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Eigenstructure of the genuine Beta operators of Lupaş and Mühlbach

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. The eigenstructure of genuine Beta operators is described, a limiting case of Beta-Jacobi operators. Its similarity to that of the classical Bernstein operators is emphasized. The significance of the mappings considered here comes, among others, from their role as a building block in genuine Bernstein-Durrmeyer operators.

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1. Introduction and historical notes

The present note deals with the eigenstructure of certain Beta-type operators introduced independently by Mühlbach and Lupaş in the early seventies of the last century (see [10],[11],[9]).

Mühlbach's definition is the more general one. For $\lambda > 0$ he defined mappings T_λ, given for $f \in C[0,1]$, $x \in [0,1]$ by

$$T_\lambda(f;x) = \begin{cases} f(0), & x = 0, \\ \displaystyle\int_0^1 f(t)K_\lambda(t,x)dt, & x \in (0,1), \\ f(1), & x = 1. \end{cases}$$

The kernel is given by

$$K_\lambda(t,x) = \frac{1}{B(\frac{x}{\lambda}, \frac{1-x}{\lambda})} t^{\frac{x}{\lambda}-1}(1-t)^{\frac{1-x}{\lambda}-1},$$

where $B(\cdot,*)$ is the Beta function, a.k.a. Euler's integral of the first kind. For more on this function see, e.g., MathWorld [16] and the references given there. Mühlbach's work was motivated by three earlier papers of Stancu, see [12] , [13], [14].

If $1/\lambda = n$ is a natural number, then we arrive at Lupaş' version of the operator, given for strictly positive integers n by

$$\bar{\mathbb{B}}_n(f;x) = \begin{cases} f(0), & x=0, \\ \dfrac{1}{B(nx, n(1-x))} \displaystyle\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)dt, & x \in (0,1), \\ f(1), & x = 1. \end{cases}$$

The $\bar{\mathbb{B}}_n$ are positive linear endomorphisms of $C[0,1]$; they reproduce linear functions and have second moments smaller than the classical Bernstein operators B_n. More precisely, see [9, Satz 2.28],

$$\bar{\mathbb{B}}_n((e_1-x)^2;x) = \frac{x(1-x)}{n+1} \le \frac{x(1-x)}{n} = B_n((e_1-x)^2;x).$$

The restrictions $\bar{\mathbb{B}}_n : \Pi_n \to \Pi_n$ and $\bar{\mathbb{B}}_n : \Pi \to \Pi$ are bijective, and $\bar{\mathbb{B}}_n : C[0,1] \to C[0,1]$ is injective. Moreover, it is known from [2] and [3] that $\bar{\mathbb{B}}_n$ preserves monotonicity and (ordinary) convexity.

Our reason to call them *genuine* Beta operators is due to the facts that they are the limiting cases of Beta operators with Jacobi weights and unique in the sense that they are the only ones among them which reproduce linear functions. Calling them *genuine* is also justified by the decomposition $B_n \circ \bar{\mathbb{B}}_n = U_n$; here U_n is the so-called genuine Bernstein-Durrmeyer operator which has been attracting much attention. Much more on Beta-Jacobi operators can be found in [6], [15], [7].

The genuine operators $\bar{\mathbb{B}}_n$ were also used in attempts to decompose the classical Bernstein operators into non-trivial building blocks. Reports on these were given by Gonska et al. [5] and by Heilmann and Rasa [8]. Aspects concerning their power series are described in [1].

2. The eigenstructure of $\bar{\mathbb{B}}_n$

The purpose of this article is to give a concise description of the eigenstructure of the Beta operators considered here. By direct computation it is easy to find the first eigenvalues and eigenpolynomials of $\bar{\mathbb{B}}_n$:

$$\begin{aligned}
\eta_0^{(n)} &= 1, & q_0^{(n)}(x) &= 1, \\
\eta_1^{(n)} &= 1, & q_1^{(n)}(x) &= x - \frac{1}{2}, \\
\eta_2^{(n)} &= \frac{n}{n+1}, & q_2^{(n)}(x) &= x(x-1), \\
\eta_3^{(n)} &= \frac{n^2}{(n+1)(n+2)}, & q_3^{(n)}(x) &= x(x-1)\left(x - \frac{1}{2}\right), \\
\eta_4^{(n)} &= \frac{n^3}{(n+1)(n+2)(n+3)}, & q_4^{(n)}(x) &= x(x-1)\left(x(x-1) + \frac{n+1}{5n+6}\right).
\end{aligned}$$

As

$$\begin{aligned} \bar{\mathbb{B}}_n e_0 &= e_0, \\ \bar{\mathbb{B}}_n e_k(x) &= \frac{nx(nx+1)\dots(nx+k-1)}{n(n+1)\dots(n+k-1)},\ k \geq 1, \end{aligned} \tag{2.1}$$

following directly from the definition of $\bar{\mathbb{B}}_n$, we conclude that the eigenvalues of $\bar{\mathbb{B}}_n : \Pi \longrightarrow \Pi$ are the numbers

$$\eta_k^{(n)} = \frac{(n-1)!}{(n+k-1)!} n^k,\ k \geq 0. \tag{2.2}$$

Let us denote by $p_k^{(n)}$ the eigenpolynomials of B_n (see [4]). Here are some examples (see [4, (9.1)]).

$$\begin{aligned} p_0^{(n)}(x) &= 1, \\ p_1^{(n)}(x) &= x - \frac{1}{2}, \\ p_2^{(n)}(x) &= x(x-1), \\ p_3^{(n)}(x) &= x(x-1)\left(x - \frac{1}{2}\right), \\ p_4^{(n)}(x) &= x(x-1)\left(x(x-1) + \frac{n-1}{5n-6}\right). \end{aligned}$$

Thus we have

$$q_k^{(n)} = p_k^{(n)},\ 0 \leq k \leq 3$$

and

$$\lim_{n\to\infty} q_k^{(n)}(x) = \lim_{n\to\infty} p_k^{(n)}(x),\ k = 4, \tag{2.3}$$

uniformly in $[0,1]$. We shall show that the eigenstructure of $\bar{\mathbb{B}}_n$ is similar to that of B_n; in particular, that (2.3) holds for all $k \geq 0$. Since the polynomials

$$\lim_{n\to\infty} p_k^{(n)}(x) := p_k^*(x),\ k \geq 0,$$

are completely described in [4], we get the same information about $\lim_{n\to\infty} q_k^{(n)}(x)$. Let $k \geq 2$ and $n \geq 1$. We want to determine $q_k^{(n)} \in \Pi_k$ such that

$$\bar{\mathbb{B}}_n q_k^{(n)} = \eta_k^{(n)} q_k^{(n)}. \tag{2.4}$$

We put

$$q_k^{(n)}(x) = \sum_{j=0}^{k} a(n,k,j) x^j, \quad \text{with} \quad a(n,k,k) = 1. \tag{2.5}$$

Hence

$$\bar{\mathbb{B}}_n(q_k^{(n)}; x) = \sum_{j=0}^{k} a(n,k,j) \bar{\mathbb{B}}_n(e_j; x).$$

With (2.1) we derive

$$\bar{\mathbb{B}}_n(q_k^{(n)};x) = \sum_{j=0}^{k} a(n,k,j)\frac{nx(nx+1)\dots(nx+j-1)}{n(n+1)\dots(n+j-1)} = \frac{n^k}{n(n+1)\dots(n+k-1)}\sum_{j=0}^{k} a(n,k,j)x^j. \tag{2.6}$$

From the definition of the Stirling numbers of first kind $s(j,i)$, we obtain immediately

$$nx(nx+1)\dots(nx+j-1) = \sum_{i=0}^{j} s(j,i)(-1)^{j-i}n^i x^i,$$

so that (2.6) becomes, after some manipulation,

$$\sum_{i=0}^{k}\left\{\sum_{j=i}^{k}\frac{s(j,i)(-1)^{j-i}n^i}{n(n+1)\dots(n+j-1)}a(n,k,j)\right\}x^i = \sum_{i=0}^{k}\frac{a(n,k,i)n^k}{n(n+1)\dots(n+k-1)}x^i.$$

This leads to

$$\sum_{j=i}^{k}\frac{s(j,i)(-1)^{j-i}}{n(n+1)\dots(n+j-1)}a(n,k,j) = \frac{n^{k-i}}{n(n+1)\dots(n+k-1)}a(n,k,i), \tag{2.7}$$

for all $i=0,1,\dots,k$. Since $s(i,i)=1$, we can solve (2.7) for $a(n,k,i)$ getting

$$a(n,k,i) = \frac{\sum_{j=i+1}^{k}(-1)^{j-i-1}s(j,i)(n+j)(n+j+1)\dots(n+k-1)a(n,k,j)}{(n+i)(n+i+1)\dots(n+k-1)-n^{k-i}}, \tag{2.8}$$

for all $i \in \{k-1,k-2,\dots,0\}$. Recalling that n and k are given, and $a(n,k,k)=1$, (2.8) represents a recurrence relation for computing $a(n,k,i)$, $i=k-1,k-2,\dots,0$. In particular, using $s(k,k-1) = -\frac{k(k-1)}{2}$, $s(k,k-2) = \frac{k(k-1)(k-2)(3k-1)}{24}$, we get

$$a(n,k,k-1) = -\frac{k}{2}, \tag{2.9}$$

$$a(n,k,k-2) = \frac{k(k-1)(k-2)}{24}\cdot\frac{6n+3k-5}{(2k-3)n+(k-1)(k-2)}. \tag{2.10}$$

Let us prove by induction that

$$a^*(k,j) := \lim_{n\to\infty} a(n,k,j) = \prod_{l=1}^{k-j}\frac{(k+1-l)(k-l)}{l(l-2k+1)}. \tag{2.11}$$

For $j=k$ (2.11) is verified because $a(n,k,k)=1$. Due to (2.9), (2.11) is verified also for $j=k-1$. Suppose now that (2.11) is true for $j=i+1$, and let's prove it for $j=i$. From (2.8) we infer

$$a(n,k,i) = \left\{(i+(i+1)+\dots+(k-1))n^{k-i-1} + \text{ terms of lower degree}\right\}^{-1} \times s(i+1,i)\left(n^{k-i-1} + \text{ terms of lower degree}\right)a(n,k,i+1),$$

so that, by the induction hypothesis,

$$\begin{aligned} a^*(k,i) &= \frac{s(i+1,i)}{i+(i+1)+\cdots+(k-1)} a^*(k,i+1) \\ &= -\frac{i(i+1)}{(k-i)(k+i-1)} \prod_{l=1}^{k-i-1} \frac{(k+1-l)(k-l)}{l(l-2k+1)} \\ &= \prod_{l=1}^{k-i} \frac{(k+1-l)(k-l)}{l(l-2k+1)}, \end{aligned}$$

and this completes the proof of (2.11).

It follows that

$$\lim_{n\to\infty} q_k^{(n)}(x) = \sum_{j=0}^{k} a^*(k,j)x^j,$$

and the coefficients $a^*(k,j)$ are equal to the coefficients $c^*(j,k)$ from [4, Theorem 4.1]. This leads to

$$\lim_{n\to\infty} q_k^{(n)}(x) = \lim_{n\to\infty} p_k^{(n)}(x) =: p_k^*(x),\ k \geq 0, \tag{2.12}$$

where (see [4, Theorem 4.5]) $p_0^*(x) = 1$, $p_1^*(x) = x - \frac{1}{2}$, and

$$p_k^*(x) = \frac{k!(k-2)!}{(2k-2)!} x(x-1) P_{k-2}^{(1,1)}(2x-1),\ k \geq 2. \tag{2.13}$$

($P_m^{(1,1)}$ are the Jacobi polynomials, orthogonal with respect to the weight $(1-t)(1+t)$ on the interval $[-1,1]$.)

Summarizing, we have proved the following

Theorem 2.1. (i) *The eigenvalues of* $\bar{\mathbb{B}}_n : \Pi \to \Pi$ *are the numbers given by (*2.2*).*

(ii) *The corresponding monic eigenpolynomials are described by (*2.5*), where the coefficients* $a(n,k,j)$ *satisfy the recurrence relation (*2.8*).*

(iii) *The eigenpolynomials satisfy the asymptotic relation (*2.12*).*

So the eigenstructure of the genuine Beta operators is similar to that of the classical Bernstein operators.

References

[1] Acar, T., Aral, A., Raşa, I., *Power series of Beta operators,* Appl. Math. Comput., **247**(2014), 815-823.

[2] Adell, J.A., German Badia, F., de la Cal, J., *Beta-type operators preserve shape properties,* Stochastic Processes and their Applications, **48**(1993), 1-8.

[3] Attalienti, A., Raşa, I., *Total positivity: an application to positive linear operators and to their limiting semigroups,* Rev. Anal. Numér. Théor. Approx., **36**(2007), 51-66.

[4] Cooper, Sh., Waldron, Sh., *The eigenstructure of the Bernstein operator,* J. Approx. Theory, **105**(2000), 133-165.

[5] Gonska, H., Heilmann, M., Lupaş, A., Raşa, I., *On the composition and decomposition of positive linear operators III: A non-trivial decomposition of the Bernstein operator,* arXiv: 1204.2723 (2012).

[6] Gonska, H., Raşa, I., Stănilă, E.-D., *Beta operators with Jacobi weights*, In: Constructive Theory of Functions, Sozopol, 2013 (K. Ivanov, G. Nikolov and R. Uluchev, Eds.), 99-112, "Prof. Marin Drinov" Academic Publishing House, Sofia, 2014.

[7] Gonska, H., Rusu, M., Stănilă, E.-D., *Inegalitaţi de tip Chebyshev-Grüss pentru operatorii Bernstein-Euler-Jacobi,* Gazeta Matematică, Seria A, **33**(62)(2015), no. 1-2, 16–28.

[8] Heilmann, M., Raşa, I., *On the decomposition of Bernstein operators,* Numerical Functional Analysis and Optimization, **36**(2015), no. 1, 72-85.

[9] Lupaş, A., *Die Folge der Betaoperatoren*, Dissertation, Universität Stuttgart, 1972.

[10] Mühlbach, G., *Verallgemeinerungen der Bernstein- und der Lagrangepolynome. Bemerkungen zu einer Klasse linearer Polynomoperatoren von D.D. Stancu*, Rev. Roumaine Math. Pures Appl., **15**(1970), 1235-1252.

[11] Mühlbach, G., *Rekursionsformeln für die zentralen Momente der Pólya- und der Beta-Verteilung,* Metrika, **19**(1972), 171-177.

[12] Stancu, D.D., *Approximation of functions by a new class of linear polynomial operators,* Rev. Roumaine Math. Pures Appl., **13**(1968), 1173-1194.

[13] Stancu, D.D., *On a new positive linear polynomial operator,* Proc. Japan Acad., **44**(1968), 221-224.

[14] Stancu, D.D., *Use of probabilistic methods in the theory of uniform approximation of continuous functions,* Rev. Roumaine Math. Pures Appl., **14**(1969), 673-691.

[15] Stănilă, E.D., *On Bernstein-Euler-Jacobi Operators*, Dissertation, Universität Duisburg-Essen, 2014.

[16] Weisstein, E.W., *Beta Function*, from MathWorld–A Wolfram Web Resource, http://mathworld.wolfram.com/BetaFunction.html (as seen on July 13, 2016).

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