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Interpolation methods for multivalued functions

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. The aim of these article is to study the interpolation problem for multivalued functions. We give some methods for the approximation of these functions.

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1. Introduction

The notion of multivalued functions appeared in the first half of the twentieth century. A multivalued function also known as multi-function, multimap, set-valued function. This is a "function" that assume two or more values for each point from the domain. These functions are not functions in the classical way because for each point assign a set of points, so there is not a one-to-one correspondence. The term of "multivalued function" is not correct, but became very popular. Multivalued functions often arise as inverse of functions which are non-injective. For example the inverse of the trigonometric, exponential, power or hyperbolic functions are multivalued functions. Also the indefinite integral can be considered as a multivalued function. These functions appears in many areas, for example in physics in the theory of defects of crystals, for vortices in superfluids and superconductors but also in optimal control theory or game theory in mathematics.

2. Interpolation problem

Let $[a, b] \subseteq \mathbb{R}$ and $f : [a, b] \to \mathbb{P}(\mathbb{R})$ be a multivalued function, where $\mathbb{P}(\mathbb{R})$ is the power set of $\mathbb{R}$, and $f(x)$ is nonempty for every $x \in [a, b]$. We say that a multivalued function is single-valued if, $f(x)$ contain only one element for every $x \in [a, b]$. Thus a common function can be considered as single-valued multifunction. Furthermore, we suppose that for each $x \in [a, b]$, $\operatorname{card}(f(x)) < \infty$. We suppose that the points $x_i \in [a, b], i = 1, 2, \ldots, l$ are given and also the set of function values on this points are known $y_{ij} \in \mathbb{R}, i = 1, 2, \ldots, l, j = 1, 2, \ldots, k$. We will interpolate the sets of points $M_j = \{(x_i, y_{ij}), i = 1, 2, \ldots, l\}, j = 1, 2, \ldots, k$ using an interpolation operator $P_j : C[a, b] \longrightarrow \mathbb{R}, j = 1, 2, \ldots, k$ and the remainder operator R_j.

Definition 2.1. *If $x \in [a, b], x \neq x_i, i = 1, 2, \ldots, k$, the value of the multivalued function in x is approximated by the following set $\{P_1(x), \ldots, P_k(x)\}$. The approximation error on the point x is given by $R_1(x) + \ldots + R_k(x)$.*

Definition 2.2. *We have the following interpolation formula:*

$$f(x) = (P_1 \cup P_2 \ldots \cup P_k)(x) + (R_1 + R_2 + \ldots + R_k)(x) \tag{2.1}$$

where $P_1 \cup \ldots \cup P_k$ is the interpolation operator and $R_1 + \ldots + R_k$ is the remainder operator.

Remark 2.3. The $P_1 \cup \ldots \cup P_k$ is an interpolation operator because the following interpolation condition are satisfied: $P_i(x_j) = y_{ji}$.

Theorem 2.4. *The interpolation operator $P_1 \cup \ldots \cup P_k$ exists and is unique.*

Proof. It is obvious, because at each set $M_j, j = 1, 2, \ldots, k$ the interpolation operators $P_j, j = 1, 2, \ldots, k$ exists and are unique. □

Furthermore let's consider the case when we have the following type of data $\{(x_i, y_{n_j j}), i = 1, 2, \ldots, l, j = 1, 2, \ldots, k, n_i \in \mathbb{N}, n_i < \infty\}$.
Let be $m = \min\{n_j, j = 1, 2, \ldots, k\}$, then we will consider the following set of data $\{(x_m, y_{mi}, i = 1, 2, \ldots, k)\}$, in this way we reduce the problem to the previous case.

3. Lagrange-type multivalued interpolation

If we considering the case when at each set M_j, the points (x_i, y_{ij}) are interpolated using Lagrange type interpolation, then the interpolation operator is $L_{l_1} \cup \ldots \cup L_{l_k}$, where $L_{l_i}, i = 1, 2, \ldots, k$ are $l - 1$ degree Lagrange polynomials, and the remainder is equal to $R_{l_1} + R_{l_2} + \ldots + R_{l_k}$ where R_{l_i} are the corresponding remainder operators.

Theorem 3.1. *The value of the multivalued Lagrange type interpolation function on the point $x \in [a, b]$, $x \neq x_i, i = 1, 2, \ldots, l$ is given by*

$$L_{l_1} \cup \ldots \cup L_{l_k}(x) = \bigcup_{i=1}^{k} \sum_{j=1}^{l-1} l_{ij}(x) y_{ij} \tag{3.1}$$

where l_{ij} are the basic Lagrange polynomials with degree $l - 1$.

Proof. From Theorem 2.4 we have that the value of the multivalued function on the point x is approximated by the following values $\{P_1(x), \ldots, P_k(x)\}$, where P_i are the corresponding interpolation operators for the data (x_i, y_{ij}), $j = 1, 2, \ldots, l$. Because now we use Lagrange-type interpolation to approximate these data, we have

$$P_i(x) = L_{l-1}(x) = \sum_{i=1}^{l} l_{ij}(x) y_{ij},$$

where l_{ij} are the corresponding basic Lagrange polynomials. □

We suppose that $y_{ij} = f_j(x_i)$ where $f_j \in C[a,b], j = 1,2,\ldots,k$.

Theorem 3.2. *If $f_j \in C^{l-1}[a,b], j = 1,2,\ldots,k,$ and $\exists f_j^{(l)} j = 1,2,\ldots,k$ on $[a,b]$ then the remainder of the multivalued interpolation formula is*

$$(R_1 + R_2 + \ldots + R_k)(x) = \sum_{j=1}^{k} \frac{u(x)}{l!} f_j^{(l)}(\xi_j) \tag{3.2}$$

were $\xi_j \in (a,b)$ and $u(x) = (x-x_1)(x-x_2)\ldots(x-x_l)$.

Proof. If we consider the Lagrange interpolation formula for each set M_j

$$f_j(x) = L_j(x) + R_j(x), j = 1,2,\ldots,k$$

where if $f_j \in C^{l-1}[a,b]$ and $\exists f_j^{(l)}$ on $[a,b]$ then there $\exists \xi_j \in (a,b), j = 1,2,\ldots,k$ such that

$$R_j(x) = \frac{u(x)}{(l!)} f_j^{(l)}(\xi_j), j = 1,2,\ldots,k$$

□

Example 3.3. If consider the multivalued function, obtained as the inverse of the function $g(x) = \sin(x)$, on the interval $[a,b] = [-1,1]$, using the method described below with Lagrange type interpolation operators on each set of points M_j, we obtain the graph from figure 1, where the dotted line is the graph of the multivalued function and the continuous line is the graph of the multivalued function obtained by interpolation.

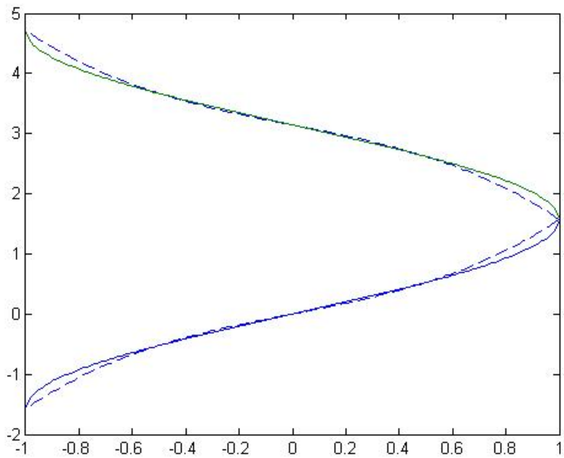

Figure 1. Interpolation of multivalued function with Lagrange operators

4. Shepard-type multivalued interpolation

We suppose that the points $x_i \in [a,b]$, $i = 1,2,\ldots,l$ are given and also the set of function values on this points are known $y_{ij} \in \mathbb{R}, i = 1,\ldots,l, j = 1,\ldots,k$. We will interpolate the sets of points $M_j = \{(x_i, y_{ij}), i = 1, \ldots, l\}, j = 1, \ldots, k$ using Shepard interpolation studied also in [2], [1], [4] and [6].

Theorem 4.1. *The Shepard-type multivalued interpolation operator is*

$$\bigcup_{i=1}^{k} S_i(x) = \bigcup_{i=1}^{k} \sum_{j=1}^{l} A_j(x) y_{ij}, \tag{4.1}$$

where S_i are the univariate Shepard operators and

$$A_j(x) = \frac{\prod\limits_{i=1, i\neq j} l|x - x_i|^{\mu}}{\sum\limits_{t=1}^{l} \prod\limits_{i=1, i\neq t} l|x - x_i|^{\mu}}$$

and $\mu \in \mathbb{R}_+$.

Remark 4.2. The basis functions A_j can be also written in the following barycentric form

$$A_j(x) = \frac{|x - x_j|^{-\mu}}{\sum\limits_{i=1}^{l} |x - x_k|^{-\mu}}, j = 1, 2, \ldots, l,$$

and they satisfy

$$\sum_{j=1}^{l} A_j(x) = 1, A_j(x_p) = \delta_{jp}, j, p = 1, 2, \ldots, l.$$

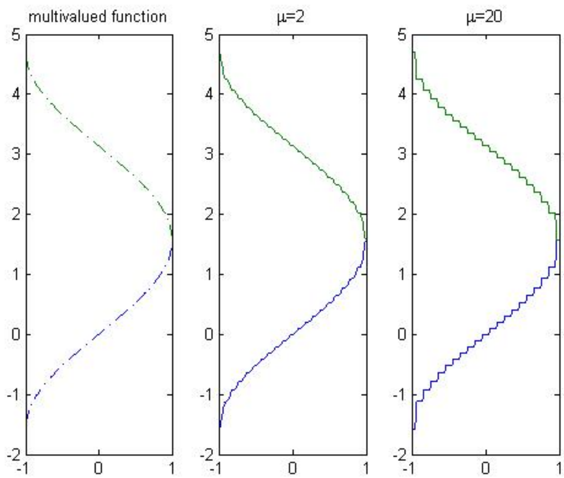

Figure 2. Interpolation of multivalued function with Shepard operators

From the remark it follows that the Shepard operators has the following properties: first of all they have the interpolation conditions $S_i(x_j) = y_{ij}$, $j = 1, 2, \dots, l$, $i = 1, 2, \dots, k$, and they have the degree of exactness $dex(S_i) = 0, i = 1, 2, \dots, k$. The graph of the function from the previous example in the case when we use Shepard operators with different parameters, is given in Figure 2.

The major disadvantage of the Shepard operator is the low degree of exactness, but this can be overcome combining the Shepard operator with another interpolation operators, for example Lagrange, Hermite, Birkhoff or other interpolation operators.

5. Spline-type multivalued interpolation

We will consider again the points $x_i \in [a, b], i = 1, 2, \dots, l$ and also the set of function values on this points $y_{ij} \in \mathbb{R}, i = 1, 2, \dots, l, j = 1, 2, \dots, k$ which are known, let $M_j = \{(x_i, y_{ij}), i = 1, 2, \dots, l\}, j = 1, 2, \dots, k$ be the set of interpolation points. In this section we will interpolate the multivalued function given by the set of points from M_j with spline interpolation function.
We suppose that the values $y_{ij} = f_j(x_i)$, where $f_j \in H^{m,2}[a, b]$ is the set of functions with $f_j \in C^{m-1}[a, b]$, $f^{(m-1)}$ absolute continuous on $[a, b]$ and $f^{(m)} \in L^2[a, b]$.

Theorem 5.1. *The multivalued interpolation operator in the case of spline interpolation is*

$$\bigcup_{i=1}^{l} S_i(x) = \bigcup_{i=1}^{l} \sum_{j=1}^{k} s_{ij}(x) y_{ij} \tag{5.1}$$

where s_{ij} are the fundamental spline interpolation functions.

Remark 5.2. The fundamental spline functions satisfies the following minimum properties $\|S_i^{(m)}\|_2 \longrightarrow \min$, in the set of all functions which satisfies the interpolation conditions.

To determine the fundamental spline functions we can use the structural characterization theorem of spline functions given also in [3] and we have

$$s_{ij}(x) = \sum_{t=0}^{m-1} a_t^{ij} x^t + \sum_{p=1}^{l} b_p^{ij} (x - x_p)_+^{2m-1}, i = 1, 2, \dots, l, j = 1, 2, \dots, k$$

with a_t^{ij}, and b_p^{ij} obtained as the solution of the following systems:

$$\begin{aligned} s_{ij}^{(r)}(\alpha) &= 0, \ r = m, \dots, 2m-1, \ \texttt{and} \ \alpha > x_l \\ s_{ij}(x_\nu) &= \delta_{j\nu}, \ \nu = 1, 2, , \dots, l \end{aligned}$$

for $j = 1, 2, \dots, k, i = 1, 2, \dots, l$.

Theorem 5.3. *If $f_j \in H^{m,2}[a, b], j = 1, 2, \dots, k$ then the remainder term of the spline-type multivalued interpolation formula is*

$$\sum_{j=1}^{k} R_j(x) = \sum_{j=1}^{k} \int_a^b \varphi_j(x, t) f_j^{(m)}(t) dt \tag{5.2}$$

where

$$\varphi_j(x,t) = \frac{(x-t)_+^{m-1}}{(m-1)!} - \sum_{i=1}^{l} s_{ij}(x)(x_i - t)_+^{m-1}, j = 1, 2, \ldots, k.$$

This follows from the representation of the error using the Peano theorem.

The graph of the function from the previous example using third degree natural spline interpolation operators is given in Figure 3.

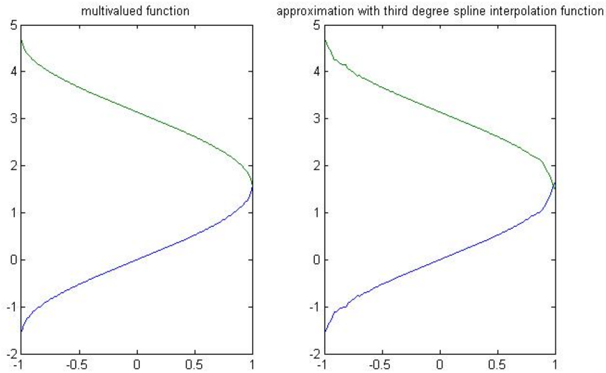

Figure 3. Interpolation of multivalued function with spline operators

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