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Optimal quadrature formulas for approximate solution of the first kind singular integral equation with Cauchy kernel

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Abstract. In the present paper in $L_2^{(m)}(-1,1)$ space the optimal quadrature formulas with derivatives are constructed for approximate solution of a singular integral equation of the first kind with Cauchy kernel. Approximate solution of the singular integral equation is obtained applying the optimal quadrature formulas. Explicit forms of coefficients for the of optimal quadrature formulas are obtained. Some numerical results are presented.

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1. Introduction. Statement of the problem

The study of various problems of mathematical physics as well as specific problems from aerodynamics, electrodynamics, elasticity theory and other areas, naturally reduces to singular integral equations [5, 16]. In this case, the plane problems [5, 12, 16, 19] are reduced to solving the characteristic singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(x)}{x-t}\mathrm{d}x = f(t), \qquad t\in(-1,1), \tag{1.1}$$

where the singular integral is understood, here in after, in the sense of the Cauchy principal value. Equation (1.1) has four complete analytical solutions corresponding

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to the values of the parameter k (see [16],pp.49-50). In particular, for $k=-1$ the only solution of (1.1) is given by the formula

$$\varphi(t)=-\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}(x-t)}\mathtt{d}x. \tag{1.2}$$

Thus, the solution of singular integral equation of the form (1.1) can be reduced to the calculation of the weighted singular integral (1.2). Therefore, the development of effective approximate methods for calculating singular integrals are of great applied importance and one of the actual problems of computational mathematics.

Quadrature and cubature formulas are one of the methods for approximation of integrals. Many methods have been developed to construct the quadrature formulas for the singular integral (1.2). See for example, [1, 5, 6, 7, 8, 9, 12, 14, 15, 16, 19, 21, 25, 26] and literature cited therein.

Particularly, in the work [11] by Eshkuvatov, Nik Long and Abdulkawi, new quadrature formulas for evaluating the singular integral of Cauchy type with unbounded weight function on the edges is constructed. The construction of the quadrature formulas is based on the modification of the discrete vortices method and linear spline interpolation over the finite interval $[-1,1]$. It is proved that the constructed quadrature formulas converge for any singular point x not coinciding with the end points of the interval $[-1,1]$. Numerical results are given to validate the accuracy of the quadrature formulas. The error bounds are found to be of order $O(h^{\alpha}|\ln h|)$ and $O(h|\ln h|)$ in the classes of functions $H^{\alpha}([-1,1]), 0<\alpha<1$ and $C^1([-1,1])$, respectively.

In [11] the authors were used modification of the discrete vortices method and linear spline methods for approximation of the singular integrals. Constructed quadrature formulas are exact only for linear functions and these formulas are not an optimal approximation technique.

In the present paper, using the functional approach, we construct optimal quadrature formulas for approximate calculation of the integral (1.2) in the space $L_2^{(m)}(-1,1)$. We recall that $L_2^{(m)}(-1,1)$ is a Hilbert space of classes of all real functions φ defined in the interval $[-1,1]$ that differ by a polynomial of degree $(m-1)$ and square integrable with derivative of order m, and equipped with the norm

$$\|\varphi\|_{L_2^{(m)}}=\left(\int_{-1}^{1}\left(\varphi^{(m)}(x)\right)^2dx\right)^{\frac{1}{2}}.$$

It should be noted that, in particular, when $m=1$ from our numerical results we get the results of the work [11] close to each other.

We consider the following quadrature formula with derivatives

$$\int_{-1}^{1}\frac{\varphi(x)}{\sqrt{1-x^2}(x-t)}\mathtt{d}x\cong\sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}C_{\alpha}[\beta]\varphi^{(\alpha)}(x_{\beta}),\quad -1<t<1, \tag{1.3}$$

in the Sobolev space $L_2^{(m)}(-1,1)$.

Here $C_\alpha[\beta]$ are the coefficients, x_β ($\in [-1,1]$) are the nodes of the quadrature formula, N is a natural number and $n = 0, 1, 2, ..., (m-1)$.

The following difference is called *the error* of the quadrature formula (1.3):

$$(\ell, \varphi) = \int\limits_{-1}^{1} \frac{\varphi(x)}{\sqrt{1-x^2}(x-t)} \mathtt{d}x - \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N} C_\alpha[\beta]\varphi^{(\alpha)}(x_\beta) = \int\limits_{-\infty}^{\infty} \ell(x)\varphi(x)\mathtt{d}x,$$

where ℓ is the error function of the formula (1.3) and has the form

$$\ell(x) = \frac{\varepsilon_{[-1,1]}(x)}{\sqrt{1-x^2}(x-t)} - \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\delta^{(\alpha)}(x-x_\beta), \tag{1.4}$$

here $\varepsilon_{[-1,1]}(x)$ is the characteristic function of the interval $[-1,1]$, δ is the Dirac delta-function.

Since the functional ℓ of the form (1.4) is defined on the space $L_2^{(m)}(-1,1)$, it belongs to the conjugate space $L_2^{(m)*}(-1,1)$, and satisfies the following equations (see [27])

$$(\ell, x^\alpha) = 0, \quad \alpha = 0, 1, 2, ..., (m-1).$$

The construction problem of optimal quadrature formulas of the form (1.3) in the sense of Sard [20] with the error functional (1.4) in the space $L_2^{(m)}(-1,1)$ for fixed x_β is to find the quantity

$$\|\mathring{\ell}|L_2^{(m)*}\|^2 = \inf_{C_\alpha[\beta]} (\ell, \psi_\ell),$$

where

$$\psi_\ell(x) \quad = \quad (-1)^m \ell(x) * G_m(x) + P_{m-1}(x),$$

here $G_m(x) = \frac{x^{2m-1}\mathtt{sgn}(x)}{2\cdot(2m-1)!}$, $P_{m-1}(x)$ is a polynomial of degree $(m-1)$, ψ_ℓ is the extremal function of the quadrature formula (1.3) in the space $L_2^{(m)}(-1,1)$ (see for instance, [4, 23, 24, 26]), $\mathtt{sgn}(x)$ is the signum function.

In the Hilbert spaces one can construct optimal quadrature formulas, optimal interpolation formulas, and splines using the Sobolev method which is based on using a discrete analogue of differential operator [27, 28]. Applying this method in the different Hilbert spaces optimal formulas and splines were constructed.

In the works [4, 2] for the norm of the error functional the following form was obtained

$$\begin{aligned} \|\ell|L_2^{(m)*}\|^2 &= (-1)^m\Bigg[\sum_{k=0}^{n}\sum_{\alpha=0}^{n}\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}(-1)^k \\ &\times C_k[\gamma]C_\alpha[\beta]\frac{(h\beta-h\gamma)^{2m-\alpha-k-1}\mathtt{sgn}(h\beta-h\gamma)}{2(2m-\alpha-k-1)!} \\ &-2\sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\int_{-1}^{1}\frac{(x-h\beta)^{2m-\alpha-1}\mathtt{sgn}(x-h\beta)}{2(2m-\alpha-1)!\sqrt{1-x^2}(x-t)}\mathrm{d}x \\ &+\int_{-1}^{1}\int_{-1}^{1}\frac{(x-y)^{2m-1}\mathtt{sgn}(x-y)}{2(2m-1)!\sqrt{1-x^2}\sqrt{1-y^2}(x-t)(y-t)}\mathrm{d}x\mathrm{d}y\Bigg]. \end{aligned} \quad (1.5)$$

The rest of the paper is organized as follows. In Section 2 we give the algorithm for construction of optimal quadrature formulas of the form (1.3). Explicit formulas for coefficients of the optimal quadrature formulas of the form (1.3) are found for any natural m. In section 3 some numerical examples are provided to illustrate the validity of the algorithm.

2. The main results

Further, we suppose that $x_\beta = h\beta - 1, h = \frac{2}{N}$ and $N + 1 \geq m$.

The idea of construction of optimal quadrature formulas of the form (1.3) is as follows: First, for $m = 1$, we minimize the norm (1.5) by coefficients $C_0[\beta]$ in the space $L_2^{(1)}(0,1)$ and get the following system for finding the optimal coefficients $\mathring{C}_0[\beta]$:

$$\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma]\frac{(h\beta-h\gamma)\mathtt{sgn}(h\beta-h\gamma)}{2}+\lambda_0=\int_{-1}^{1}\frac{(x-h\beta+1)\ \mathtt{sgn}(x-h\beta+1)}{2\sqrt{1-x^2}(x-t)}\mathrm{d}x,$$

$$\beta = 0, 1, ..., N,$$

$$\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma]=\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}(x-t)}\mathrm{d}x.$$

We note that the obtained system was solved in the work [15], i.e. there the optimal coefficients $\mathring{C}_0[\beta]$ were found in the space $L_2^{(1)}(-1,1)$.

Further, in the case $m = 2$, putting the optimal coefficients $\mathring{C}_0[\beta]$ to the expression (1.5) we minimize this norm by coefficients $C_1[\beta]$ in the space $L_2^{(2)}(-1,1)$ and find the optimal coefficients $\mathring{C}_1[\beta]$. Continuing by this manner for the cases $m = 3, 4, ..., k-1$, i.e. putting the obtained optimal coefficients $\mathring{C}_0[\beta], \mathring{C}_1[\beta], ..., \mathring{C}_{k-2}[\beta]$ to the expression of the norm (1.5) and minimizing this norm by coefficients $C_{k-1}[\beta]$ in the space $L_2^{(k)}(-1,1)$, we get the following system for finding the optimal coefficients $\mathring{C}_{k-1}[\beta]$:

$$\sum_{\gamma=0}^{N} \mathring{C}_{k-1}[\gamma] \frac{(h\beta - h\gamma)\mathtt{sgn}(h\beta - h\gamma)}{2} + (-1)^{k-1}(k-1)!\lambda_{k-1} = F_{k-1}[\beta], \tag{2.1}$$

$\beta = 0, 1, ..., N,$

$$\sum_{\gamma=0}^{N} \mathring{C}_{k-1}[\gamma] = \frac{g_{k-1}}{(k-1)!} - \sum_{i=0}^{k-2}\sum_{\gamma=0}^{N} \mathring{C}_i[\gamma] \frac{(h\gamma - 1)^{k-i-1}}{(k-i-1)!}. \tag{2.2}$$

Here

$$F_{k-1}[\beta] = f_{k-1}[\beta] - \sum_{l=0}^{k-2}\sum_{\gamma=0}^{N} (-1)^{l+k-1} \mathring{C}_l[\gamma] \frac{(h\beta - h\gamma)^{k-l}\mathtt{sgn}(h\beta - h\gamma)}{2(k-l)!}, \tag{2.3}$$

where

$$\begin{aligned} f_{k-1}[\beta] &= \int_{-1}^{1} \frac{(x - h\beta + 1)^k \mathtt{sgn}(x - h\beta + 1)}{2 \cdot k!\sqrt{1-x^2}(x-t)} \mathrm{d}x \\ &= -\frac{1}{k!}\left[\sum_{i=1}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (t - h\beta + 1)^{k-i} (A_1 + A_2) - \frac{(t - h\beta + 1)^k}{\sqrt{1-t^2}} A_3\right], \end{aligned}$$

$$\begin{aligned} A_1 &= \sum_{j=1}^{\left[\frac{i-1}{2}\right]} \begin{pmatrix} i-1 \\ 2j \end{pmatrix} (-t)^{i-2j-1} \Bigg(-\frac{\sqrt{1-(h\beta-1)^2}}{2j} \Bigg[(h\beta - 1)^{2j-1} \\ &\quad + \sum_{l=1}^{j-1} \frac{(2j-1)(2j-3)...(2j-2l+1)}{2^l (j-1)(j-2)...(j-l)} (h\beta - 1)^{2j-2l-1} \Bigg] \\ &\quad + \frac{(2j-1)!!}{2^j j!} \arcsin(h\beta - 1) \Bigg), \\ A_2 &= \sum_{j=0}^{\left[\frac{i-2}{2}\right]} \begin{pmatrix} i-1 \\ 2j+1 \end{pmatrix} \sum_{l=0}^{j} \frac{(-t)^{i-2j-2}(-1)^{l+1}}{(2l+1)} \begin{pmatrix} j \\ l \end{pmatrix} \left(\sqrt{1-(h\beta-1)^2}\right)^{2l+1} \\ &\quad + (-t)^{i-1} \arcsin(h\beta - 1), \\ A_3 &= \ln\left| \frac{1 - t(h\beta - 1) + \sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta - 1 - t} \right|, \end{aligned}$$

$$g_{k-1} = \int_{-1}^{1} \frac{x^{k-1}}{\sqrt{1-x^2}(x-t)} \mathtt{d}x \tag{2.4}$$
$$= \pi \sum_{i=1}^{k-1} \begin{pmatrix} k-1 \\ i \end{pmatrix} t^{k-1-i} \left(\sum_{j=1}^{\left[\frac{i-1}{2}\right]} \begin{pmatrix} i-1 \\ 2j \end{pmatrix} (-t)^{i-1-j} \frac{(2j-1)!!}{2^j j!} + (-t)^{i-1} \right).$$

Now we solve the system (2.1)-(2.2). The solution of the system (2.1)-(2.2) we find by the following way.
We denote

$$u(h\beta) = \sum_{\gamma=0}^{N} \mathring{C}_{k-1}[\beta] \frac{(h\beta - h\gamma)\mathtt{sgn}(h\beta - h\gamma)}{2} + (-1)^{k-1}(k-1)!\lambda_{k-1}. \tag{2.5}$$

Assume $\beta \leq 0$, then from (2.5) we have

$$u(h\beta) = -\frac{h\beta}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right) - \mu_{k-1} + (-1)^{k-1}(k-1)!\lambda_{k-1},$$

where

$$\nu_{k-2} = \sum_{i=0}^{k-2}\sum_{\gamma=0}^{N} C_i[\gamma]\frac{(h\gamma)^{k-i-1}}{(k-i-1)!}, \quad \mu_{k-1} = -\frac{1}{2}\sum_{\gamma=0}^{N} C_{k-1}[\gamma](h\gamma).$$

Suppose $\beta \geq N$, then taking into account (2.5), we get

$$u(h\beta) = \frac{h\beta}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right) + \mu_{k-1} + (-1)^{k-1}(k-1)!\lambda_{k-1}.$$

We introduce the following denotations

$$a_{k-1}^{-} = \mu_{k-1} - (k-1)!(-1)^{k-1}\lambda_{k-1} \text{ and } a_{k-1}^{+} = \mu_{k-1} + (k-1)!(-1)^{k-1}\lambda_{k-1}.$$

Then we obtain that

$$u(h\beta) = \begin{cases} -\frac{h\beta}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right) - a_{k-1}^{-}, & \beta \leq 0, \\ F_{k-1}[\beta], & 0 \leq \beta \leq N, \\ \frac{h\beta}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right) + a_{k-1}^{+}, & \beta \geq N, \end{cases} \tag{2.6}$$

where a_{k-1}^{-} and a_{k-1}^{+} are unknowns.
Hence, taking into account the values of the function $u(h\beta)$ at the points $\beta = 0$ and $\beta = N$, we get

$$a_{k-1}^{-} = F_{k-1}[0], \quad a_{k-1}^{+} = F_{k-1}[N] - \frac{1}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right).$$

Now, using the following known equality from [24]

$$h\sum_{\gamma=-\infty}^{\infty} D_1[\gamma]\frac{(h\beta - h\gamma)\mathtt{sgn}(h\beta - h\gamma)}{2} = \delta[\beta], \tag{2.7}$$

where in [22]

$$D_1[\beta]=\begin{cases}0, & |\beta|\geq 2,\\ h^{-2}, & |\beta|=1,\\ -2h^{-2}, & \beta=0,\end{cases}\tag{2.8}$$

$\delta[\beta]=\begin{cases}1, & \beta=0,\\ 0, & \beta\neq 0,\end{cases}$ taking account of (2.6) and (2.7), for the optimal coefficients $\mathring{C}_{k-1}[\beta]$, when $0\leq\beta\leq N$, we get the following

$$\begin{aligned}\mathring{C}_{k-1}[\beta] &= h\sum_{\gamma=-\infty}^{\infty}D_1[\beta-\gamma]u(h\gamma)=h\Bigg[\sum_{\gamma=0}^{N}D_1[\beta-\gamma]F_{k-1}[\gamma]\\ &+\sum_{\gamma=1}^{\infty}D_1[\beta+\gamma]\left(\frac{h\gamma}{2}\left(\frac{g_{k-1}}{(k-1)!}-\nu_{k-2}\right)-a_{k-1}^{-}\right)\\ &+\sum_{\gamma=1}^{\infty}D_1[N+\gamma-\beta]\left(\frac{1+h\gamma}{2}\left(\frac{g_{k-1}}{(k-1)!}-\nu_{k-2}\right)+a_{k-1}^{+}\right)\Bigg].\end{aligned}$$

Hence, by virtue of (2.8), we have the following.

Theorem 2.1. *The coefficients for the optimal quadrature formulas of the form (*1.3*) in the Sobolev space* $L_2^{(m)}(-1,1)$ *are defined as follows*

$$\mathring{C}_{k-1}[0] = h^{-1}\left[F_{k-1}[1]-F_{k-1}[0]+\frac{h}{2}\left(\frac{g_{k-1}}{(k-1)!}-\nu_{k-2}\right)\right],\tag{2.9}$$

$$\mathring{C}_{k-1}[\beta] = h^{-1}\Big[F_{k-1}[\beta-1]-2F_{k-1}[\beta]+F_{k-1}[\beta+1]\Big],\tag{2.10}$$

for $\beta=1,...,N-1$

$$\mathring{C}_{k-1}[N] = h^{-1}\left[F_{k-1}[N-1]-F_{k-1}[N]+\frac{h}{2}\left(\frac{g_{k-1}}{(k-1)!}-\nu_{k-2}\right)\right],\tag{2.11}$$

$k=0,1,2,...,m-1,$ *where for* $t\neq h\gamma-1,\ \gamma=0,1,2,...,N,$

$$F_{k-1}[\beta]=f_{k-1}[\beta]-\sum_{l=0}^{k-2}\sum_{\gamma=0}^{N}(-1)^{l+k-1}\mathring{C}_l[\gamma]\frac{(h\beta-h\gamma)^{k-l}\,sgn(h\beta-h\gamma)}{2(k-l)!},$$

$$f_{k-1}[\beta]=-\frac{1}{k!}\left[\sum_{i=1}^{k}\binom{k}{i}(t-h\beta+1)^{k-i}(A_1+A_2)-\frac{(t-h\beta+1)^k}{\sqrt{1-t^2}}A_3\right],$$

and for $t=h\gamma-1,\ \gamma=0,1,2,...,N,$

$$F_{k-1}[\beta]=\overline{f}_{k-1}[\beta]-\sum_{l=0}^{k-2}\sum_{\gamma=0}^{N}(-1)^{l+k-1}\mathring{C}_l[\gamma]\frac{(h\beta-h\gamma)^{k-l}\,sgn(h\beta-h\gamma)}{2(k-l)!},$$

$$\overline{f}_{k-1}[\beta]=-\frac{1}{k!}\left[\sum_{i=1}^{k}\binom{k}{i}(t-h\beta+1)^{k-i}(A_1+A_2)\right],$$

here

$$
\begin{aligned}
A_1 &= \sum_{j=1}^{\left[\frac{i-1}{2}\right]} \binom{i-1}{2j} (-t)^{i-2j-1} \Bigg(-\frac{\sqrt{1-(h\beta-1)^2}}{2j} \Bigg[(h\beta-1)^{2j-1} \\
&\quad + \sum_{l=1}^{j-1} \frac{(2j-1)(2j-3)...(2j-2l+1)}{2^l (j-1)(j-2)...(j-l)} (h\beta-1)^{2j-2l-1} \Bigg] \\
&\quad + \frac{(2j-1)!!}{2^j j!} \arcsin(h\beta-1) \Bigg), \\
A_2 &= \sum_{j=0}^{\left[\frac{i-2}{2}\right]} \binom{i-1}{2j+1} \sum_{l=0}^{j} \frac{(-t)^{i-2j-2}(-1)^{l+1}}{(2l+1)} \binom{j}{l} \left(\sqrt{1-(h\beta-1)^2}\right)^{2l+1} \\
&\quad + (-t)^{i-1} \arcsin(h\beta-1), \\
A_3 &= \ln \left| \frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t} \right|,
\end{aligned}
$$

$$
g_{k-1} = \pi \sum_{i=1}^{k-1} \binom{k-1}{i} t^{k-1-i} \left(\sum_{j=1}^{\left[\frac{i-1}{2}\right]} \binom{i-1}{2j} (-t)^{i-1-j} \frac{(2j-1)!!}{2^j j!} + (-t)^{i-1} \right).
$$

From Theorem 2.1 in particular, when $m=1$, $m=2$ and $m=3$. We have the following.

For the case $m=1$.

Corollary 2.2. *For $t \neq h\gamma - 1$, coefficients of the optimal quadrature formula (1.3) with equally spaced nodes in the space $L_2^{(1)}(-1,1)$ have the following form*

$$
\begin{aligned}
\mathring{C}_0[0] &= h^{-1}\left(F_0[1] - \frac{\pi}{2}\right), \\
\mathring{C}_0[\beta] &= h^{-1}\left(F_0[\beta-1] - 2F_0[\beta] + F_0[\beta+1]\right), \quad \beta = 1,2,...,N-1, \\
\mathring{C}_0[N] &= h^{-1}\left(F_0[N-1] + \frac{\pi}{2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
F_0[\beta] &= -\arcsin(h\beta-1) \\
&\quad + \frac{t-(h\beta-1)}{\sqrt{1-t^2}} \ln \left| \frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t} \right|.
\end{aligned}
$$

Corollary 2.3. *For $t = h\gamma - 1$, coefficients of the optimal quadrature formula (1.3) with equally spaced nodes in the space $L_2^{(1)}(-1,1)$ have the following form* *if* $\gamma = 1$, *i.e. for* $t = h - 1$*:*

$$\begin{aligned}
\mathring{C}_0[0] &= h^{-1}\left(\overline{F}_0[1] - \frac{\pi}{2}\right),\\
\mathring{C}_0[1] &= h^{-1}\left(F_0[0] - 2\overline{F}_0[1] + F_0[2]\right),\\
\mathring{C}_0[2] &= h^{-1}\left(\overline{F}_0[1] - 2F_0[2] + F_0[2]\right),\\
\mathring{C}_0[\beta] &= h^{-1}\left(F_0[\beta-1] - 2F_0[\beta] + F_0[\beta+1]\right), \quad \beta = 3, 4, ..., N-1,\\
\mathring{C}[N] &= h^{-1}\left(F_0[N-1] + \frac{\pi}{2}\right),
\end{aligned}$$

if $\gamma = 2, 3, 4, ..., N-2$*:*

$$\begin{aligned}
\mathring{C}_0[0] &= h^{-1}\left(F_0[1] - \frac{\pi}{2}\right),\\
\mathring{C}_0[\beta] &= h^{-1}\left(F_0[\beta-1] - 2F_0[\beta] + F_0[\beta+1]\right),\\
& \qquad \beta = 1, 2, ..., \gamma-2 \quad and \quad \beta = \gamma+2, \gamma+3, ..., N-1,\\
\mathring{C}_0[\gamma-1] &= h^{-1}\left(F_0[\gamma-2] - 2F_0[\gamma-1] + \overline{F}_0[\gamma]\right),\\
\mathring{C}_0[\gamma] &= h^{-1}\left(F_0[\gamma-1] - 2\overline{F}_0[\gamma] + F_0[\gamma+1]\right),\\
\mathring{C}_0[\gamma+1] &= h^{-1}\left(\overline{F}_0[\gamma] - 2F_0[\gamma+1] + F_0[\gamma+2]\right),\\
\mathring{C}_0[N] &= h^{-1}\left(F_0[N-1] + \frac{\pi}{2}\right),
\end{aligned}$$

if $\gamma = N - 1$, *i.e. for* $t = 1 - h$*:*

$$\begin{aligned}
\mathring{C}_0[0] &= h^{-1}\left(F_0[1] - \frac{\pi}{2}\right),\\
\mathring{C}_0[\beta] &= h^{-1}\left(F_0[\beta-1] - 2F_0[\beta] + F_0[\beta+1]\right), \quad \beta = 1, 2, ..., N-3,\\
\mathring{C}_0[N-2] &= h^{-1}\left(F_0[N-3] - 2F_0[N-2] + \overline{F}_0[N-1]\right),\\
\mathring{C}_0[N-1] &= h^{-1}\left(F_0[N-2] - 2\overline{F}_0[N-1] + F_0[N]\right),\\
\mathring{C}_0[N] &= h^{-1}\left(\overline{F}_0[N-1] + \frac{\pi}{2}\right),
\end{aligned}$$

where $\overline{F}_0[\beta] = -\arcsin(h\beta - 1)$, $F_0[\beta]$ *is given in Corollary* 2.2

The case $m = 2$. In this case we have the following result of the work [2] as immediate corollary of Theorem 2.1.

Corollary 2.4. *For* $t \neq h\gamma - 1$, *coefficients of the optimal quadrature formula (*1.3*) with equally spaced nodes in the space* $L_2^{(2)}(-1,1)$ *take the form*

$$\begin{aligned}
\mathring{C}_1[0] &= h^{-1}\left(F_1[1] - F_1[0] + \frac{h}{2}\left(\pi - \sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right),\\
\mathring{C}_1[\beta] &= h^{-1}\Big(F_1[\beta-1] - 2F_1[\beta] + F_1[\beta+1]\Big), \quad \beta = \overline{1, N-1}\\
\mathring{C}_1[N] &= h^{-1}\left(F_1[N-1] - F_1[N] + \frac{h}{2}\left(\pi - \sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right),
\end{aligned}$$

where

$$F_1[\beta] = f_1[\beta] + \frac{h^2}{4}\sum_{\gamma=0}^{N} \mathring{C}_0[\gamma](h\beta - h\gamma)^2 \mathbf{sgn}(h\beta - h\gamma),$$

$$\begin{aligned} f_1[\beta] &= -\frac{1}{2}\Bigg[-\sqrt{1-(h\beta-1)^2} + \Big(t - 2h\beta + 2\Big)\arcsin(h\beta - 1) \\ &\quad - \frac{(t-(h\beta-1))^2}{\sqrt{1-t^2}} \ln\left| \frac{1 - t(h\beta-1) + \sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta - 1 - t} \right| \Bigg] \end{aligned}$$

and $\mathring{C}_0[\beta]$, $\beta = 0, 1, 2, ..., N$ *are defined in Corollary* 2.2.

Corollary 2.5. *For* $t = h\gamma - 1$*, coefficients of the optimal quadrature formula* (1.3) *with equally spaced nodes in the space* $L_2^{(2)}(-1,1)$ *take the form*

if $\gamma = 1$*, i.e. for* $t = h - 1$*:*

$$\begin{aligned} \mathring{C}_1[0] &= h^{-1}\left(\overline{F}_1[1] - F_1[0] + \frac{h}{2}\left(\pi - \sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right), \\ \mathring{C}_1[1] &= h^{-1}\left(F_1[0] - 2\overline{F}_1[1] + F_1[2]\right), \\ \mathring{C}_1[2] &= h^{-1}\left(\overline{F}_1[1] - 2F_1[2] + F_1[2]\right), \\ \mathring{C}_1[\beta] &= h^{-1}\left(F_1[\beta-1] - 2F_1[\beta] + F_1[\beta+1]\right), \quad \beta = 3, 4, ..., N-1, \\ \mathring{C}_1[N] &= h^{-1}\left(F_1[N-1] - F_1[N] + \frac{h}{2}\left(\pi - \sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right), \end{aligned}$$

if $\gamma = 2, 3, 4, ..., N-2$*:*

$$\begin{aligned} \mathring{C}_1[0] &= h^{-1}\left(F_1[1] - F_1[0] + \frac{h}{2}\left(\pi - \sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right), \\ \mathring{C}_1[\beta] &= h^{-1}\left(F_1[\beta-1] - 2F_1[\beta] + F_1[\beta+1]\right), \\ &\quad \beta = 1, 2, ..., \gamma-2 \quad and \quad \beta = \gamma+2, \gamma+3, ..., N-1, \\ \mathring{C}_1[\gamma-1] &= h^{-1}\left(F_1[\gamma-2] - 2F_1[\gamma-1] + \overline{F}_1[\gamma]\right), \\ \mathring{C}_1[\gamma] &= h^{-1}\left(F_1[\gamma-1] - 2\overline{F}_1[\gamma] + F_1[\gamma+1]\right), \\ \mathring{C}_1[\gamma+1] &= h^{-1}\left(\overline{F}_1[\gamma] - 2F_1[\gamma+1] + F_1[\gamma+2]\right), \\ \mathring{C}_1[N] &= h^{-1}\left(F_1[N-1] - F_1[N] + \frac{h}{2}\left(\pi - \sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right), \end{aligned}$$

if $\gamma = N-1$, i.e. for $t = 1-h$:

$$\begin{aligned}
\overset{\circ}{C}_1[0] &= h^{-1}\left(F_1[1]-F_1[0]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\overset{\circ}{C}_0[\gamma](h\gamma)\right)\right),\\
\overset{\circ}{C}_1[\beta] &= h^{-1}\left(F_1[\beta-1]-2F_1[\beta]+F_1[\beta+1]\right), \quad \beta=1,2,...,N-3,\\
\overset{\circ}{C}_1[N-2] &= h^{-1}\left(F_1[N-3]-2F_1[N-2]+\overline{F}_1[N-1]\right),\\
\overset{\circ}{C}_1[N-1] &= h^{-1}\left(F_1[N-2]-2\overline{F}_1[N-1]+F_1[N]\right),\\
\overset{\circ}{C}_1[N] &= h^{-1}\left(\overline{F}_1[N-1]-F_1[N]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\overset{\circ}{C}_0[\gamma](h\gamma)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\overline{F}_1[\beta] &= -\frac{1}{2}\left[-\sqrt{1-(h\beta-1)^2}+\left(t-2h\beta+2\right)\arcsin(h\beta-1)\right]\\
&\quad +\frac{h^2}{4}\sum_{\gamma=0}^{N}\overset{\circ}{C}_0(h\beta-h\gamma)^2\,\boldsymbol{sgn}(h\beta-h\gamma).
\end{aligned}$$

$\overset{\circ}{C}_0[\beta]$, $F_1[\beta]$, $\beta=0,1,2,...,N$ are given in Corollaries 2.3 and 2.4.

In the case $m=3$ we have the following results of the work [3] as immediate corollary of Theorem 2.1.

Corollary 2.6. *For $t \neq h\gamma-1$, coefficients of the optimal quadrature formula (1.3) with equally spaced nodes in the space $L_2^{(3)}(-1,1)$ have the following form*

$$\begin{aligned}
\overset{\circ}{C}_2[0] &= h^{-1}\left(F_2[1]-F_2[0]\right.\\
&\quad \left.+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\overset{\circ}{C}_0[\gamma](h\gamma-1)^2+2\overset{\circ}{C}_1[\gamma](h\gamma-1)\right)\right)\right),\\
\overset{\circ}{C}_2[\beta] &= h^{-1}\left(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\right), \quad \beta=\overline{1,N-1},\\
\overset{\circ}{C}_2[N] &= h^{-1}\left(F_2[N-1]-F_2[N]\right.\\
&\quad \left.+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\overset{\circ}{C}_0[\gamma](h\gamma-1)^2+2\overset{\circ}{C}_1[\gamma](h\gamma-1)\right)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
F_2[\beta] &= f_2[\beta]-\frac{h^3}{12}\sum_{\gamma=0}^{N}\overset{\circ}{C}_0[\gamma](h\beta-h\gamma)^3\,\boldsymbol{sgn}(h\beta-h\gamma)\\
&\quad +\frac{h^2}{4}\sum_{\gamma=0}^{N}\overset{\circ}{C}_1[\gamma](h\beta-h\gamma)^2\,\boldsymbol{sgn}(h\beta-h\gamma),
\end{aligned}$$

$$
\begin{aligned}
f_2[\beta] &= \frac{1}{12}\Bigg(\Big(2t-5(h\beta-1)\Big)\sqrt{1-(h\beta-1)^2} \\
&\quad -\Big(1+2t^2-6t(h\beta-1)+6(h\beta-1)^2\Big)\arcsin(h\beta-1) \\
&\quad -\frac{2(t-(h\beta-1))^3}{\sqrt{1-t^2}}\ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\Bigg).
\end{aligned}
$$

and $\mathring{C}_0[\beta], \mathring{C}_1[\beta], \quad \beta=\overline{0,N}$ *are given in Corollaries* 2.2 *and* 2.4.

Corollary 2.7. *For* $t=h\gamma-1$, *coefficients of the optimal quadrature formula* (1.3) *with equally spaced nodes in the space* $L_2^{(3)}(-1,1)$ *have the following form when* $\gamma=1$*:*

$$
\begin{aligned}
\mathring{C}_2[0] &= h^{-1}\left(\overline{F}_2[1]-F_2[0]\right. \\
&\quad \left.+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\right), \\
\mathring{C}_2[1] &= h^{-1}\left(F_2[0]-2\overline{F}_2[1]+F_2[2]\right), \\
\mathring{C}_2[2] &= h^{-1}\left(\overline{F}_2[1]-2F_2[2]+F_2[2]\right), \\
\mathring{C}_2[\beta] &= h^{-1}\left(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\right), \quad \beta=3,4,...,N-1, \\
\mathring{C}_2[N] &= h^{-1}\left(F_2[N-1]-F_2[N]\right. \\
&\quad \left.+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\right),
\end{aligned}
$$

when $\gamma=2,3,4,...,N-2$*:*

$$
\begin{aligned}
\mathring{C}_2[0] &= h^{-1}\left(F_2[1]-F_2[0]\right. \\
&\quad \left.+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\right), \\
\mathring{C}_2[\beta] &= h^{-1}\left(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\right), \quad \beta=1,2,...,\gamma-2, \\
\mathring{C}_2[\gamma-1] &= h^{-1}\left(F_2[\gamma-2]-2F_2[\gamma-1]+\overline{F}_2[\gamma]\right), \\
\mathring{C}_2[\gamma] &= h^{-1}\left(F_2[\gamma-1]-2\overline{F}_2[\gamma]+F_2[\gamma+1]\right), \\
\mathring{C}_2[\gamma+1] &= h^{-1}\left(\overline{F}_2[\gamma]-2F_2[\gamma+1]+F_2[\gamma+2]\right), \\
\mathring{C}_2[\beta] &= h^{-1}\left(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\right), \quad \beta=\gamma+2,\gamma+3,...,N-1, \\
\mathring{C}_2[N] &= h^{-1}\left(F_2[N-1]-F_2[N]\right. \\
&\quad \left.+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\right),
\end{aligned}
$$

when $\gamma = N-1$*:*

$$\begin{aligned}
\mathring{C}_2[0] &= h^{-1}\Big(F_2[1]-F_2[0] \\
&\quad +\frac{h}{4}\Big(\pi t-\sum_{\gamma=0}^{N}\Big(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\Big)\Big)\Big), \\
\mathring{C}_2[\beta] &= h^{-1}\left(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\right), \quad \beta=1,2,...,N-3, \\
\mathring{C}_2[N-2] &= h^{-1}\left(F_2[N-3]-2F_2[N-2]+\overline{F}_2[N-1]\right), \\
\mathring{C}_2[N-1] &= h^{-1}\left(F_2[N-2]-2\overline{F}_2[N-1]+F_2[N]\right), \\
\mathring{C}_2[N] &= h^{-1}\Big(\overline{F}_2[N-1]-F_2[N] \\
&\quad +\frac{h}{4}\Big(\pi t-\sum_{\gamma=0}^{N}\Big(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\Big)\Big)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\overline{F}_2[\beta] &= \overline{f}_2[\beta]-\frac{h^3}{12}\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\beta-h\gamma)^3\,\mathbf{sgn}(h\beta-h\gamma) \\
&\quad +\frac{h^2}{4}\sum_{\gamma=0}^{N}\mathring{C}_1[\gamma](h\beta-h\gamma)^2\,\mathbf{sgn}(h\beta-h\gamma),
\end{aligned}$$

$$\begin{aligned}
\overline{f}_2[\beta] &= \frac{1}{12}\Big(\Big(2t-5(h\beta-1)\Big)\sqrt{1-(h\beta-1)^2} \\
&\quad -\Big(1+2t^2-6t(h\beta-1)+6(h\beta-1)^2\Big)\arcsin(h\beta-1)\Big).
\end{aligned}$$

$\mathring{C}_0[\beta], \mathring{C}_1[\beta]$, $F_2[\beta]$, $\beta=0,1,2,...,N$ *are given in Corollaries* 2.3*,* 2.5 *and* 2.6

3. Numerical results

In this section we give some numerical results in order to show numerically convergence of the optimal quadrature formulas (1.3), with coefficients given in Theorem 2.1, in dependence on the values of N and m. Furthermore, here we compare numerical results of the quadrature formulas (1.3) with numerical results of the quadrature formula constructed in [11] in the space $L_2^{(1)}(-1,1)$.

Let us consider (1.2) and $f(t)=t^5+t^3+20t$. The corresponding exact solution of (1.1) is

$$\varphi(x)=\sqrt{1-x^2}\left(x^4+1.5x^2+\frac{167}{8}\right).$$

Tables 1-8 compare the exact solutions of singular integral equation in the form (1.1), the error rates of approximate solutions of quadrature formulas (16), (20), (21), (22) of the work [11], with the proposed (1.3), in which the approximate solutions of optimal quadrature formulas when $m=1$.

These tables show that the proposed method, when $m = 1$, outperforms the results of [11], four quadrature formulas proposed in two way approaches. Our proposed theorem is applicable for arbitrary m and N. This means that the proposed optimal quadrature formulas are exact for any polynomial of degree $(m-1)$. The error rates shown in Table 9 show that the proposed method, when $N = 20$, $m = 1$, $m = 2$, and $m = 3$ in singular integral equations, confirms the previous statement. The combination of the results illustrated in Table 9 and constructed optimal quadrature formulas by increasing N and m, allows the approximate calculations of the Fredholm singular integral equation of the first kind with high accuracy.

4. Conclusion

In the present paper, in the Sobolev space $L_2^{(m)}(-1,1)$ we constructed the optimal quadrature formula for approximate solution of singular integral equations with Cauchy kernel. Here we found analytical forms for coefficients of the constructed optimal quadrature formulas. We applied these coefficients to approximate solution of the Fredholm singular integral equation of the first kind. We showed that singular integral equations can be solved with higher accuracy using the optimal quadrature formulas which are constructed based on Sobolev method.

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Table 1. Error terms for OQF (1.3).

N=20				
$t \neq h\gamma - 1$	Exact	Error QF(16)in [11]	OQF(1.3), m=1	Error
-0.887	10.4702332992	0.0417283096	10.4705320346	0.0002987354
-0.695	15.6980321754	0.0897051922	15.7084204293	0.0103882539
-0.495	18.5096575257	0.0818203672	18.5217620154	0.0121044897
-0.293	20.0890157504	0.0788584125	20.1010005193	0.0119847689
-0.095	20.7941454186	0.0915690660	20.8061508094	0.0120053908
0.095	20.7941454186	0.0915691610	20.8061508094	0.0120053908
0.293	20.0890157504	0.0788588154	20.1010005193	0.0119847689
0.495	18.5096575257	0.0818187549	18.5217620154	0.0121044897
0.695	15.6980321754	0.0897054275	15.7084204293	0.0103882539
0.887	10.4702332992	0.0417277253	10.4705320346	0.0002987354

Table 2. Error terms for OQF (1.3).

N=200				
$t \neq h\gamma - 1$	Exact	Error QF(16)in [11]	OQF(1.3), m=1	Error
-0.987	3.7424126959	0.0096756670	3.7423362651	0.0000764308
-0.935	8.1393802207	0.0045964403	8.1394671385	0.0000869179
-0.887	10.4702332992	0.0091676216	10.4702698056	0.0000365064
-0.695	15.6980321754	0.0026300570	15.6981798537	0.0001476783
-0.495	18.5096575257	0.0054628196	18.5098039531	0.0001464275
-0.293	20.0890157504	0.0061221725	20.0891616458	0.0001458954
-0.095	20.7941454186	0.0069000509	20.7942755288	0.0001301102
0.095	20.7941454186	0.0069001771	20.7942755288	0.0001301102
0.293	20.0890157504	0.0061240901	20.0891616457	0.0001458954
0.495	18.5096575257	0.0054671436	18.5098039531	0.0001464275
0.695	15.6980321754	0.0026330865	15.6981798537	0.0001476783
0.887	10.4702332992	0.0091552088	10.4702698056	0.0000365064
0.935	8.1393802206	0.0046156414	8.1394671385	0.0000869179
0.987	3.7424126959	0.0096747715	3.7423362651	0.0000764308

Table 3. Error terms for OQF (1.3).

N=20				
$t = h\gamma - 1$	Exact	Error QF(20)in [11]	OQF(1.3), m=1	Error
-0.9	9.9147951260	0.0434208684	9.9246302672	0.0098351412
-0.7	15.6040925306	0.0454488972	15.6182823480	0.0141898173
-0.5	18.4571664181	0.0694506386	18.4712004187	0.0140340005
-0.3	20.0499895293	0.0788768819	20.0629747128	0.0129851834
-0.1	20.7853870599	0.0814265589	20.7976199210	0.0122328611
0.1	20.7853870599	0.0814266037	20.7976199210	0.0122328611
0.3	20.0499895293	0.0788768802	20.0629747128	0.0129851834
0.5	18.4571664182	0.0694507351	18.4712004187	0.0140340005
0.7	15.6040925307	0.0454491744	15.6182823480	0.0141898173
0.9	9.9147951260	0.0434206055	9.9246302673	0.0098351412

Table 4. Error terms for OQF (1.3).

N=20				
$t = h\gamma - 1$	Exact	Error QF(20)in [11]	OQF(1.3), m=1	Error
-0.98	4.6242972766	0.0058168646	4.6243374562	0.0000401796
-0.96	6.4698890368	0.0053701073	6.4699564868	0.0000674500
-0.94	7.8405806772	0.0047221016	7.8406659308	0.0000852536
-0.90	9.9147951260	0.0032612171	9.9149040083	0.0001088823
-0.70	15.6040925306	0.0025401403	15.6042411554	0.0001486248
-0.50	18.4571664182	0.0054238405	18.4573137570	0.0001473389
-0.30	20.0499895293	0.0065655622	20.0501270539	0.0001375246
-0.10	20.7853870599	0.0068972439	20.7855174868	0.0001304269
0.10	20.7853870599	0.0068972228	20.7855174868	0.0001304269
0.30	20.0499895293	0.0065654914	20.0501270540	0.0001375247
0.50	18.4571664182	0.0054241103	18.4573137571	0.0001473389
0.70	15.6040925307	0.0025399121	15.6042411554	0.0001486248
0.90	9.9147951260	0.0032614564	9.9149040083	0.0001088823
0.94	7.8405806772	0.0047223289	7.8406659308	0.0000852536
0.96	6.4698890368	0.0053701392	6.4699564868	0.0000674500
0.98	4.6242972766	0.0058168227	4.6243374562	0.0000401796

Table 5. Error terms for OQF (1.3).

N=20				
t	Exact	Error QF(21)in [11]	OQF(1.3), m=1	Error
-0.9999	0.3305542576	0.0016697528	0.3288666955	0.0016875620
-0.9980	1.4767423383	0.0070573417	1.4696058654	0.0071364729
-0.9450	7.5265525954	0.0073648393	7.5337473788	0.0071947834
-0.9150	9.2115705777	0.0189073179	9.2305812184	0.0190106407

Table 6. Error terms for OQF (1.3).

N=200				
t	Exact	Error QF(21)in [11]	OQF(1.3), m=1	Error
-0.9999	0.3305542576	0.0001051563	0.3304914855	0.0000627721
-0.999	1.0447877703	0.0002953748	1.0446330797	0.0001546905
-0.998	1.4767423383	0.0003658651	1.4765870900	0.0001552483
-0.997	1.8076410236	0.0003931256	1.8075212773	0.0001197463
-0.995	2.3310993510	0.0004063873	2.3310939480	0.0000054030
-0.993	2.7551786054	0.0004410670	2.7552845634	0.0001059580
-0.991	3.1206688242	0.0005965699	3.1208008580	0.0001320338

Table 7. Error terms for OQF (1.3).

N=20				
t	Exact	Error QF(22)in [11]	OQF(1.3), m=1	Error
0.9999	0.3305542576	0.0016696240	0.3288666955	0.0016875620
0.9980	1.4767423383	0.0070573629	1.4696058654	0.0071364729
0.9450	7.5265525954	0.0073644315	7.5337473788	0.0071947834
0.9150	9.2115705777	0.0189067192	9.2305812184	0.0190106407

Table 8. Error terms for OQF (1.3).

N=200				
t	Exact	Error QF(22)in [11]	OQF(1.3), m=1	Error
0.9999	0.3305542576	0.0001051332	0.3304914855	0.0000627721
0.999	1.0447877703	0.0002953269	1.0446330797	0.0001546905
0.998	1.4767423383	0.0003656582	1.4765870900	0.0001552483
0.997	1.8076410236	0.0003931503	1.8075212773	0.0001197463
0.995	2.3310993510	0.0004062447	2.3310939480	0.0000054030
0.993	2.7551786054	0.0004408404	2.7552845634	0.0001059580
0.991	3.1206688242	0.0005964761	3.1208008580	0.0001320338

Table 9. Error terms for OQF (1.3).

N=20				
t	Exact	Error of OQF $m = 1$	Error of OQF $m = 2$	Error of OQF $m = 3$
-0.887	10.4702332992	0.0002987354	0.0001764502	0.0000053844
-0.695	15.6980321754	0.0103882539	0.0002530201	0.0000066140
-0.495	18.5096575257	0.0121044897	0.0001493195	0.0000104882
-0.293	20.0890157504	0.0119847689	0.0000694601	0.0000126264
-0.095	20.7941454186	0.0120053908	0.0000449824	0.0000149919
0.095	20.7941454186	0.0120053908	0.0000449824	0.0000149919
0.293	20.0890157504	0.0119847689	0.0000694601	0.0000126264
0.495	18.5096575257	0.0121044897	0.0001493195	0.0000104882
0.695	15.6980321754	0.0103882539	0.0002530201	0.0000066140
0.887	10.4702332992	0.0002987354	0.0001764502	0.0000053844

References

[1] Akhmedov, D.M., *Optimal quadrature formulas for the Cauchy type singular integral in the Sobolev space* $L_2^{(2)}(0,1)$, Uzbek Mathematical Journal, **2**(2011), 48-56.

[2] Akhmedov, D.M., *Optimal quadrature formulas of Hermite type for calculating singular integrals with weight functions*, (in Russian), Uzbek Mathematical Journal, **3**(2012), 31-42.

[3] Akhmedov, D.M., *Numerical integration of singular integrals of Cauchy type*, (in Russian), Uzbek Mathematical Journal, **2**(2014), 3-16.

[4] Akhmedov, D.M., Hayotov, A.R., Shadimetov, Kh.M., *Optimal quadrature formulas with derivatives for Cauchy type singular integrals*, Applied Mathematics and Computation, **317**(2018), 150-159.

[5] Belotserkovskii, S.M., Lifanov, I.K., *Numerical Methods in Singular Integral Equations*, (in Russian), Nauka, Moscow, 1985.

[6] Blaga, P., Coman, Gh., *Some problems on optimal quadrature*, Stud. Univ. Babeş-Bolyai Math., **52**(2007), no. 4, 21-44.

[7] Cătinaş, T., Coman, Gh., *Optimal quadrature formula based on* ϕ*-function method*, Stud. Univ. Babeş-Bolyai Math., **52**(2006), no. 1, 49-64.

[8] Dagninno, C., Santi, E., *On the convergance of spline product quadrature formulas for Cauchy principal value integrals*, Journal of Computational and Applied Mathematics, **36**(1991), 181-187.

[9] Diethelm, K., *Gaussian quadrature formulas of the first kind for Cauchy principal value integrals: Basic properties an error estimates*, Journal of Computational and Applied Mathematics, **65**(1995), 97-114.

[10] Eshkuvatov, Z.K., *Modification of quadrature formulas of Belotserkovsky for weight singular integrals*, (in Russian), Uzbek Mathematical Journal, **5**(1998), 91-100.

[11] Eshkuvatov, Z.K., Nik Long, N.M.A., Abdulkawi, M., *Quadrature formula for approximating the singular integral of Cauchy type with unbounded weight function on the edges*, Journal of Computational and Applied Mathematics, **233**(2009), no. 8, 334-345.

[12] Gakhov, F.D., *Boundary problems*, (in Russian), Nauka, Moscow, 1977.

[13] Iovane, G.F., Lifanov, I.K., Sumbatyan, M.A., *On direct numerical treatment of hypersingular integral equations arising in mechanics and acoustics*, Acta Mech., **162**(2003), 99-110.

[14] Isroilov, M.I., *Approximate-analytical solution of singular integral equation of first kind using quadrature formula*, (in Russian), in: Numerical Integration and Adjacent Problems. Colecction of Articles Academy of Republic of Uzbekistan, Press FAN, Tashkent, (1990), 7-23.

[15] Israilov, M.I., Shadimetov, Kh.M., *Weight optimal quadrature formulas for singular integrals of the Cauchy type*, Doklady AN RUz, **8**(1991), 10-11.

[16] Lifanov, I.K., *The method of singular equations and numerical experiments*, (in Russian), TOO Yanus, Moscow, 1995.

[17] Lifanov, I.K., *Singular Integral Equation and Discrete Vortices*, VSP, Utrecht, Netherlands, 1996.

[18] Lifanov, I.K., Polonskii, I.E., *Proof of the numerical method of discrete vortices for solving singular integral equations*, PMM, **39**(1975), no. 4, 742-746.

[19] Muskhelishvili, N.I., *Singular Integral Equations*, (in Russian), Nauka, Moscow, 1968.

[20] Sard, A., *Best approximate integration formulas; best approximation formulas*, Amer. J. Math., **71**(1949), 80-91.

[21] Shadimetov, Kh.M., *Optimal quadrature formulas for singular integrals of the Cauchy type*, (in Russian), Doklady AN RUz, **6**(1987), 9-11.

[22] Shadimetov, Kh.M., *The discrete analogue of the operator and its properties*, Voprosy Vichislitelnoy i Prikladnoy Matematiki, Tashkent, 1985, 22-35.

[23] Shadimetov, Kh.M., *On an extremal function of quadrature formulas*, Doklady AN RUz, **9-10**(1995), 3-5.

[24] Shadimetov, Kh.M., *Optimal lattice quadrature and cubature formulas in Sobolev spaces*, Fan va Texnologiya, Tashkent, (2019), 218.

[25] Shadimetov, Kh.M., Hayotov, A.R., Akhmedov, D.M., *Optimal quadrature formulas for the Cauchy type singular integral in the Sobolev space* $L_2^{(2)}(-1,1)$, American Journal of Numerical Analysis, **1**(2013), no. 1, 22-31.

[26] Shadimetov, Kh.M., Hayotov, A.R., Akhmedov, D.M., *Optimal quadrature formulas for Cauchy type singular integrals in Sobolev space*, Applied Mathematics and Computation, **263**(2015), 302-314.

[27] Sobolev, S.L., *Introduction to the Theory of Cubature Formulas*, (in Russian), Nauka, Moscow, 1974.

[28] Sobolev, S.L., Vaskevich, V.L., *The Theory of Cubature Formulas*, Kluwer Academic Publishers Group, Dordrecht, 1997.

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