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On some classes of Fleming-Viot type differential operators on the unit interval

Francesco Altomare

Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. Of concern are some classes of initial-boundary value differential problems associated with one-dimensional Fleming-Viot differential operators. Among other things, these operators occur in some models from population genetics to study the fluctuation of gene frequency under the influence of mutation and selection. The main aim of this survey paper is to discuss old and more recent results about the existence, uniqueness and continuous dependence from initial data of the solutions to these problems through the theory of the C_0-semigroups of operators. Other additional aspects which will be highlighted, concern the approximation of the relevant semigroups in terms of positive linear operators. The given approximation formulae allow to infer several preservation properties of the semigroups together with their asymptotic behavior. The analysis is carried out in the context of the space $C([0,1])$ as well as, in some particular cases, in $L^p([0,1])$ spaces, $1 \leq p < +\infty$. Finally, some open problems are also discussed.

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1. Introduction

In the present paper we shall discuss initial-boundary value differential problems associated with differential operators of the form

$$A(u)(x) := \frac{\alpha(x)}{2}u''(x) + (p(1-x) - qx)u'(x) \quad (0 < x < 1) \tag{1.1}$$

acting on suitable subspaces of $C^2_*([0,1])$, the linear space of all real-valued continuous functions on $[0,1]$ which are twice continuously differentiable on $]0,1[$.
Here, $\alpha \in C([0,1]), 0 \leq \alpha(x)$ for every $x \in [0,1]$, $p \geq 0$ and $q \geq 0$.

The differential operators (1.1) are referred to as the one-dimensional Fleming-Viot operators and they occur in some models from population genetics to study the fluctuation of gene frequency under the influence of mutation and selection ([15]).

Setting

$$a := p+q \text{ and } b := \begin{cases} 1 & \text{if } p = q = 0, \\ p/(p+q) & \text{if } p+q > 0, \end{cases}$$

the operator (1.1) turns into the operator

$$A(u)(x) := \frac{\alpha(x)}{2}u''(x) + a(b-x)u'(x) \quad (0 < x < 1) \tag{1.2}$$

with $a \geq 0$ and $0 \leq b \leq 1$, which, to our purposes, is more convenient to handle.

We begin to state the first main problem we shall deal with.

Problem 1.1. Determine a linear subspace $D(A)$ of $C^2_*([0,1])$ such that

(i) For every $u \in D(A)$, $A(u)$ continuously extends to $[0,1]$.
(ii) The operator $A : D(A) \to C([0,1])$ generates a strongly continuous Markov semigroup $(T(t))_{t\geq 0}$ on $C([0,1])$.

For some details concerning the theory of strongly continuous (Markov) semigroup and for unexplained terminology the reader is referred, e.g., to [6, Chapter 2]. If $(A, D(A))$ generates a strongly continuous semigroup, then, given $u_0 \in C([0,1])$, the following Cauchy problem is well-posed

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \alpha(x)\frac{\partial^2 u(x,t)}{\partial x^2} + a(b-x)\frac{\partial u(x,t)}{\partial x} & 0 < x < 1, t \geq 0, \\ u(\cdot, t) \in D(A) & t \geq 0, \\ \lim\limits_{t\to 0^+} u(x,t) = u_0(x) & \text{uniformly w.r. to } 0 \leq x \leq 1, \end{cases} \tag{1.3}$$

if and only if $u_0 \in D(A)$.

Moreover, the unique solution to (1.3) is given by

$$u(x,t) = T(t)u_0(x) \quad (0 \leq x \leq 1,\ t \geq 0). \tag{1.4}$$

and it continuously depends on the initial datum $u_{0.}$
The subspace $D(A)$ (if any) is referred to as a well-posed domain for A.

Note also that (1.3) is, indeed, an initial-boundary value problem since the boundary conditions are usually included in the definition of $D(A)$.

The partial differential equation which appears in (1.3) is the so-called *backward equation* of a normal Markov process

$$(\Omega, \mathcal{U}, (P^x)_{x\in[0,1]}, (Z_t)_{t\geq 0})$$

having $[0,1]$ as state space, with mean instantaneous velocity $a(b-x)$ and variance instantaneous velocity $\alpha(x)$ at the position $x \in [0,1]$ (see, e.g., [6, Section 2.3.2]

Having determined $D(A)$, we shall discuss the next subsequent problem:

Problem 1.2. Introduce (if any) a sequence of positive linear operators $(L_n)_{n\geq 1}$ on $C([0,1])$ such that for every $t \geq 0$, for some sequence $(k(n))_{n\geq 1}$ of positive integers and for every $f \in C([0,1])$,

$$T(t)f = \lim_{n\to\infty} L_n^{k(n)} f \quad \text{uniformly on } [0,1]. \tag{1.5}$$

In such a case, we say that the sequence $(L_n)_{n\geq 1}$ is an admissible sequence for the semigroup $(T(t))_{t\geq 0}$.

In principle, from formula (1.5) it is possible to infer some preservation properties of the semigroup which have their counterparts in terms of regularity properties (with respect to the spatial variable $x \in [0,1]$) of the solutions

$$u(x,t) = T(t)u_0(x)$$

to the initial-boundary value problems (1.3).

Moreover, estimates of the quantities $\|T(t)f - L_n^{k(n)} f\|$ could give numerical approximations of the solutions themselves.

According to a theorem of H. F. Trotter ([6, Corollary 2.2.3]), a natural way to get the approximation formula (1.5), is to show that

(i) $\|L_n^k\| \leq M \exp(\omega nk)$ for some $M \geq 1$ and $\omega \in \boldsymbol{R}$, and for every $n, k \geq 1$,
and, in addition, to determine (if any) a linear subspace D_0 of $D(A)$ such that

(ii) D_0 is a core for $D(A)$, i.e., D_0 is dense in $D(A)$ for the graph norm

$$\|u\|_A := \|u\| + \|A(u)\| \ (u \in D(A)),$$

and

(iii) For every $u \in D_0$,

$$\lim_{n\to\infty} n(L_n(u) - u) = A(u) \text{ uniformly on } [0,1].$$

In such a case, formula (1.5) holds true for every $t \geq 0$, for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$ and for every $f \in C([0,1])$.
Moreover, $\|T(t)\| \leq M \exp(\omega t)$ for every $t \geq 0$.

When conditions (i)–(iii) are satisfied, we say that the sequence $(L_n)_{n\geq 1}$ is a strong admissible sequence for the semigroup $(T(t))_{t\geq 0}$.

In the subsequent section we shall survey some old and more recent results about these two problems. However, we point out that for the case $\alpha(x) = x(1-x)$ ($x \in [0,1]$), rather satisfactory results have been obtained (see, e.g., [11], [12], [13], [16] and the references therein). For the general case, parts of the results we are discussing in the present paper are taken from [8].

We also point out that in the paper [8] as well as in the monograph [6], similar problems have been treated for general convex compact subsets K of $\boldsymbol{R}^d, d \geq 1$, having non-empty interior.

In these contexts the differential operators are of the form

$$A(u)(x) := \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} a(b_i - x_i)\frac{\partial u}{\partial x_i}(x). \tag{1.6}$$

($u \in C^2(K)$, $x \in K$).

However, in the framework of the unit interval more complete results can be shown. For additional results concerning Fleming-Viot type differential operators we refer, e.g., to [1], [14] and the references therein.

2. Generation results and approximation

On account of the Feller theory developed in the 1950s (see, e.g., [6, Section 2.3.3]), we shall describe four groups of boundary conditions which allow to determine well-posed domains for A.

From now on we shall assume that

(i) $0 < \alpha(x)$ for each $0 < x < 1$ and $\alpha(0) = \alpha(1) = 0$.
(ii) α is differentiable at 0 and at 1 and $\alpha'(0) \neq 0 \neq \alpha'(1)$;

From conditions (i) and (ii) it follows that

$$0 < \alpha(x) \leq Mx(1-x) \text{ for each } 0 < x < 1 \text{ and for some } M > 0.$$

There is no loss of generality in assuming that $M = 1$ because, if $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ denote the Markov semigroups generated by the differential operators associated with α, a and b, and $\frac{\alpha}{M}$, $\frac{a}{M}$ and b respectively, then

$$T(t) = S(Mt) \quad \text{ for every } t \geq 0.$$

Thus, from now on we shall assume that

(iii) $0 < \alpha(x) \leq x(1-x)$ for each $0 < x < 1$.

The special case $\alpha(x) = x(1-x)$ for every $x \in [0,1]$, will be referred to as the maximal case.

Finally, we also assume that

(iv) the function

$$r(x) := \begin{cases} \dfrac{ab}{2\alpha'(0)} & \text{if } x = 0, \\ \dfrac{a(b-x)x(1-x)}{2\alpha(x)} & \text{if } 0 < x < 1, \\ \dfrac{a(1-b)}{2\alpha'(1)} & \text{if } x = 1, \end{cases} \tag{2.1}$$

is Hölder continuous at 0 and at 1.

Condition (iv) is satisfied, for instance, if α is differentiable in [0, 1]. Moreover, note also that $\alpha'(0) \leq 1$ and $-1 \leq \alpha'(1)$.

It is also useful to consider the function

$$\lambda(x) := \frac{a(b-x)}{r(x)} = \begin{cases} 2\alpha'(0) & \text{if } x = 0, \\ \dfrac{x(1-x)}{2\alpha(x)} & \text{if } 0 < x < 1, \\ -2\alpha'(1) & \text{if } x = 1. \end{cases} \tag{2.2}$$

Then, $A = \lambda B$, where B denotes the differential operator

$$B(u)(x) := \frac{x(1-x)}{2} u''(x) + r(x)u'(x) \qquad (0 < x < 1) \tag{2.3}$$

$(u \in C^2_*([0,1]))$.

Thus, on account of a well-known multiplicative perturbation generation result (see, e.g., [6, Theorem 2.3.11]) the generation problems for A can be solved by studying similar ones for B.

2.1. The case $a = 0$ and the case $0 < ab < \alpha'(0)/2$ and $0 < a(1-b) < -\alpha'(1)/2$

In these cases a well-posed domain for A is the so-called Ventcel' domain of A. For a proof of the next result it is enough to combine [6, Theorem 5.7.2] and [13, pp. 120-121, item (2)], taking the formula $A = \lambda B$ into account.

Theorem 2.1. *If $a = 0$ or if $0 < ab < \alpha'(0)/2$ and $0 < a(1-b) < -\alpha'(1)/2$, then a well-posed domain for A is*

$$D_V(A) := \left\{ u \in C^2_*([0,1]) \mid \lim_{x \to 0^+} A(u)(x) = 0 = \lim_{x \to 1^-} A(u)(x) = 0 \right\}, \tag{2.4}$$

The capital letter V refers to the mathematician Ventcel' who extended the Feller work to multidimensional settings. Moreover, the Ventcel' conditions, i.e., the boundary conditions incorporated in $D_V(A)$, imply that, once the Markov process reaches 0 or 1, then it stops there for ever ([6, Subsection 2.3.3].

As regards the construction of a strong admissible sequence for the semigroup, we are able to provide a solution for the case $a = 0$ only and we leave as an open problem the second subcase $0 < ab < \alpha'(0)/2$ *and* $0 < a(1-b) < -\alpha'(1)/2$. However, at least in the maximal case $\alpha(x) = x(1-x)$ $(x \in [0,1])$, it is possible to describe the asymptotic behaviour of the semigroup also for the second subcase (for the case $a = 0$ see the subsequent results).

We have indeed (see [11, Theorem 4.2]) that for every $f \in C([0,1])$

$$\lim_{t \to +\infty} T(t)f = f(0)(1-\varphi) + f(1)\varphi \quad \textit{uniformly on } [0,1],$$

where, for every $x \in [0,1]$,

$$\varphi(x) := \frac{\int_0^x t^{-2ab}(1-t)^{-2a(1-b)}dt}{\int_0^1 t^{-2ab}(1-t)^{-2a(1-b)}dt}.$$

In particular,

$$\lim_{t \to \infty} T(t)(f) = 0 \quad \text{uniformly on } [0,1]$$

if and only if $f(0) = f(1) = 0$.

We proceed now to study the case $a = 0$. According to [6, Remark 4.5.5], consider a Markov operator T on $C([0,1])$ such that $T(e_1) = e_1$ and

$$\alpha = T(e_2) - e_2,$$

where $e_1(x) = x$ and $e_2(x) = x^2$ $(x \in [0,1])$.

By appealing to the Riesz representation theorem, consider the family $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ such that

$$T(f)(x) := \int_0^1 f d\mu_x, \quad (0 \leq x \leq 1 \textit{ and } f \in C([0,1])). \tag{2.5}$$

Definition 2.2. *For every $n \geq 1$, the n-th Bernstein-Schnabl operator associated with T is the positive linear operator $B_n : C([0,1]) \longrightarrow C([0,1])$ defined for every $f \in C([0,1])$ and $x \in [0,1]$ as*

$$B_n(f)(x) := \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\mu_x(x_1)\cdots d\mu_x(x_n). \tag{2.6}$$

For a detailed analysis on these operators and for a proof of the results below we refer to the monographs [3, Chapter 6] and [6, Chapter 3] and the references therein.

Theorem 2.3. *The sequence $(B_n)_{n\geq 1}$ of Bernstein-Schnabl operators associated with T is a strong admissible sequence for the semigroup generated by the operator $(A, D_V(A))$ for the case $a = 0$, i.e.,*

$$A(u)(x) := \frac{\alpha(x)}{2} u''(x) \qquad (0 < x < 1)$$

and

$$D_V(A) := \left\{ u \in C_*^2([0,1]) \;\mid\; \lim_{x\to 0^+} A(u)(x) = 0 = \lim_{x\to 1^-} A(u)(x) = 0 \right\}.$$

Moreover, $C^2([0,1])$ is a core for $D_V(A)$.

From the theorem above it is possible to infer some preservation properties of the semigroup which have their counterparts in terms of regularity properties (with respect to the spatial variable $x \in [0,1]$) of the solutions

$$u(x,t) = T(t)u_0(x)$$

to the relevant initial-boundary value problems.

For given $M > 0$ and $0 < \sigma \leq 1$ we set

$$\begin{aligned} &Lip(M,\sigma) := \\ &\{f \in C([0,1]) \mid\mid f(x) - f(y) \mid\leq M \mid x - y \mid^\sigma \textit{ for every } x, y \in [0,1]\}. \end{aligned} \tag{2.7}$$

Corollary 2.4. *The following statements hold true:*

(1) *$T(t)f = f$ on 0 and 1 for every $f \in C([0,1])$.*
(2) *If the operator T maps continuous increasing functions into (continuous) increasing functions, then each $T(t)$ maps continuous increasing functions into increasing functions.*
(3) *If $T(Lip(1,1)) \subset Lip(1,1)$, then for every $M > 0$, $0 < \sigma \leq 1$ and $t \geq 0$,*

$$T(t)(Lip(M,\sigma)) \subset Lip(M,\sigma).$$

(4) *If $f \in C([0,1])$, the following statements are equivalent:*
   (i) *f is convex;*
   (ii) *$B_{n+1}(f) \leq B_n(f)$ for every $n \geq 1$;*
   (iii) *$f \leq B_n(f)$ for every $n \geq 1$;*

(iv) $f \leq T(t)f$ *for every* $t \geq 0$.

(5) *For every* $f \in C([0,1])$

$$\lim_{t\to\infty} T(t)(f) = (1-e_1)f(0) + e_1 f(1)$$

uniformly on $[0,1]$ *and hence*

$$\lim_{t\to\infty} T(t)(f) = 0 \qquad \textit{uniformly on } [0,1]$$

if and only if $f(0) = f(1) = 0$.

In order to show the behaviour of the semigroup $(T(t))_{t\geq 0}$ on convex functions, for every $f \in C([0,1])$ and $x, y \in [0,1]$, consider

$$\Delta(f;x,y) := B_2(f)(x) + B_2(f)(y) - 2\iint_{[0,1]^2} f\left(\frac{s+t}{2}\right) d\mu_x(s)d\mu_y(t),$$

where the operators B_2 is the Bernstein-Schnabl operator of order 2.

Theorem 2.5. *Suppose that*

(i) T *maps continuous convex functions into (continuous) convex functions;*

(ii) $\Delta(f;x,y) \geq 0$ *for every convex function* $f \in C([0,1])$ *and for every* $x, y \in [0,1]$.

If $f \in C([0,1])$ *is convex, then* $T(t)f$ *is convex for every* $t \geq 0$ *and* $(T(t)f)_{t\geq 0}$ *is increasing.*

For additional results concerning Bernstein-Schnabl operators we also refer to [2].

2.2. The case $ab \geq \alpha'(0)/2$ and $a(1-b) \geq -\alpha'(1)/2$

For all the results shown in this subsection the reader is referred to [8, Sections 3 and 4]

Theorem 2.6. *If* $ab \geq \alpha'0)/2$ *and* $a(1-b) \geq -\alpha'(1)/2$, *then a well-posed domain for* A *is*

$$D_M(A) := \left\{u \in C^2_*([0,1]) \mid \lim_{x\to 0^+} A(u)(x) \in \mathbf{R} \text{ and } \lim_{x\to 1^-} A(u)(x) \in \mathbf{R}\right\}.$$

The domain $D_M(A)$ is also referred to as the maximal domain for A. Moreover, the maximal boundary conditions incorporated in the domain $D_M(A)$, imply that the probability that the Markov process reaches 0 or 1 in a finite time is zero ([6, Subsection 2.3.3].

As regards the construction of a strong admissible sequence for the semigroup, consider again a Markov operator T on $C([0,1])$ such that $T(e_1) = e_1$ and $\alpha = T(e_2) - e_2$, along with the family $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ representing T (see (2.5)). Finally let μ be a probability Borel measure on $[0,1]$.

Then, for every $n \geq 1$, consider the positive linear operator C_n defined by setting

$$C_n(f)(x) = \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \ldots + x_n + ax_{n+1}}{n+a}\right) d\mu_x(x_1)\cdots d\mu_x(x_n)d\mu(x_{n+1}) \tag{2.8}$$

for every $x \in [0,1]$ and for every $f \in C([0,1])$.

The germ of the idea of the above definition goes back to [9] (see also [10]). Subsequently, in [4] (see also [5]) the authors considered a natural generalization to multidimensional settings such as hypercubes and simplices, obtaining, as a particular case, the multidimensional Kantorovich operators on these frameworks.

The general definition (introduced in the context of general convex compact subsets) has been set in the recent paper [7], obtaining a new class of positive linear operators which encompasses not only several well-known approximation processes both in univariate and multivariate settings, but also new ones in finite and infinite dimensional frameworks as well.

Clearly, in the special case $a = 0$, the operators C_n correspond to the B_n ones. Moreover, introducing the auxiliary continuous function

$$I_n(f)(x) := \int_0^1 f\left(\frac{n}{n+a}\,x + \frac{a}{n+a}\,t\right) d\mu(t)$$

($f \in C([0,1])$, $x \in [0,1]$, $n \geq 1$), then

$$C_n(f) = B_n(I_n(f)).$$

Therefore $C_n(f) \in C([0,1])$ and the operator $C_n : C([0,1]) \to C([0,1])$, being linear and positive, is continuous with norm equal to 1, because $C_n(\mathbf{1}) = \mathbf{1}$.

We proceed to show some specific examples.

Example 2.7. Consider the maximal case $\alpha(x) = x(1-x)$ which corresponds to the Markov operator $T_1 : C([0,1]) \longrightarrow C([0,1])$ defined, for every $f \in C([0,1])$ and $0 \leq x \leq 1$, by

$$T_1(f)(x) := (1-x)f(0) + xf(1).$$

Then, the Bernstein-Schnabl operators associated with T_1 are the classical Bernstein operators

$$B_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

($n \geq 1$, $f \in C([0,1])$, $x \in [0,1]$).

Considering, as above, $a \geq 0$ along with a probability Borel measure μ on $[0,1]$, we get

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 f\left(\frac{k+at}{n+a}\right) d\mu(t)$$

($n \geq 1$, $f \in C([0,1])$, $x \in [0,1]$).

In particular, if μ is the Borel-Lebesgue measure λ_1 on $[0,1]$, then we get

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 f\left(\frac{k+at}{n+a}\right) dt$$

For $a = 1$, this formula gives the classical Kantorovich operators. Moreover, as already remarked, for $a = 0$ we obtain the Bernstein operators; thus, by means of the previous formula, we obtain a link between these fundamental sequences of approximating operators in terms of a continuous parameter $a \in [0,1]$. For other examples we refer to [7].

Theorem 2.8. *Assume that b is the baricenter of the measure μ. If $ab \geq \alpha'(0)/2$ and $a(1-b) \geq -\alpha'(1)/2$, then the sequence $(C_n)_{n\geq 1}$ is a strong admissible sequence for the semigroup generated by the operator $(A, D_M(A))$.*
Moreover, $C^2([0,1])$ is a core for $D_M(A)$.

Now we proceed to show some properties of the semigroup $(T(t))_{t\geq 0}$ generated by $(A, D_M(A))$ which have their counterparts in terms of regularity properties (with respect to the spatial variable $x \in [0,1]$) of the solutions

$$u(x,t) = T(t)u_0(x)$$

to the initial-boundary value problems (1.1).

However, the next property concerns the sequence $(C_n)_{n\geq 1}$ and it seems to be not devoid of interest. It is related to some saturation aspects for these operators (see [6, Remark 2.2.12])

Theorem 2.9. *If $u, v \in C([0,1])$ and if $\lim_{n\to\infty} n(C_n(u)-u) = v$ uniformly on $[0,1]$, then $u \in D_M(A)$ and $A(u) = v$.*
In particular, if $\lim_{n\to\infty} n(C_n(u) - u) = 0$ uniformly on $[0,1]$, then $u \in D_M(A)$ and $A(u) = 0$, i.e.,

$$\frac{\alpha(x)}{2}u''(x) + a(b-x)u'(x) = 0 \ (x \in]0,1[).$$

From now on we refer again to a Markov operator T on $C([0,1])$ generating the coefficient α, i.e., $T(e_1) = e_1$ and $\alpha = T(e_2) - e_2$. For every $m \geq 1$, denote by $P_m([0,1])$ the subset of all polynomials on $[0,1]$ of degree no greater than m.

Theorem 2.10. *If $T(P_m([0,1])) \subset P_m([0,1])$ for every $m \geq 1$, then*

$$T(t)(P_m([0,1])) \subset P_m([0,1]) \text{ for every } m \geq 1 \text{ and } t \geq 0.$$

Theorem 2.11. *If $T(\mathrm{Lip}(1,1)) \subset \mathrm{Lip}(1,1)$, then*

$$T(t)(\mathrm{Lip}(M,1)) \subset \mathrm{Lip}(M,1) \text{ for every } t \geq 0 \text{ and } M \geq 0.$$

(see (2.7)).

Theorem 2.12. *Suppose that conditions (i) and (ii) of Theorem 2.5 are satisfied. If $f \in C([0,1])$ is convex, then $T(t)f$ is convex for every $t \geq 0$.*

Additional results can be shown for the maximal case $\alpha(x) = x(1-x)$ $(x \in [0,1])$. Thus, $\alpha'(0) = 1$ and $\alpha'(1) = -1$ so that

$$ab \geq 1/2 \text{ and } a(1-b) \geq 1/2.$$

Combining results of [11] and [12], it is possible to show that the semigroup $(T(t))_{t\geq 0}$ can be also expressed as a limit of iterates of Bernstein-Durrmeyer operators with Jacobi weights which are defined as

$$M_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} a_{n,k}(f) \tag{2.9}$$

$(n \geq 1,\ f \in C([0,1]),\ x \in [0,1])$, where

$$a_{n,k}(f) := \frac{1}{B(k+\gamma+1, n-k+\delta+1)} \int_0^1 t^{k+\gamma}(1-t)^{n-k+\delta} f(t)dt,$$
$$\gamma = 2ab - 1 \text{ and } \delta = 2a(1-b) - 1,$$

and B denotes the usual Euler's *Beta* function.

By means of such operators it is possible to show that (see [12, Section 3.2]

Theorem 2.13. *For every $p \geq 1$, $(T(t))_{t\geq 0}$ extends to a positive contraction C_0-semigroup $(\widetilde{T}(t))_{t\geq 0}$ on $L^p([0,1],\mu)$, where μ is the absolutely continuous measure having the normalized Jacobi weight*

$$w_{\gamma,\delta} := \frac{x^\gamma(1-x)^\delta}{\int_0^1 t^\gamma(1-t)^\delta f(t)dt}$$

as density with respect to the Borel-Lebesgue measure on $[0,1]$.
Moreover, the generator $(\widetilde{A}, D(\widetilde{A}))$ of the semigroup $(\widetilde{T}(t))_{t\geq 0}$ is an extension of $(A, D_M(A))$ and $C^2([0,1])$ is a core for $(\widetilde{A}, D(\widetilde{A}))$.
Therefore, $(\widetilde{A}, D(\widetilde{A}))$ is the closure of $(A, D_M(A))$ in $L^p([0,1],\mu)$ as well.
Furthermore, if $t \geq 0$ and if $(k(n)_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} k(n)/n = t$, then for every $f \in L^p([0,1],\mu)$,

$$\widetilde{T}(t)(f) = \lim_{n\to\infty} M_n^{k(n)}(f) \qquad \text{in } L^p([0,1],\mu).$$

Finally, for every $f \in C([0,1])$,

$$\lim_{t\to+\infty} T(t)(f) = \int_0^1 f(x)\, d\mu(x)$$

uniformly on $[0,1]$, and for every $f \in L^p([0,1],\mu), 1 \leq p < +\infty)$

$$\lim_{t\to+\infty} \widetilde{T}(t)(f) = \int_0^1 f(x)\, d\mu(x) \quad \text{in } L^p([0,1],\mu).$$

We also point out that, in the particular case $b = 1/2$ (and hence $a \geq 1$), then the previous results continue to hold true in the space $L^p([0,1])$, $(1 \leq p < +\infty)$, and with the generalized Kantorovich operators as strong admissible sequence (see [8, Section 4]).

2.3. The case $ab < \alpha'(0)/2$ and $a(1-b) \geq -\alpha'(1)/2$ and the case $ab \geq \alpha'(0)/2$ and $a(1-b) < -\alpha'(1)/2$

In these cases the well-posed domain for A are the so-called mixed domain of A. For a proof of the next generation results we refer to [6, Theorem 5.7.7] and [13, pp. 120-121, item (2)], taking the formula $A = \lambda B$ into account.

Theorem 2.14.

(i) *If $ab < \alpha'(0)/2$ and $a(1-b) \geq -\alpha'(1)/2$, then a well-posed domain for A is*

$$D_{VM}(A) := \left\{ u \in C^2_*([0,1]) \ \middle|\ \lim_{x\to 0^+} A(u)(x) = 0 \text{ and } \lim_{x\to 1^-} A(u)(x) \in \mathbf{R} \right\}.$$

(ii) *If* $ab \geq \alpha'(0)/2$ *and* $a(1-b) < -\alpha'(1)/2$, *then a well-posed domain for* A *is*

$$D_{MV}(A) := \left\{ u \in C_*^2([0,1]) \mid \lim_{x \to 0^+} A(u)(x) \in \mathbf{R} \text{ and } \lim_{x \to 1^-} A(u)(x) = 0 \right\}.$$

The domains $D_{VM}(A)$ and $D_{MV}(A)$ are referred to as the mixed domains for A. The relevant boundary conditions imply that the probability that the Markov process reaches 1, resp. 0, in a finite time is zero whereas the probability that the Markov process reaches 0, resp. 1, in a finite time is strictly positive and, when it reaches that point, then it remains there for ever.

As regards the construction of a strong admissible sequence for the semigroup generated by the mixed domains $D_{VM}(A)$ and $D_{MV}(A)$, we are able to provide a solution only for the case

$$b = 0 \text{ and } a \geq -\alpha'(1)/2,$$

as well as for the case

$$b = 1 \text{ and } a \geq \alpha'(0)/2,$$

and we leave the remaining cases as an open problem.

In both the previous special cases, a strong admissible sequence is given by particular generalized Kantorovich operators (2.8) obtained with μ being the unit mass concentrated at 0, resp. at 1, namely,

$$C_n(f)(x) = \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \ldots + x_n}{n+a}\right) d\mu_x(x_1) \cdots d\mu_x(x_n)$$

and, respectively,

$$C_n(f)(x) = \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \ldots + x_n + a}{n+a}\right) d\mu_x(x_1) \cdots d\mu_x(x_n)$$

for every $n \geq 1$, $x \in [0,1]$ and $f \in C([0,1])$.
Moreover, $C^2([0,1]) \bigcap D_{VM}(A)$, resp. $C^2([0,1]) \bigcap D_{MV}(A)$, is a core for $D_{VM}(A)$, resp. $D_{MV}(A)$.

All the shape preserving properties described for the maximal domains continue to hold true in these case, except that the asymptotic behaviour of the semigroup (see [8, Theorem 3.9]).

To this respect we have indeed (see [11, Theorem 4.2]) that, in the maximal case $\alpha(x) = x(1-x)$ $(x \in [0,1])$, for every $f \in C([0,1])$

$$\lim_{t \to +\infty} T(t)f := \begin{cases} f(0) & if \;\; ab < \frac{1}{2} \;\; and \;\; a(1-b) \geq \frac{1}{2}, \\ f(1) & if \;\; ab \geq \frac{1}{2} \;\; and \;\; a(1-b) < \frac{1}{2}. \end{cases}$$

Remarks 2.15. 1. It is worth pointing out that, because of Theorem 2.13, an initial-boundary value problem like (1.3) also holds true in the setting of $L^p([0,1],\mu)$ spaces other than in the space $C([0,1])$. Accordingly, it would be desirable to investigate whether a result similar to Theorem 2.13 holds true also when α is not maximal.

Perhaps, the analysis of such a problem might lead to the need to introduce a new sequence of positive linear operators generalizing Bernstein-Durrmeyer operators (2.9).

2. Apart from the Ventcel' domain (with $a = 0$) (see Corollary 2.4, statement (v)), all the results concerning the asymptotic behaviour of the semigroups have been established when α is maximal. It should be interesting to get similar results in the non-maximal case.

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Francesco Altomare
Dipartimento di Matematica
Università degli Studi di Bari "A. Moro"
Campus Universitario, Via E. Orabona n. 4
70125 Bari, Italy
e-mail: francesco.altomare@uniba.it