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Oscillation criteria for third-order semi-canonical differential equations with unbounded neutral coefficients

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Abstract. In this paper, we investigate the oscillatory behavior of solutions to a class of third-order differential equations of the form

$$\mathcal{L}z(t) + f(t)y^{\beta}(\sigma(t)) = 0,$$

where $\mathcal{L}z(t) = (p(t)(q(t)z'(t))')'$ is a semi-canonical operator and $z(t) = y(t) + g(t)y(\tau(t))$. The main idea is to convert the semi-canonical operator into canonical form and then obtain some new sufficient conditions for the oscillation of all solutions. The obtained results essentially improve and complement to the known results. Examples are provided to illustrate the main results.

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1. Introduction

In this paper, we are concerned with the oscillation of solutions of the semi-canonical third-order neutral differential equation

$$\mathcal{L}z(t) + f(t)y^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $\mathcal{L}$ is the differential operator defined by

$$\mathcal{L}z(t) = (p(t)(q(t)z'(t))')', \quad z(t) = y(t) + g(t)y(\tau(t)),$$

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and β is the ratio of odd positive integers. Throughout the paper, and without further mention, we will always assume that:

(H_1) $f, g \in C([t_0, \infty), \mathbb{R})$, $g(t) \geq 1$, $g(t) \not\equiv 1$ for large t, and $f(t) \geq 0$ is not identically zero for large t,

(H_2) $\tau, \sigma \in C^1([t_0, \infty), \mathbb{R})$, $\tau(t) \leq t$, τ is strictly increasing, σ is nondecreasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(H_3) the operator $\mathcal{L}$ is in semi-canonical form, that is,

$$\int_{t_0}^{\infty} \frac{1}{p(t)} dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{q(t)} dt = \infty,$$

where $p, q \in C([t_0, \infty), (0, \infty))$.

By a *solution* of (1.1), we mean a function $y \in C([t_y, \infty), \mathbb{R})$ for some $t_y \geq t_0$ such that $z \in C^1([t_y, \infty), \mathbb{R})$, $qz' \in C^1([t_y, \infty), \mathbb{R})$, $p(qz')' \in C^1([t_y, \infty), \mathbb{R})$ and y satisfies (1.1) on $[t_y, \infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_y, \infty)$ and satisfy the condition

$$\sup\{|y(t)| : T_1 \leq t < \infty\} > 0 \quad \text{for any} \quad T_1 \geq t_y;$$

we tacitly assume that (1.1) possesses such solutions. Such a solution $y(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_y, \infty)$, and it is called *nonoscillatory* otherwise. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

In the recent years many papers appeared in the literature dealing with the oscillatory and asymptotic behavior of solutions of various classes of third-order neutral type differential equations; see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 17, 19, 20] and the references cited therein. However, except for the papers [5, 6, 12, 19, 20], all the papers mentioned above were dealing with the case when $g(t)$ is bounded, that is, the cases when $0 \leq g(t) \leq g_0 < 1$, $-1 < g_0 \leq g(t) \leq 0$ and $0 < g(t) \leq g_0 < \infty$ were studied and so the criteria obtained in these papers cannot be applied to the case $g(t) \to \infty$ as $t \to \infty$.

Moreover, very recently in [5, 6, 20] the authors studied equation (1.1) and obtained oscillation criteria where $q(t) \equiv 1$ and $p(t) \equiv 1$ or $\int_{t_0}^{\infty} \frac{1}{p(t)} dt = \infty$. Based on these observations, the aim of this paper is to obtain some oscillation criteria that can be applied not only to the case where $g(t) \to \infty$ as $t \to \infty$ but also to the cases when $g(t)$ is bounded, $\int_{t_0}^{\infty} \frac{1}{p(t)} dt < \infty$ and $\int_{t_0}^{\infty} \frac{1}{q(t)} dt = \infty$. The main idea is to connect the semi-canonical equation (1.1) with that of canonical equations and then we obtain oscillation criteria for (1.1).

In the sequel, we deal only with positive solutions of (1.1), since if $y(t)$ is a solution of (1.1), then $-y(t)$ is also a solution.

2. Main results

Throughout the paper we employ the following notations:

$$A(t) := \int_{t}^{\infty} \frac{1}{p(s)} ds, \quad a(t) := p(t)A^2(t), \quad b(t) := \frac{q(t)}{A(t)},$$

$$F(t) := A(t)f(t), \quad \Pi(t) := \int_{t_0}^{t} \frac{1}{a(s)} ds, \quad B(t) := \int_{t_0}^{t} \frac{\Pi(s)}{b(s)} ds,$$

$$c(t) := \exp\left(\int_{t_1}^{t} \frac{\Pi(s)}{b(s)B(s)} ds\right) \quad \text{for } t \geq t_1 \quad \text{for some } t_1 \geq t_0,$$

$$h(t) := \tau^{-1}(\sigma(t)), \quad \lambda(t) := \tau^{-1}(\eta(t)), \quad \eta \in C^1([t_0, \infty), \mathbb{R}),$$

$$\psi_1(t) := \frac{1}{g(\tau^{-1}(t))}\left[1 - \frac{c(\tau^{-1}(\tau^{-1}(t)))}{g(\tau^{-1}(\tau^{-1}(t)))c(\tau^{-1}(t))}\right],$$

$$\psi_2(t) := \frac{1}{g(\tau^{-1}(t))}\left[1 - \frac{1}{g(\tau^{-1}(\tau^{-1}(t)))}\right],$$

and

$$R(t) := \int_{h(t)}^{\lambda(t)} \left(\frac{1}{b(u)} \int_{u}^{\lambda(t)} \frac{1}{a(v)} dv\right) du.$$

In order to ensure the nonnegativity of $\psi_1(t)$, we assume the following condition also holds:

(H_4) There exists a $t_1 \in [t_0, \infty)$ such that

$$\frac{c(\tau^{-1}(\tau^{-1}(t)))}{g(\tau^{-1}(\tau^{-1}(t)))c(\tau^{-1}(t))} \leq 1 \quad \text{for all } t \geq t_1. \tag{2.1}$$

Theorem 2.1. *Assume that*

$$\int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty. \tag{2.2}$$

Then the semi-canonical operator $\mathcal{L}$ has the following unique canonical representation

$$\mathcal{L}z(t) = \frac{1}{A(t)}\left(p(t)A^2(t)\left(\frac{q(t)}{A(t)} z'(t)\right)'\right)'. \tag{2.3}$$

Proof. Direct calculation shows that

$$\begin{aligned}\left(p(t)A^2(t)\left(\frac{q(t)}{A(t)} z'(t)\right)'\right)' &= (A(t)p(t)(q(t)z'(t))' + q(t)z'(t))' \\ &= A(t)(p(t)(q(t)z'(t))')'.\end{aligned}$$

Therefore

$$\frac{1}{A(t)}\left(p(t)A^2(t)\left(\frac{q(t)}{A(t)} z'(t)\right)'\right)' = (p(t)(q(t)z'(t))')'.$$

Taking (2.2) into account, we see that

$$\int_{t_0}^{\infty} \frac{A(t)}{q(t)} dt = \infty,$$

and since

$$\int_{t_0}^{\infty} \frac{1}{p(t)A^2(t)} dt = \lim_{t \to \infty}\left(\frac{1}{A(t)} - \frac{1}{A(t_0)}\right) = \infty,$$

we say that (2.3) is in the canonical form. However, Trench proved in [18] that there exists only one canonical representation of $\mathcal{L}$ (up to multiplicative constants with product 1) and so our canonical form is unique. This completes the proof. □

From Theorem 2.1, it follows that (1.1) can be written in the canonical form as

$$(a(t)(b(t)z'(t))')' + F(t)y^{\beta}(\sigma(t)) = 0 \tag{2.4}$$

and the next result is immediate.

Theorem 2.2. *Assume that* (2.2) *holds. Then semi-canonical equation* (1.1) *possesses solution* $y(t)$ *if and only if canonical equation* (2.4) *has the solution* $y(t)$.

Corollary 2.3. *Assume that* (2.2) *holds. Then semi-canonical differential equation* (1.1) *has an eventually positive solution if and only if canonical equation* (2.4) *has an eventually positive solution.*

Corollary 2.3 clearly simplifies investigation of (1.1) since for (2.4) if $y(t)$ is an eventually positive solution, then the corresponding function $z(t)$ satisfies either

(I) $z(t) > 0$, $b(t)z'(t) > 0$, $a(t)(b(t)z'(t))' > 0$, $(a(t)(b(t)z'(t))')' < 0$, or

(II) $z(t) > 0$, $b(t)z'(t) < 0$, $a(t)(b(t)z'(t))' > 0$, $(a(t)(b(t)z'(t))')' < 0$

for sufficiently large t.

Lemma 2.4. *Assume that $z(t)$ satisfies case (I) for all $t \geq t_1$ for some $t_1 \geq t_0$. Then*

$$z'(t) \geq \frac{\Pi(t)}{b(t)} a(t)(b(t)z'(t))', \tag{2.5}$$

$$z(t) \geq B(t)a(t)(b(t)z'(t))', \tag{2.6}$$

$$z(t) \geq \frac{B(t)}{\Pi(t)} b(t)z'(t), \tag{2.7}$$

and

$$\frac{z(t)}{c(t)} \quad \textit{is nonincreasing} \tag{2.8}$$

for all $t \geq t_1$.

Proof. Since $a(t)(b(t)z'(t))'$ is positive and decreasing, we see that

$$b(t)z'(t) = b(t_1)z'(t_1) + \int_{t_1}^{t} a(s)\frac{(b(s)z'(s))'}{a(s)} ds$$

or

$$z'(t) \geq \frac{a(t)}{b(t)}(b(t)z'(t))'\Pi(t),$$

i.e., (2.5) holds. Integrating the last inequality from t_1 to t yields

$$z(t) \geq a(t)(b(t)z'(t))' \int_{t_1}^{t} \frac{\Pi(s)}{b(s)} ds = B(t)a(t)(b(t)z'(t))',$$

i.e., (2.6) holds. From (2.5), we see that $b(t)z'(t)/\Pi(t)$ is decreasing for $t \geq t_2$ for some $t_2 \geq t_1$, and therefore

$$z(t) = z(t_2) + \int_{t_2}^{t} \frac{b(s)z'(s)}{\Pi(s)} \frac{\Pi(s)}{b(s)} ds \geq \frac{B(t)}{\Pi(t)} b(t)z'(t).$$

From the last inequality, we see that

$$\left(\frac{z(t)}{c(t)}\right)' = \frac{\left(z'(t) - \frac{\Pi(t)}{b(t)B(t)}z(t)\right)}{c(t)} \leq 0$$

for $t \geq t_3$ for some $t_3 \geq t_2$. Hence, $z(t)/c(t)$ is non-increasing. This completes the proof. □

Theorem 2.5. *Let* (2.2) *holds. Assume that there exists a nondecreasing function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that* $\sigma(t) \leq \eta(t) < \tau(t)$ *for all* $t \geq t_0$. *If both first-order delay differential equations*

$$X'(t) + F(t)\Psi_1^\beta(\sigma(t))B^\beta(h(t))X^\beta(h(t)) = 0 \tag{2.9}$$

and

$$W'(t) + F(t)\Psi_2^\beta(\sigma(t))R^\beta(t)W^\beta(\lambda(t)) = 0 \tag{2.10}$$

oscillate, then (1.1) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. From Corollary 2.3, $y(t)$ is also a positive solution of (2.4) for $t \geq t_1$. Then the corresponding function $z(t)$ satisfies either case (I) or case (II) for $t \geq t_2$ for some $t_2 \geq t_1$.

First, we consider case (I). From the definition of z, we get

$$\begin{aligned} y(t) &= \frac{1}{g(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - y(\tau^{-1}(t))\right] \\ &\geq \frac{z(\tau^{-1}(t))}{g(\tau^{-1}(t))} - \frac{z(\tau^{-1}(\tau^{-1}(t)))}{g(\tau^{-1}(t))g(\tau^{-1}(\tau^{-1}(t)))}. \end{aligned} \tag{2.11}$$

Now $\tau(t) \leq t$ and τ is strictly increasing, so τ^{-1} is increasing and $t \leq \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t)).$$

From this and the fact that $z(t)/c(t)$ is nonincreasing, we see that

$$z\left(\tau^{-1}(\tau^{-1}(t))\right) \leq \frac{c(\tau^{-1}(\tau^{-1}(t)))z(\tau^{-1}(t))}{c(\tau^{-1}(t))}. \tag{2.12}$$

Using (2.12) in (2.11) yields

$$y(t) \geq \psi_1(t)z(\tau^{-1}(t)). \tag{2.13}$$

Since $\lim_{t\to\infty}\sigma(t) = \infty$, we can choose $t_3 \geq t_2$ such that $\sigma(t) \geq t_2$ for all $t \geq t_3$. Thus, it follows from (2.13) that

$$y(\sigma(t)) \geq \psi_1(\sigma(t))z(h(t)) \quad \text{for } t \geq t_3. \tag{2.14}$$

Combining (2.14) with (2.4) yields

$$(a(t)(b(t)z'(t))')' + F(t)\psi_1^\beta(\sigma(t))z^\beta(h(t)) \leq 0 \quad \text{for } t \geq t_3. \tag{2.15}$$

From (2.6), we have

$$z(h(t)) \geq B(h(t))a(h(t))(b(h(t))z'(h(t)))'. \tag{2.16}$$

Using (2.16) in (2.15) and letting $X(t) = a(t)(b(t)z'(t))'$, we see that $X(t)$ is a positive solution of the first-order delay differential inequality

$$X'(t) + F(t)\psi_1^\beta(\sigma(t))B^\beta(h(t))X^\beta(h(t)) \leq 0. \tag{2.17}$$

Therefore, by Corollary 1 of [14], we conclude that (2.9) also has a positive solution, which is a contradiction.

Next, we consider case (II). Since z is strictly decreasing and $\tau(t) \leq t$, we have

$$z(\tau^{-1}(t)) \geq z(\tau^{-1}(\tau^{-1}(t)))$$

and using this in (2.11), we obtain

$$y(t) \geq \psi_2(t)z(\tau^{-1}(t)).$$

Hence,

$$y(\sigma(t)) \geq \psi_2(\sigma(t))z(h(t)) \tag{2.18}$$

for $t \geq t_3$ for some $t_3 \geq t_2$. Using (2.18) in (2.4) yields

$$(a(t)(b(t)z'(t))')' + F(t)\psi_2^\beta(\sigma(t))z^\beta(h(t)) \leq 0 \quad \text{for } t \geq t_3. \tag{2.19}$$

For $t \geq s \geq t_3$, we have

$$b(t)z'(t) - b(s)z'(s) = \int_s^t \frac{a(u)(b(u)z'(u))'}{a(u)}du,$$

or

$$-z'(s) \geq \left(\frac{1}{b(s)}\int_s^t \frac{1}{a(u)}du\right) a(t)(b(t)z'(t))'.$$

Again integrating, we have

$$-z(t) + z(s) \geq \left(\int_s^t \frac{1}{b(u)}\left(\int_u^t \frac{1}{a(v)}dv\right)du\right) a(t)(b(t)z'(t))',$$

or

$$z(s) \geq \left[\int_s^t \frac{1}{b(u)}\left(\int_u^t \frac{1}{a(v)}dv\right)du\right] a(t)(b(t)z'(t))'. \tag{2.20}$$

Since $\sigma(t) \leq \eta(t)$ and the fact that τ is strictly increasing, we have

$$\tau^{-1}(\sigma(t)) \leq \tau^{-1}(\eta(t)).$$

Setting $s = \tau^{-1}(\sigma(t))$ and $t = \tau^{-1}(\eta(t))$ into (2.20), we obtain

$$z(h(t)) \geq \left(\int_{h(t)}^{\lambda(t)} \frac{1}{b(u)}\left(\int_u^{\lambda(t)} \frac{1}{a(v)}dv\right)du\right) a(\lambda(t))(b(\lambda(t))z'(\lambda(t)))'. \tag{2.21}$$

Using (2.21) in (2.19) and letting $W(t) = a(t)(b(t)z'(t))'$, we see that W is a positive solution of the first-order delay differential inequality

$$W'(t) + F(t)\psi_2^\beta(\sigma(t))R^\beta(t)W^\beta(\lambda(t)) \leq 0. \tag{2.22}$$

The remaining part of the proof is similar to the case (I) and hence the details are not repeated. This completes the proof. □

Corollary 2.6. *Let* (2.2) *holds and* $\beta = 1$. *Assume that there exists a nondecreasing function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that* $\sigma(t) \le \eta(t) < \tau(t)$ *for all* $t \ge t_0$. *If*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} F(s)\psi_1(\sigma(s))B(h(s))ds > \frac{1}{e} \tag{2.23}$$

and

$$\liminf_{t\to\infty} \int_{\lambda(t)}^{t} F(s)\psi_2(\sigma(s))R(s)ds > \frac{1}{e}, \tag{2.24}$$

then (1.1) *is oscillatory.*

Proof. The proof follows from a well-known result in [11] and Theorem 2.5, and hence the details are omitted. □

Corollary 2.7. *Let* (2.2) *holds and* $0 < \beta < 1$. *Assume that there exists a nondecreasing function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that* $\sigma(t) \le \eta(t) < \tau(t)$ *for all* $t \ge t_0$. *If*

$$\int_{T}^{\infty} F(t)\psi_1^{\beta}(\sigma(t))B^{\beta}(h(t))dt = \infty \tag{2.25}$$

and

$$\int_{T}^{\infty} F(t)\psi_2^{\beta}(\sigma(t))R^{\beta}(t)dt = \infty \tag{2.26}$$

for all sufficiently large $T \in [t_0, \infty)$ *with* $\sigma(t) \ge t_0$ *for all* $t \ge T$, *then* (1.1) *oscillates.*

Proof. Proceeding exactly as in the proof of Theorem 2.5, we again arrive at (2.17) and (2.22) for $t \ge t_3$. Since $h(t) < t$ and $X(t)$ is positive and decreasing, inequality (2.17) takes the form

$$X'(t) + F(t)\psi_1^{\beta}(\sigma(t))B^{\beta}(h(t))X^{\beta}(t) \le 0,$$

or

$$\frac{X'(t)}{X^{\beta}(t)} + F(t)\psi_1^{\beta}(\sigma(t))B^{\beta}(h(t)) \le 0. \tag{2.27}$$

Integrating (2.27) from t_3 to t yields

$$\int_{t_3}^{t} F(s)\psi_1^{\beta}(\sigma(s))B^{\beta}(h(s))ds \le \frac{X^{1-\beta}(t_3)}{1-\beta} < \infty \quad \text{as } t \to \infty,$$

which contradicts (2.25). The remainder of the proof follows from $\lambda(t) < t$ and inequality (2.22). The proof is complete. □

In our final result, assume that $\sigma(t) = t - \delta_1$, $\tau(t) = t - \delta_3$ and $\eta(t) = t - \delta_2$, where δ_1, δ_2 and δ_3 are positive real numbers.

Corollary 2.8. *Let* (2.2) *holds and* $\beta > 1$. *If* $\delta_1 \ge \delta_2 > \delta_3$,

$$\liminf_{t\to\infty} \beta^{-t/(\delta_1-\delta_3)} \log\Big(F(t)\psi_1^{\beta}(t-\delta_1)B^{\beta}(t+\delta_3-\delta_1)\Big) > 0 \tag{2.28}$$

and

$$\liminf_{t\to\infty} \beta^{-t/(\delta_2-\delta_3)} \log\Big(F(t)\psi_2^{\beta}(t-\delta_1)R^{\beta}(t)\Big) > 0, \tag{2.29}$$

then (1.1) *oscillates.*

Proof. Application of (2.28) and (2.29) and Corollary 1.2 of [15] imply that (2.9) and (2.10) oscillate. Hence, by Theorem 2.5, equation (1.1) oscillates. □

3. Examples

In this section, we present some examples to show the importance of the main results.

Example 3.1. Consider the third-order linear neutral differential equation

$$\left(t^2\left(\frac{1}{t}\left(y(t)+16y\left(\frac{t}{2}\right)\right)'\right)'\right)' + \frac{f_0}{t^2}y\left(\frac{t}{4}\right) = 0, \quad t \geq 1. \tag{3.1}$$

Here $p(t) = t^2$, $q(t) = 1/t$, $g(t) = 16$, $f(t) = f_0/t^2$ with $f_0 > 0$, $\tau(t) = t/2$, $\sigma(t) = t/4$ and $\beta = 1$. Then $A(t) = 1/t$, $a(t) = 1$, $b(t) = 1$, $F(t) = f_0/t^3$ and the transformed equation is

$$\left(y(t)+16y\left(\frac{t}{2}\right)\right)''' + \frac{f_0}{t^3}y\left(\frac{t}{4}\right) = 0, \quad t \geq 1, \tag{3.2}$$

which is in canonical form. Simple calculation show that

$$\Pi(t) = t-1, \; B(t) = (t-1)^2/2, \; c(t) = (t-1)^2, \text{ and } \psi_2(t) = 15/256.$$

Since (2.1) holds, we have $\psi_1(t) \geq 0$ and

$$\psi_1(t) = \frac{1}{16}\left[1 - \frac{(4t-1)^2}{16(2t-1)^2}\right] \geq \frac{7}{256}.$$

By choosing $\eta(t) = t/3$, we see that $h(t) = t/2$, $\lambda(t) = 2t/3$ and $R(t) = t^2/72$. It is clear that condition (2.2) holds. Condition (2.23) becomes

$$\liminf_{t\to\infty} \int_{t/2}^{t} \frac{f_0}{2^9}\left(\frac{3}{s} - \frac{14}{s^2} + \frac{15}{s^3}\right) ds = \frac{3 f_0 \ln 2}{2^9},$$

and so condition (2.22) is satisfied if $f_0 > \dfrac{2^9}{3e\ln 2}$.

Condition (2.24) becomes

$$\liminf_{t\to\infty} \int_{2t/3}^{t} \frac{5 f_0}{3\times 2^{11}}\frac{1}{s} ds = \frac{5 f_0 \ln 3/2}{3\times 2^{11}},$$

that is, (2.24) is satisfied if $f_0 > \dfrac{3\times 2^{11}}{5e\ln 3/2}$. Thus, by Corollary 2.6, equation (3.1) is oscillatory if $f_0 > \dfrac{3\times 2^{11}}{5e\ln 3/2}$.

Note that canonical equation (3.2) is considered in [20] and proved that (3.2) is oscillatory if $f_0 > \frac{3\times 2^{11}}{5\ln 3/2}$. Hence, Corollary 2.6 improves Theorem 2.7 of [20].

Example 3.2. Consider the third-order sublinear neutral differential equation

$$\left(t^2\left(\frac{1}{t}\left(y(t)+ty\left(\frac{t}{2}\right)\right)'\right)'\right)'+\frac{f_0}{t^\alpha}y^{3/5}\left(\frac{t}{10}\right)=0,\quad t\geq 16. \tag{3.3}$$

Here $p(t)=t^2$, $q(t)=1/t$, $g(t)=t$, $f(t)=f_0/t^\alpha$ with $f_0>0$ and $\alpha\leq 3/5$, $\tau(t)=t/2$, $\sigma(t)=t/10$ and $\beta=3/5$. Then $A(t)=1/t$, $a(t)=1$, $b(t)=1$, $F(t)=f_0/t^{\alpha+1}$ and the transformed equation is

$$\left(y(t)+ty\left(\frac{t}{2}\right)\right)'''+\frac{f_0}{t^{\alpha+1}}y^{3/5}\left(\frac{t}{10}\right)=0,\quad t\geq 16, \tag{3.4}$$

which is in canonical form. Simple calculation shows that

$$\Pi(t)=t-16,\ B(t)=(t-16)^2/2,\ c(t)=(t-16)^2,\ \text{and}\ \psi_2(t)=\frac{4t-1}{8t^2}>0.$$

Since (2.1) holds, we have $\psi_1(t)\geq 0$ and $\psi_1(t)\geq\dfrac{4t-9}{8t^2}$. By choosing $\eta(t)=t/8$, we see that $h(t)=t/5$, $\lambda(t)=t/4$ and $R(t)=t^2/800$. It is clear that condition (2.2) holds. For any $T\geq t_0$ with $\sigma(t)\geq t_0$, condition (2.25) becomes

$$\int_T^\infty\frac{f_0}{t^{\alpha+1}}\left(\frac{10t-225}{2t^2}\right)^{3/5}\left(\frac{t-80}{\sqrt{50}}\right)^{6/5}dt\geq d_1\int_{T_1}^\infty\frac{1}{t^{\alpha+2/5}}dt=\infty,$$

where $d_1>0$ is a constant and $T_1\geq T$.

Condition (2.26) becomes

$$\int_T^\infty\frac{f_0}{t^{\alpha+1}}\left(\frac{10t-25}{2t^2}\right)^{3/5}\left(\frac{t^2}{800}\right)^{3/5}dt\geq d_2\int_{T_1}^\infty\frac{1}{t^{\alpha+2/5}}=\infty,$$

where $d_2>0$ is a constant and $T_1\geq T$. Thus, by Corollary 2.7, equation (3.3) is oscillatory if $\alpha\leq 3/5$.

Note that canonical equation (3.4) is considered in [20] and proved that (3.4) is oscillatory if $\alpha=\frac{1}{5}$. Hence, Corollary 2.7 improves Theorem 2.8 of [20].

Example 3.3. Consider the third-order superlinear neutral differential equation

$$\left(t^2\left(\frac{1}{t}\left(y(t)+ty\left(t-2\right)\right)'\right)'\right)'+t\exp(4^t)y^3(t-4)=0,\quad t\geq 2. \tag{3.5}$$

Here $p(t)=t^2$, $q(t)=1/t$, $g(t)=t$, $f(t)=t\exp(4^t)$, $\tau(t)=t-2$, $\sigma(t)=t-4$ and $\beta=3$. Then $A(t)=1/t$, $a(t)=1$, $b(t)=1$, $F(t)=\exp(4^t)$ and the transformed equation is

$$(y(t)+ty(t-2))'''+\exp(4^t)y^3(t-4)=0, \tag{3.6}$$

which is in canonical form. A simple calculation show that

$$\Pi(t)=t-2,\ B(t)=(t-2)^2/2,\ c(t)=(t-2)^2,$$

$$\psi_1(t)=\frac{1}{t+2}\left[1-\frac{(t+2)^2}{(t+4)t^2}\right]\geq\frac{t}{(t+2)(t+4)}\geq 0\ \text{and}\ \psi_2(t)=\frac{t+3}{(t+2)(t+4)}\geq 0.$$

By choosing $\eta(t) = t - 3$, we see that $h(t) = t - 2$, $\lambda(t) = t - 1$, $R(t) = 1/2$, $\delta_1 = 4$, $\delta_2 = 3$, $\delta_3 = 2$. As in Examples 3.1 and 3.2, it is easy to see that conditions (2.2), (2.28) and (2.29) are satisfied. Thus, by Corollary 2.8, equation (3.5) is oscillatory.

4. Conclusion

In this paper, we have established some new oscillation criteria for (1.1). The results are obtained by converting (1.1) into canonical type equation. Hence, the results are new and complement to those in [5, 6, 12, 20]. Also we have shown that the results obtained here improve those in [20].

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