\quad + \|f\|_p \left| m_{\xi_n,W} - 1 \right|. \end{aligned} \tag{1.41}$$

2. Main Results

We start with the general global smoothness preservation results for the operators $P_{r,n}^{*\,[m]}$, $W_{r,n}^{*\,[m]}$, and $Q_{r,n}^{*\,[m]}$, defined as in (1.8), (1.10), and (1.12).

Theorem 2.1. *Let $h > 0$, $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$.*

i) Assume $\omega_{\bar{m}}(f,h) < \infty$. Then

$$\omega_{\bar{m}}\left(P_{r,n}^{*\,[m]} f, h\right) \leq \left(\sum_{j=0}^{r} \left| \alpha_{j,r}^{[\bar{m}]} \right| \right) \omega_{\bar{m}}(f,h), \tag{2.1}$$

$$\omega_{\bar{m}}\left(W_{r,n}^{*\,[m]} f, h\right) \leq \left(\sum_{j=0}^{r} \left| \alpha_{j,r}^{[\bar{m}]} \right| \right) \omega_{\bar{m}}(f,h), \tag{2.2}$$

$$\omega_{\bar{m}}\left(Q_{r,n}^{*\,[m]} f, h\right) \leq \left(\sum_{j=0}^{r} \left| \alpha_{j,r}^{[\bar{m}]} \right| \right) \omega_{\bar{m}}(f,h). \tag{2.3}$$

ii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p \geq 1$. Then

$$\omega_{\bar{m}}\left(P_{r,n}^{*\,[m]} f, h\right)_p \leq \left(\sum_{j=0}^{r} \left| \alpha_{j,r}^{[\bar{m}]} \right| \right) \omega_{\bar{m}}(f,h)_p, \tag{2.4}$$

$$\omega_{\bar{m}}\left(W_{r,n}^{*\,[m]}f,h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}\left(f,h\right)_p, \tag{2.5}$$

$$\omega_{\bar{m}}\left(Q_{r,n}^{*\,[m]}f,h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}\left(f,h\right)_p. \tag{2.6}$$

Proof. By [1], Chapter 3. □

Next, we give

Remark 2.2. Let $r=1$, then we calculate that $\alpha_{0,1}^{[m]}=0$, $\alpha_{1,1}^{[m]}=1$. Now, denote

$$P_{1,n}^{*\,[m]}\left(f;x\right) := P_{n}^{*\,[m]}\left(f;x\right), \tag{2.7}$$

$$W_{1,n}^{*\,[m]}\left(f;x\right) := W_{n}^{*\,[m]}\left(f;x\right), \tag{2.8}$$

$$Q_{1,n}^{*\,[m]}\left(f;x\right) := Q_{n}^{*\,[m]}\left(f;x\right). \tag{2.9}$$

By Theorem 2.1 and Remark 2.2, we obtain

Theorem 2.3. *Let $h>0$, $f\in C\left(\mathbb{R}^N\right)$, $N\geq 1$.*
i) Assume $\omega_{\bar{m}}\left(f,h\right)<\infty$. Then

$$\omega_{\bar{m}}\left(P_{n}^{*\,[m]}f,h\right) \leq \omega_{\bar{m}}\left(f,h\right), \tag{2.10}$$

$$\omega_{\bar{m}}\left(W_{n}^{*\,[m]}f,h\right) \leq \omega_{\bar{m}}\left(f,h\right), \tag{2.11}$$

$$\omega_{\bar{m}}\left(Q_{n}^{*\,[m]}f,h\right) \leq \omega_{\bar{m}}\left(f,h\right). \tag{2.12}$$

ii) Assume $f\in\left(C\left(\mathbb{R}^N\right)\cap L_p\left(\mathbb{R}^N\right)\right)$, $p\geq 1$. Then

$$\omega_{\bar{m}}\left(P_{n}^{*\,[m]}f,h\right)_p \leq \omega_{\bar{m}}\left(f,h\right)_p, \tag{2.13}$$

$$\omega_{\bar{m}}\left(W_{n}^{*\,[m]}f,h\right)_p \leq \omega_{\bar{m}}\left(f,h\right)_p, \tag{2.14}$$

$$\omega_{\bar{m}}\left(Q_{n}^{*\,[m]}f,h\right)_p \leq \omega_{\bar{m}}\left(f,h\right)_p. \tag{2.15}$$

We present the our general global smoothness preservation results for the non-unitary operators $P_{r,n}^{[m]}$ and $W_{r,n}^{[m]}$ as follows

Theorem 2.4. *Let $h>0$, $f\in C\left(\mathbb{R}^N\right)$, $N\geq 1$.*
i) Assume $\omega_{\bar{m}}\left(f,h\right)<\infty$. Then

$$\begin{aligned}&\omega_{\bar{m}}\left(P_{r,n}^{[m]}f,h\right) \\ \leq\;& \left(\frac{1+2e^{-1/\xi_n}\left(\xi_n+1\right)}{1+2\xi_n e^{-1/\xi_n}}\right)^N\left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}\left(f,h\right),\end{aligned} \tag{2.16}$$

$$\omega_{\bar{m}}\left(W_{r,n}^{[m]}f,h\right) \tag{2.17}$$
$$\leq \left\{1+\frac{2e^{-1/\xi_n}}{\sqrt{\pi\xi_n}\left[1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right]+1}\right\}^N\left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f,h).$$

ii) Assume $f\in\left(C\left(\mathbb{R}^N\right)\cap L_p\left(\mathbb{R}^N\right)\right)$, $p\geq 1$. *Then*

$$\omega_{\bar{m}}\left(P_{r,n}^{[m]}f,h\right)_p \tag{2.18}$$
$$\leq \left(\frac{1+2e^{-1/\xi_n}\left(\xi_n+1\right)}{1+2\xi_n e^{-1/\xi_n}}\right)^N\left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f,h)_p,$$

$$\omega_{\bar{m}}\left(W_{r,n}^{[m]}f,h\right)_p \tag{2.19}$$
$$\leq \left\{1+\frac{2e^{-1/\xi_n}}{\sqrt{\pi\xi_n}\left[1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right]+1}\right\}^N\left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f,h)_p.$$

Proof. We see that

$$P_{r,n}^{[m]}(f;x)=\lambda_1(\xi_n)\,P_{r,n}^{*[m]}(f;x), \tag{2.20}$$

and

$$W_{r,n}^{[m]}(f;x)=\lambda_2(\xi_n)\,W_{r,n}^{*[m]}(f;x), \tag{2.21}$$

where

$$\lambda_1(\xi_n) := \frac{\sum_{\nu_1=-\infty}^{\infty}\cdots\sum_{\nu_N=-\infty}^{\infty}e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\left(1+2\xi_n e^{-1/\xi_n}\right)^N} = \prod_{i=1}^{N}\left(\frac{\sum_{\nu_i=-\infty}^{\infty}e^{-\frac{|\nu_i|}{\xi_n}}}{1+2\xi_n e^{-1/\xi_n}}\right), \tag{2.22}$$

and

$$\lambda_2(\xi_n) := \frac{\sum_{\nu_1=-\infty}^{\infty}\cdots\sum_{\nu_N=-\infty}^{\infty}e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\left[\sqrt{\pi\xi_n}\left(1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right)+1\right]^N} = \prod_{i=1}^{N}\left(\frac{\sum_{\nu_i=-\infty}^{\infty}e^{-\frac{\nu_i^2}{\xi_n}}}{\sqrt{\pi\xi_n}\left[1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right]+1}\right). \tag{2.23}$$

Additionally, in [2], the author showed that

$$\frac{\sum\limits_{\nu_i=-\infty}^{\infty} e^{-\frac{|\nu_i|}{\xi_n}}}{1+2\xi_n e^{-1/\xi_n}} \leq \frac{1+2e^{\frac{-1}{\xi}}\left(\xi+1\right)}{1+2\xi e^{\frac{-1}{\xi}}}, \tag{2.24}$$

and

$$\frac{\sum\limits_{\nu_i=-\infty}^{\infty} e^{-\frac{\nu_i^2}{\xi_n}}}{\sqrt{\pi\xi_n}\left[1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right]+1} \leq 1+\frac{2e^{\frac{-1}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}. \tag{2.25}$$

Thus, by (2.22), (2.23), (2.24), (2.25), and Theorem 2.1 the proof is complete. □

Now, we demonstrate the following optimality result

Proposition 2.5. *Above inequalities* (2.10)-(2.12) *are sharp. The equalities are attained by any*

$$g_j\left(x\right)=x_j^{\bar{m}},\quad j=1,...,N,\quad x=\left(x_1,...,x_j,...,x_N\right)\in\mathbb{R}^N.$$

Proof. By [1], Chapter 3. □

In [6], the authors observed

Theorem 2.6. *Let* $f \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. *Here* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$, $\xi_n > 0$, $\left(\xi_n\right)_{n\in\mathbb{N}}$ *a bounded sequence. Let* $\tilde{\beta} := \left(\tilde{\beta}_1, ..., \tilde{\beta}_N\right)$, $\tilde{\beta}_i \in \mathbb{Z}^+$, $i = 1, ..., N$; $\left|\tilde{\beta}\right| := \sum\limits_{i=1}^{N} \tilde{\beta}_i = l$. *Here* $f\left(x+\nu j\right)$, $x \in \mathbb{R}^N$, $\nu \in \mathbb{Z}^N$, *is* μ_{ξ_n}*-integrable with respect to* ν, *for* $j = 1, ..., r$. *There exist* μ_{ξ_n}*-integrable functions* $h_{i_1,j}$, $h_{\tilde{\beta}_1,i_2,j}$, $h_{\tilde{\beta}_1,\tilde{\beta}_2,i_3,j}, ..., h_{\tilde{\beta}_1,\tilde{\beta}_2,...,\tilde{\beta}_{N-1},i_N,j} \geq 0$ *(*$j = 1, ..., r$*) on* $\mathbb{R}^N$ *such that*

$$\left|\frac{\partial^{i_1} f\left(x+\nu j\right)}{\partial x_1^{i_1}}\right| \leq h_{i_1,j}\left(\nu\right),\quad i_1=1,...,\tilde{\beta}_1, \tag{2.26}$$

$$\left|\frac{\partial^{\tilde{\beta}_1+i_2} f\left(x+\nu j\right)}{\partial x_2^{i_2}\partial x_1^{\tilde{\beta}_1}}\right| \leq h_{\tilde{\beta}_1,i_2,j}\left(\nu\right),\quad i_2=1,...,\tilde{\beta}_2,$$

$$\vdots$$

$$\left|\frac{\partial^{\tilde{\beta}_1+\tilde{\beta}_2+...+\tilde{\beta}_{N-1}+i_N} f\left(x+\nu j\right)}{\partial x_N^{i_N}\partial x_{N-1}^{\tilde{\beta}_{N-1}}...\partial x_2^{\tilde{\beta}_2}\partial x_1^{\tilde{\beta}_1}}\right| \leq h_{\tilde{\beta}_1,\tilde{\beta}_2,...,\tilde{\beta}_{N-1},i_N,j}\left(\nu\right),\quad i_N=1,...,\tilde{\beta}_N,$$

$\forall\ x \in \mathbb{R}^N$, $\nu \in \mathbb{Z}^N$.

i) When

$$\mu_{\xi_n}\left(\nu\right)=\frac{e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty}...\sum\limits_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}, \tag{2.27}$$

then both of the next exist and

$$\left(P_{r,n}^{*\ [m]}\left(f;x\right)\right)_{\tilde{\beta}}=P_{r,n}^{*\ [m]}\left(f_{\tilde{\beta}};x\right). \tag{2.28}$$

ii) When

$$\mu_{\xi_n}\left(\nu\right)=\frac{e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty}\cdots\sum_{\nu_N=-\infty}^{\infty}e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}, \tag{2.29}$$

then both of the next exist and

$$\left(W_{r,n}^{*\ [m]}\left(f;x\right)\right)_{\tilde{\beta}}=W_{r,n}^{*\ [m]}\left(f_{\tilde{\beta}};x\right). \tag{2.30}$$

iii) Let $\hat{\alpha}\in\mathbb{N}$ *and* $\beta>\frac{1}{\hat{\alpha}}$. *When*

$$\mu_{\xi_n}\left(\nu\right)=\frac{\prod_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}{\sum_{\nu_1=-\infty}^{\infty}\cdots\sum_{\nu_N=-\infty}^{\infty}\prod_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}, \tag{2.31}$$

then both of the next exist and

$$\left(Q_{r,n}^{*\ [m]}\left(f;x\right)\right)_{\tilde{\beta}}=Q_{r,n}^{*\ [m]}\left(f_{\tilde{\beta}};x\right). \tag{2.32}$$

Corollary 2.7. *When* $r=1$, *by the Theorem* 2.6, *we observe that*

$$\left(P_{n}^{*\ [m]}\left(f;x\right)\right)_{\tilde{\beta}}=P_{n}^{*\ [m]}\left(f_{\tilde{\beta}};x\right), \tag{2.33}$$

$$\left(W_{n}^{*\ [m]}\left(f;x\right)\right)_{\tilde{\beta}}=W_{n}^{*\ [m]}\left(f_{\tilde{\beta}};x\right), \tag{2.34}$$

and

$$\left(Q_{n}^{*\ [m]}\left(f;x\right)\right)_{\tilde{\beta}}=Q_{n}^{*\ [m]}\left(f_{\tilde{\beta}};x\right). \tag{2.35}$$

For the non-unitary operators $P_{r,n}^{[m]}$ and $W_{r,n}^{[m]}$ we have

Theorem 2.8. *Let the assumption of Theorem* 2.6 *be true. Then we have*

$$\left(P_{r,n}^{[m]}\left(f;x\right)\right)_{\tilde{\beta}}=P_{r,n}^{[m]}\left(f_{\tilde{\beta}};x\right), \tag{2.36}$$

and

$$\left(W_{r,n}^{[m]}\left(f;x\right)\right)_{\tilde{\beta}}=W_{r,n}^{[m]}\left(f_{\tilde{\beta}};x\right). \tag{2.37}$$

Proof. By (2.20), (2.21), and Theorem 2.6, we obtain

$$\begin{aligned}\left(P_{r,n}^{[m]}\left(f;x\right)\right)_{\tilde{\beta}} &= \lambda_1\left(\xi_n\right)\left(P_{r,n}^{*[m]}\left(f;x\right)\right)_{\tilde{\beta}} \\ &= \lambda_1\left(\xi_n\right)P_{r,n}^{*[m]}\left(f_{\tilde{\beta}};x\right)=P_{r,n}^{[m]}\left(f_{\tilde{\beta}};x\right),\end{aligned} \tag{2.38}$$

and

$$\begin{aligned}\left(W_{r,n}^{[m]}(f;x)\right)_{\tilde{\beta}} &= \lambda_2(\xi_n)\left(W_{r,n}^{*[m]}(f;x)\right)_{\tilde{\beta}} \\ &= \lambda_2(\xi_n)\, W_{r,n}^{*[m]}\left(f_{\tilde{\beta}};x\right) = W_{r,n}^{[m]}\left(f_{\tilde{\beta}};x\right).\end{aligned} \tag{2.39}$$

□

Next, we get

Theorem 2.9. *Let $h > 0$, $\gamma = 0, \tilde{\beta}$, and the assumptions of the Theorem 2.6 be true.*
i) Assume $\omega_{\bar{m}}(f_\gamma, h) < \infty$. Then

$$\omega_{\bar{m}}\left(\left(P_{r,n}^{*\,[m]}f\right)_\gamma, h\right) \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h), \tag{2.40}$$

$$\omega_{\bar{m}}\left(\left(W_{r,n}^{*\,[m]}f\right)_\gamma, h\right) \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h), \tag{2.41}$$

$$\omega_{\bar{m}}\left(\left(Q_{r,n}^{*\,[m]}f\right)_\gamma, h\right) \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h). \tag{2.42}$$

ii) Assume $f_\gamma \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p \geq 1$. Then

$$\omega_{\bar{m}}\left(\left(P_{r,n}^{*\,[m]}f\right)_\gamma, h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h)_p, \tag{2.43}$$

$$\omega_{\bar{m}}\left(\left(W_{r,n}^{*\,[m]}f\right)_\gamma, h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h)_p, \tag{2.44}$$

$$\omega_{\bar{m}}\left(\left(Q_{r,n}^{*\,[m]}f\right)_\gamma, h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h)_p. \tag{2.45}$$

Proof. By Theorem 2.1 and Theorem 2.6. □

Additionally, as a quick result of Theorem 2.3 and Theorem 2.6, we have

Corollary 2.10. *Let $h > 0$, $\gamma = 0, \tilde{\beta}$, and the assumptions of the Theorem 2.6 be true.*
i) Assume $\omega_{\bar{m}}(f, h) < \infty$. Then

$$\omega_{\bar{m}}\left(\left(P_n^{*\,[m]}f\right)_\gamma, h\right) \leq \omega_{\bar{m}}(f_\gamma, h), \tag{2.46}$$

$$\omega_{\bar{m}}\left(\left(W_n^{*\,[m]}f\right)_\gamma, h\right) \leq \omega_{\bar{m}}(f_\gamma, h), \tag{2.47}$$

$$\omega_{\bar{m}}\left(\left(Q_n^{*\,[m]}f\right)_\gamma, h\right) \leq \omega_{\bar{m}}(f_\gamma, h). \tag{2.48}$$

ii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p \geq 1$. *Then*

$$\omega_{\bar{m}}\left(\left(P_n^{*\,[m]} f\right)_\gamma, h\right)_p \leq \omega_{\bar{m}}\left(f_\gamma, h\right)_p, \tag{2.49}$$

$$\omega_{\bar{m}}\left(\left(W_n^{*\,[m]} f\right)_\gamma, h\right)_p \leq \omega_{\bar{m}}\left(f_\gamma, h\right)_p, \tag{2.50}$$

$$\omega_{\bar{m}}\left(\left(Q_n^{*\,[m]} f\right)_\gamma, h\right)_p \leq \omega_{\bar{m}}\left(f_\gamma, h\right)_p. \tag{2.51}$$

Additionally for the non-unitary operators, $P_{r,n}^{[m]}$ and $W_{r,n}^{[m]}$, we obtain

Theorem 2.11. *Let* $h > 0$, $\gamma = 0, \tilde{\beta}$, *and the assumptions of the Theorem* 2.6 *be true.*
i) Assume $\omega_{\bar{m}}\left(f_\gamma, h\right) < \infty$. *Then*

$$\omega_{\bar{m}}\left(\left(P_{r,n}^{[m]} f\right)_\gamma, h\right) \tag{2.52}$$

$$\leq \left(\frac{1 + 2e^{-1/\xi_n}\left(\xi_n + 1\right)}{1 + 2\xi_n e^{-1/\xi_n}}\right)^N \left(\sum_{j=0}^r \left|\alpha_{j,r}^{[\bar{m}]}\right|\right) \omega_{\bar{m}}\left(f_\gamma, h\right),$$

$$\omega_{\bar{m}}\left(\left(W_{r,n}^{[m]} f\right)_\gamma, h\right) \tag{2.53}$$

$$\leq \left\{1 + \frac{2e^{-1/\xi_n}}{\sqrt{\pi}\xi_n\left[1 - \operatorname{erf}\left(\frac{1}{\xi_n}\right)\right] + 1}\right\}^N \left(\sum_{j=0}^r \left|\alpha_{j,r}^{[\bar{m}]}\right|\right) \omega_{\bar{m}}\left(f_\gamma, h\right).$$

ii) Assume $f_\gamma \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p \geq 1$. *Then*

$$\omega_{\bar{m}}\left(\left(P_{r,n}^{[m]} f\right)_\gamma, h\right)_p \tag{2.54}$$

$$\leq \left(\frac{1 + 2e^{-1/\xi_n}\left(\xi_n + 1\right)}{1 + 2\xi_n e^{-1/\xi_n}}\right)^N \left(\sum_{j=0}^r \left|\alpha_{j,r}^{[\bar{m}]}\right|\right) \omega_{\bar{m}}\left(f_\gamma, h\right)_p,$$

$$\omega_{\bar{m}}\left(\left(W_{r,n}^{[m]} f\right)_\gamma, h\right)_p \tag{2.55}$$

$$\leq \left\{1 + \frac{2e^{-1/\xi_n}}{\sqrt{\pi}\xi_n\left[1 - \operatorname{erf}\left(\frac{1}{\xi_n}\right)\right] + 1}\right\}^N \left(\sum_{j=0}^r \left|\alpha_{j,r}^{[\bar{m}]}\right|\right) \omega_{\bar{m}}\left(f_\gamma, h\right)_p.$$

Proof. By Theorem 2.4 and Theorem 2.8. □

Now we show our simultaneous approximation results.
We start with

Theorem 2.12. *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l \in \mathbb{N}$, $N \geq 1$, $x \in \mathbb{R}^N$. Let the assumptions of Theorem 2.6 is true and $\gamma = 0, \tilde{\beta}$. Assume $\|f_{\gamma+\alpha}\|_\infty < \infty$. Then for all $x \in \mathbb{R}^N$, we have*

i)

$$\left\| \left(P_{r,n}^{*\,[m]} f\right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha|=\widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_\infty \tag{2.56}$$

$$\leq \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha|=m}\right)} \frac{\left(\omega_r\left(f_{\gamma+\alpha}, \xi_n\right)\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} u_{P,\xi_n}^{*},$$

for $\xi_n \in (0,1]$.

ii)

$$\left\| \left(W_{r,n}^{*\,[m]} f\right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha|=\widetilde{j}}} \frac{p_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_\infty \tag{2.57}$$

$$\leq \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha|=m}\right)} \frac{\left(\omega_r\left(f_{\gamma+\alpha}, \xi_n\right)\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} u_{W,\xi_n}^{*},$$

for $\xi_n \in (0,1]$.

iii)

$$\left\| \left(Q_{r,n}^{*\,[m]} f\right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha|=\widetilde{j}}} \frac{q_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_\infty \tag{2.58}$$

$$\leq \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha|=m}\right)} \frac{\left(\omega_r\left(f_{\gamma+\alpha}, \xi_n\right)\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} u_{Q,\xi_n}^{*},$$

for $\xi_n \in (0,1]$, and $\hat{\alpha} \in \mathbb{N}$, $\beta > \max\left\{\frac{1+r+\alpha_i}{2\hat{\alpha}},\ \frac{r+2}{2\hat{\alpha}}\right\}$.

Proof. By [4] and Theorem 2.6. □

Next, when $m = 0$, we obtain

Theorem 2.13. *Let $f \in C_B^l\left(\mathbb{R}^N\right)$, $l \in \mathbb{N}$, $N \geq 1$. Let the assumptions of Theorem 2.6 is true and $\gamma = 0, \tilde{\beta}$. Then for all $x \in \mathbb{R}^N$, we have* *i)*

$$\left\| \left(P^{*\,[0]}_{r,n} f \right)_{\gamma} - f_{\gamma} \right\|_{\infty} \leq \Phi^{*}_{P,\xi_n} \omega_r(f_{\gamma}, \xi_n), \tag{2.59}$$

for $\xi_n \in (0,1]$.

ii)

$$\left\| \left(W^{*\,[0]}_{r,n} f \right)_{\gamma} - f_{\gamma} \right\|_{\infty} \leq \Phi^{*}_{W,\xi_n} \omega_r(f_{\gamma}, \xi_n), \tag{2.60}$$

for $\xi_n \in (0,1]$.

iii)

$$\left\| \left(Q^{*\,[0]}_{r,n} f \right)_{\gamma} - f_{\gamma} \right\|_{\infty} \leq \Phi^{*}_{Q,\xi_n} \omega_r(f_{\gamma}, \xi_n), \tag{2.61}$$

for $\xi_n \in (0,1]$*, and* $\hat{\alpha} \in \mathbb{N}$, $\beta > \frac{r+2}{2\hat{\alpha}}$.

Proof. By [4] and Theorem 2.6. □

For the non-unitary cases we have

Theorem 2.14. *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$*,* $m, l \in \mathbb{N}$*,* $N \geq 1$*. Let the assumptions of Theorem* 2.6 *is true and* $\gamma = 0, \tilde{\beta}$*. Assume* $\|f_{\gamma+\alpha}\|_{\infty} < \infty$*. Then for all* $x \in \mathbb{R}^N$*, we have*

i)

$$\begin{aligned} &\left\| \left(E^{[m]}_{n,P}(f) \right)_{\gamma} \right\|_{\infty} \\ \leq\ & m_{\xi_n,P} \sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0 \\ |\alpha| = m}} \frac{\omega_r\left(f_{\alpha+\gamma}, \xi_n\right)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} u^{*}_{P,\xi_n} \\ & + \|f_{\gamma}\|_{\infty} \left| m_{\xi_n,P} - 1 \right|, \end{aligned} \tag{2.62}$$

and

$$\begin{aligned} &\left\| \left(E^{[m]}_{n,W}(f) \right)_{\gamma} \right\|_{\infty} \\ \leq\ & m_{\xi_n,W} \sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0 \\ |\alpha| = m}} \frac{\omega_r\left(f_{\alpha+\gamma}, \xi_n\right)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} u^{*}_{W,\xi_n} \\ & + \|f_{\gamma}\|_{\infty} \left| m_{\xi_n,W} - 1 \right|. \end{aligned} \tag{2.63}$$

ii) Let $f \in C^{l}_{B}\left(\mathbb{R}^N\right)$*,* $l \in \mathbb{N}$*,* $N \geq 1$*. Let the assumptions of Theorem* 2.6 *is true and* $\gamma = 0, \tilde{\beta}$*. Then for all* $x \in \mathbb{R}^N$*, we have*

$$\begin{aligned} &\left\| \left(E^{[0]}_{n,P}(f) \right)_{\gamma} \right\|_{\infty} \\ \leq\ & m_{\xi_n,P} \Phi^{*}_{P,\xi_n} \omega_r(f_{\gamma}, \xi_n) + \|f_{\gamma}\|_{\infty} \left| m_{\xi_n,P} - 1 \right|, \end{aligned} \tag{2.64}$$

and

$$\left\| \left(E_{n,W}^{[0]}(f;x) \right)_\gamma \right\|_\infty \tag{2.65}$$
$$\leq m_{\xi_n,W}\Phi^*_{W,\xi_n}\omega_r(f_\gamma,\xi_n) + \|f_\gamma\|_\infty |m_{\xi_n,W}-1| .$$

Proof. By [4], (1.38) – (1.41), and by the equalities $\left(E_{n,P}^{[m]}(f,x)\right)_\gamma = E_{n,P}^{[m]}(f_\gamma,x)$ and $\left(E_{n,W}^{[m]}(f,x)\right)_\gamma = E_{n,W}^{[m]}(f_\gamma,x)$ for $m \in \mathbb{Z}^+$. □

Now, we give our L_p results. We begin with

Theorem 2.15. *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m,l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_{\gamma+\alpha} \in L_p\left(\mathbb{R}^N\right)$, $|\alpha| = m$, $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true. Then*

i)

$$\left\| \left(P^{*\,[m]}_{r,n} f\right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \tag{2.66}$$
$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(S^{p,m}_{P^*,\xi_n} \right)^{\frac{1}{p}} \omega_r\left(f_{\gamma+\alpha},\xi_n\right)_p .$$

ii)

$$\left\| \left(W^{*\,[m]}_{r,n} f\right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{p_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \tag{2.67}$$
$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(S^{p,m}_{W^*,\xi_n} \right)^{\frac{1}{p}} \omega_r\left(f_{\gamma+\alpha},\xi_n\right)_p .$$

iii)

$$\left\| \left(Q^{*\,[m]}_{r,n} f\right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{q_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \tag{2.68}$$
$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(S^{p,m}_{Q^*,\xi_n} \right)^{\frac{1}{p}} \omega_r\left(f_{\gamma+\alpha},\xi_n\right)_p ,$$

where $\hat{\alpha} \in \mathbb{N}$, $\beta > \max\left\{\frac{1+\lceil \alpha_i p\rceil + \lceil rp\rceil}{2\hat{\alpha}}, \frac{2+\lceil rp\rceil}{2\hat{\alpha}}\right\}$ *for all* $i = 1, ..., N$.

Proof. By [5] and Theorem 2.6. □

Next, we present our results for the case of $m = 0$ and $p > 1$.

Theorem 2.16. *Let* $f \in C^l(\mathbb{R}^N)$, $l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_\gamma \in L_p(\mathbb{R}^N), x \in \mathbb{R}^N$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true. Then*

i)

$$\left\|\left(P_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma\right\|_p \leq \left(S_{P^*,\xi_n}^{p,0}\right)^{\frac{1}{p}} \omega_r(f_\gamma, \xi_n)_p . \tag{2.69}$$

ii)

$$\left\|\left(W_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma\right\|_p \leq \left(S_{W^*,\xi_n}^{p,0}\right)^{\frac{1}{p}} \omega_r(f_\gamma, \xi_n)_p . \tag{2.70}$$

iii)

$$\left\|\left(Q_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma\right\|_p \leq \left(S_{Q^*,\xi_n}^{p,0}\right)^{\frac{1}{p}} \omega_r(f_\gamma, \xi_n)_p , \tag{2.71}$$

where $\hat{\alpha} \in \mathbb{N}$, $\beta > \frac{2+\lceil rp\rceil}{2\hat{\alpha}}$.

Proof. By [5] and Theorem 2.6. □

For the case of $m = 0$ and $p = 1$, we have

Theorem 2.17. *Let* $f \in C^l(\mathbb{R}^N)$, $l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_\gamma \in L_1(\mathbb{R}^N), x \in \mathbb{R}^N$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true.*

i)

$$\left\|\left(P_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma\right\|_1 \leq S_{P^*,\xi_n}^{1,0} \omega_r(f_\gamma, \xi_n)_1 . \tag{2.72}$$

ii)

$$\left\|\left(W_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma\right\|_1 \leq S_{W^*,\xi_n}^{1,0} \omega_r(f_\gamma, \xi_n)_1 . \tag{2.73}$$

iii)

$$\left\|\left(Q_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma\right\|_1 \leq S_{Q^*,\xi_n}^{1,0} \omega_r(f_\gamma, \xi_n)_1 , \tag{2.74}$$

where $\hat{\alpha} \in \mathbb{N}$, $\beta > \frac{2+r}{2\hat{\alpha}}$.

Proof. By [5] and Theorem 2.6. □

Next, we give the case of $m \in \mathbb{N}$ and $p = 1$ as

Theorem 2.18. *Let* $f \in C^{m+l}(\mathbb{R}^N)$, $m, l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_{\gamma+\alpha} \in L_1(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true. Then* *i)*

$$\left\| \left(P_{r,n}^{*\,[m]} f \right)_{\gamma} - f_{\gamma} - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_1 \tag{2.75}$$

$$\leq \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S_{P^*,\xi_n}^{1,m} \omega_r \left(f_{\gamma+\alpha}, \xi_n \right)_1 .$$

ii)

$$\left\| \left(W_{r,n}^{*\,[m]} f \right)_{\gamma} - f_{\gamma} - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{p_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_1 \tag{2.76}$$

$$\leq \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S_{W^*,\xi_n}^{1,m} \omega_r \left(f_{\gamma+\alpha}, \xi_n \right)_1 .$$

iii)

$$\left\| \left(Q_{r,n}^{*\,[m]} f \right)_{\gamma} - f_{\gamma} - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{q_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_1 \tag{2.77}$$

$$\leq \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S_{Q^*,\xi_n}^{1,m} \omega_r \left(f_{\gamma+\alpha}, \xi_n \right)_1 ,$$

where $\hat{\alpha} \in \mathbb{N}$, $\beta > \max \left\{ \frac{1+\alpha_i+r}{2\hat{\alpha}}, \ \frac{2+r}{2\hat{\alpha}} \right\}$ *for all* i.

Proof. By [5] and Theorem 2.6. □

Finally, we give our L_p results for the error quantities $E_{n,P}^{[0]}(f;x)$, $E_{n,P}^{[0]}(f;x)$, and the errors $E_{n,P}^{[m]}(f;x)$, $E_{n,P}^{[m]}(f;x)$. We begin with

Theorem 2.19. *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_{\gamma+\alpha} \in L_p\left(\mathbb{R}^N\right)$, $|\alpha| = m$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem*

2.6 *be true. Then*

$$\left\| \left(E_{n,P}^{[m]}(f) \right)_{\gamma} \right\|_{p} \leq m_{\xi_n,P} \left(\frac{m \left(S_{P^*,\xi_n}^{p,m} \right)^{\frac{1}{p}} \omega_r \left(f_{\alpha+\gamma}, \xi_n \right)_p}{\left(q\left(m-1 \right) +1 \right)^{\frac{1}{q}}} \right) \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} + \| f_{\gamma} \|_p \left| m_{\xi_n,P} - 1 \right|, \tag{2.78}$$

and

$$\left\| \left(E_{n,W}^{[m]}(f) \right)_{\gamma} \right\|_{p} \leq m_{\xi_n,W} \left(\frac{m \left(S_{W^*,\xi_n}^{p,m} \right)^{\frac{1}{p}} \omega_r \left(f_{\alpha+\gamma}, \xi_n \right)_p}{\left(q\left(m-1 \right) +1 \right)^{\frac{1}{q}}} \right) \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} + \| f_{\gamma} \|_p \left| m_{\xi_n,W} - 1 \right|. \tag{2.79}$$

Proof. By [5], (1.40), (1.41), and by the equalities $\left(E_{n,P}^{[m]}(f,x) \right)_{\gamma} = E_{n,P}^{[m]}(f_{\gamma},x)$ and $\left(E_{n,W}^{[m]}(f,x) \right)_{\gamma} = E_{n,W}^{[m]}(f_{\gamma},x)$ for $m \in \mathbb{Z}^{+}$. □

Next, we present the following results for the case of $m = 0$ and $p > 1$ as

Theorem 2.20. *Let* $f \in C^{l}\left(\mathbb{R}^{N}\right)$, $l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_{\gamma} \in L_p\left(\mathbb{R}^{N}\right)$, $p, q > 1 :$ $\frac{1}{p} + \frac{1}{q} = 1$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true. Then*

$$\left\| \left(E_{n,P}^{[0]}(f) \right)_{\gamma} \right\|_{p} \leq m_{\xi_n,P} \left(S_{P^*,\xi_n}^{p,0} \right)^{\frac{1}{p}} \omega_r \left(f_{\gamma}, \xi_n \right)_p + \| f_{\gamma} \|_p \left| m_{\xi_n,P} - 1 \right|, \tag{2.80}$$

and

$$\left\| \left(E_{n,W}^{[0]}(f) \right)_{\gamma} \right\|_{p} \leq m_{\xi_n,W} \left(S_{W^*,\xi_n}^{p,0} \right)^{\frac{1}{p}} \omega_r \left(f_{\gamma}, \xi_n \right)_p + \| f_{\gamma} \|_p \left| m_{\xi_n,W} - 1 \right|. \tag{2.81}$$

Proof. By [5], (1.38), (1.39), and by the equalities $\left(E_{n,P}^{[m]}(f,x) \right)_{\gamma} = E_{n,P}^{[m]}(f_{\gamma},x)$ and $\left(E_{n,W}^{[m]}(f,x) \right)_{\gamma} = E_{n,W}^{[m]}(f_{\gamma},x)$ for $m \in \mathbb{Z}^{+}$. □

For the case of $m = 0$ and $p = 1$, we obtain

Theorem 2.21. *Let* $f \in C^{l}\left(\mathbb{R}^{N}\right)$, $l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_{\gamma} \in L_1\left(\mathbb{R}^{N}\right)$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true. Then*

$$\left\| \left(E_{n,P}^{[0]}(f) \right)_{\gamma} \right\|_{1} \leq m_{\xi_n,P} S_{P^*,\xi_n}^{1,0} \omega_r \left(f_{\gamma}, \xi_n \right)_1 + \| f_{\gamma} \|_1 \left| m_{\xi_n,P} - 1 \right|, \tag{2.82}$$

and

$$\left\| \left(E_{n,W}^{[0]}(f) \right)_{\gamma} \right\|_1 \leq m_{\xi_n,W} S_{W^*,\xi_n}^{1,0} \omega_r \left(f_\gamma, \xi_n \right)_1 + \| f_\gamma \|_1 \left| m_{\xi_n,W} - 1 \right| . \tag{2.83}$$

Proof. By [5], (1.38) , (1.39), and by the equalities $\left(E_{n,P}^{[m]}(f,x)\right)_{\gamma} = E_{n,P}^{[m]}(f_\gamma, x)$ and $\left(E_{n,W}^{[m]}(f,x)\right)_{\gamma} = E_{n,W}^{[m]}(f_\gamma, x)$ for $m \in \mathbb{Z}^+$. □

Our final result is for the case of $m \in \mathbb{N}$ and $p = 1$

Theorem 2.22. *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_{\gamma+\alpha} \in L_1\left(\mathbb{R}^N\right)$, $|\alpha| = m$, and $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. Let the assumptions of Theorem 2.6 be true. Then*

$$\begin{aligned} \left\| \left(E_{n,P}^{[m]}(f) \right)_{\gamma} \right\|_1 \leq \; & m_{\xi_n,P} \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S_{P^*,\xi_n}^{1,m} \omega_r \left(f_{\alpha+\gamma}, \xi_n \right)_1 \\ & + \| f_\gamma \|_1 \left| m_{\xi_n,P} - 1 \right| , \end{aligned} \tag{2.84}$$

and

$$\begin{aligned} \left\| \left(E_{n,W}^{[m]}(f) \right)_{\gamma} \right\|_1 \leq \; & m_{\xi_n,W} \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S_{W^*,\xi_n}^{1,m} \omega_r \left(f_{\alpha+\gamma}, \xi_n \right)_1 \\ & + \| f_\gamma \|_1 \left| m_{\xi_n,W} - 1 \right| . \end{aligned} \tag{2.85}$$

Proof. By [5], (1.40) , (1.41), and by the equalities $\left(E_{n,P}^{[m]}(f,x)\right)_{\gamma} = E_{n,P}^{[m]}(f_\gamma, x)$ and $\left(E_{n,W}^{[m]}(f,x)\right)_{\gamma} = E_{n,W}^{[m]}(f_\gamma, x)$ for $m \in \mathbb{Z}^+$. □

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