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Porosity-based methods for solving stochastic feasibility problems

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. The notion of porosity is well known in Optimization and Nonlinear Analysis. Its importance is brought out by the fact that the complement of a σ-porous subset of a complete pseudo-metric space is a residual set, while the existence of the latter is essential in many problems which apply the generic approach. Thus, under certain circumstances, some refinements of known results can be achieved by looking for porous sets. In 2001 Gabour, Reich and Zaslavski developed certain generic methods for solving stochastic feasibility problems. This topic was further investigated in 2021 by Barshad, Reich and Zaslavski, who provided more general results in the case of unbounded sets. In the present paper we introduce and examine new generic methods that deal with the aforesaid problems, in which, in contrast with previous studies, we consider sigma-porous sets instead of meager ones.

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1. Introduction and background

We consider (generalized) stochastic feasibility problems from the point of view of the generic approach (for more applications of this approach, see, for example, [7]). These are the problems of finding almost common fixed points of measurable (with respect to a probability measure) families of mappings. Namely, we provide generic methods for finding almost common fixed points by using the notion of porosity. Our results are applicable to both the consistent case (that is, the case where the aforesaid

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almost common fixed points exist) and the inconsistent case (that is, the case where there are no common fixed points at all).

We begin by recalling the definitions of porosity and local convexity.

Given a pseudo-metric space (Y, ρ), we denote by $B_\rho(y, r)$, for each $y \in Y$ and $r > 0$, the open ball in (Y, ρ) of center y and radius r. Recall that a subset E of a complete pseudo-metric space (Y, ρ) is called a *porous* subset of Y if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists a point $z \in Y$ for which

$$B_\rho(z, \alpha r) \subset B_\rho(y, r) \setminus E.$$

A subset of Y is called a *σ-porous* subset of Y if it is a countable union of porous subsets of Y. Note that since a porous set is nowhere dense, any σ-porous set is of the first category and hence its complement is residual in (Y, ρ), that is, it contains a countable intersection of open and dense subsets of (Y, ρ). For this reason, there is a considerable interest in σ-porous sets while searching for generic solutions to optimization problems. More information concerning the notion of porosity and its applications can be found, for example, in [3], [6], [7] and [8].

Recall that a topological vector space V with the topology T is said to be a *locally convex space* if there exists a family $\mathscr{P}$ of pseudo-norms on V such that the family of open balls $\{B_\rho(x_0, \varepsilon) : x_0 \in V, \varepsilon > 0, \rho \in \mathscr{P}\}$ is a subbasis for T and $\cap_{\rho \in \mathscr{P}} Z_\rho = \{0\}$, where $Z_\rho = \{x \in V : \rho(x) = 0\}$ for each $\rho \in \mathscr{P}$. Clearly, every normed space (as a topological vector space with respect to its norm) is a locally convex space. In the sequel we use the following result (see Theorem 3.9 in [2]).

Theorem 1.1. *Let V be a real locally convex topological vector space, and let A and B be two disjoint closed and convex subsets of V. If either A or B is compact, then A and B are strictly separated, that is, there is $\alpha \in \mathbb{R}$ and a continuous linear functional $\phi : V \to \mathbb{R}$ such that $\phi(a) > \alpha$ for each $a \in A$ and $\phi(b) < \alpha$ for each $b \in B$.*

Now we introduce the spaces for which we investigate the stochastic feasibility problem. Other spaces which can be considered regarding this problem, can be found, for example, in [1] and [5].

Suppose that $(X, \|\cdot\|)$ is a normed vector space with norm $\|\cdot\|$, F is a nonempty, closed, convex and bounded subset of X, $(\Omega, \mathcal{A}, \mu)$ is a probability measure space (more information on measure spaces and measurable mappings can be found, for example, in [3]) and K is a subset of X which contains F. Denote by $\mathcal{N}$ the set of all nonexpansive mappings $A : K \to F$, that is, all mappings $A : K \to F$ such that $\|Ax - Ay\| \le \|x - y\|$ for each $x, y \in K$. For the set $\mathcal{N}$, define a metric $\rho_{\mathcal{N}} : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ by

$$\rho_{\mathcal{N}}(A, B) := \sup\{\|Ax - Bx\| : x \in K\}, \ A, B \in \mathcal{N}.$$

Clearly, the metric space $(\mathcal{N}, \rho_{\mathcal{N}})$ is complete if $(X, \|\cdot\|)$ is a Banach space.

Denote by $\mathcal{N}_\Omega$ the set of all mappings $T : \Omega \to \mathcal{N}$ such that for each $x \in K$, the mapping $T'_x : \Omega \to F$, defined, for each $\omega \in \Omega$, by $T'_x(\omega) := T(\omega)(x)$, is measurable. It is not difficult to see that if $T \in \mathcal{N}_\Omega$, then T'_x is integrable on Ω. For each $T \in \mathcal{N}_\Omega$, define an operator $\widetilde{T} : K \to F$ by $\widetilde{T}x = \int_\Omega T'_x(\omega)\, d\mu(\omega)$ for each $x \in K$. By Theorem 1.1, this is indeed a mapping the image of which is contained in F. Note that the mapping defined on $\mathcal{N}_\Omega$ by $T \mapsto \widetilde{T}$ is onto $\mathcal{N}$. Clearly, for each $T \in \mathcal{N}_\Omega$, we have $\widetilde{T} \in \mathcal{N}$. Thus we consider the topology defined by the following pseudo-metric on $\mathcal{N}_\Omega$:

$$\rho_{\mathcal{N}_\Omega}(T,S) := \rho_{\mathcal{N}}\left(\widetilde{T},\widetilde{S}\right),\ T,S \in \mathcal{N}_\Omega.$$

It is not difficult to see that the pseudo-metric space $(\mathcal{N}_\Omega, \rho_{\mathcal{N}_\Omega})$ is complete if $(X, \|\cdot\|)$ is a Banach space.

Denote by $\mathcal{M}_\Omega$ the set of all sequences $\{T_n\}_{n=1}^\infty \subset \mathcal{N}_\Omega$. We define a pseudo-metric $\rho_{\mathcal{M}_\Omega} : \mathcal{M}_\Omega \times \mathcal{M}_\Omega \to \mathbb{R}$ on $\mathcal{N}_\Omega$ in the following way:

$$\rho_{\mathcal{M}_\Omega}\left(\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty\right) := \sup\left\{\rho_{\mathcal{N}_\Omega}(T_n,S_n) :\ n = 1,2\dots\right\},$$
$$\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty \in \mathcal{M}_\Omega.$$

Obviously, this space is complete if $(X, \|\cdot\|)$ is a Banach space.

The rest of the paper is organized as follows. In Section 2 we state our main results. Two auxiliary assertions are presented in Section 3. In Section 4 we provide the proofs of our main results.

In all our results we also assume that $(X, \|\cdot\|)$ is a Banach space.

2. Statements of the main results

In this section we state our main results. We establish them in Section 4 below.

Recall that for each $T \in \mathcal{N}_\Omega$, a point $x \in K$ is an *almost common fixed point* of the family $\{T(\omega)\}_{\omega\in\Omega}$ if $T(\omega)x = x$ for almost all $\omega \in \Omega$. Similarly, for each sequence $\{T_n\}_{n=1}^\infty \in \mathcal{M}_\Omega$, a point $x \in K$ is an *almost common fixed point* of the family $\{T_n(\omega)\}_{\omega\in\Omega,\, n=1,2\dots}$ if $T_n(\omega)x = x$ for all $n = 1,2,\dots$ and almost all $\omega \in \Omega$.

Theorem 2.1. *There exists a set $\mathcal{F} \subset \mathcal{M}_\Omega$ such that $\mathcal{M}_\Omega \setminus \mathcal{F}$ is a σ-porous subset of $\mathcal{M}_\Omega$ and for each $\{T_n\}_{n=1}^\infty \in \mathcal{F}$, the following assertion holds true:*

For each $\varepsilon > 0$, there is a positive integer N such that for each integer $n \geq N$ and each mapping $s : \{1,2,\dots\} \to \{1,2,\dots\}$, we have

$$\left\|\widetilde{T_{s(n)}}\dots\widetilde{T_{s(1)}}x - \widetilde{T_{s(n)}}\dots\widetilde{T_{s(1)}}y\right\| < \varepsilon$$

for each $x, y \in K$. Consequently, if there is an almost common fixed point of the family $\{T_n(\omega)\}_{\omega\in\Omega,\, n=1,2\dots}$, then it is unique and for each $x \in K$, the sequence $\left\{\widetilde{T_{s(n)}}\dots\widetilde{T_{s(1)}}x\right\}_{n=1}^\infty$ converges to it as $n \to \infty$, uniformly on K, for each mapping $s : \{1,2,\dots\} \to \{1,2,\dots\}$.

Theorem 2.2. *There exists a set $\mathcal{F} \subset \mathcal{N}_\Omega$ such that the set $\mathcal{G} := \mathcal{N}_\Omega \setminus \mathcal{F}$ a σ-porous subset of $\mathcal{N}_\Omega$, and for each $T \in \mathcal{F}$, the following assertion holds true:*

There exists $x_T \in K$ which is the unique fixed point of the operator $\widetilde{T}$ such that for each $x \in K$, the sequence $\left\{\widetilde{T}^n x\right\}_{n=1}^\infty$ converges to x_T as $n \to \infty$, uniformly on K. Moreover, the set $\mathfrak{F}$ of all almost common fixed points of the family $\{T(\omega)\}_{\omega\in\Omega}$ is contained in $\{x_T\}$. As a result, if $\mathfrak{F} \neq \emptyset$, then x_T is the unique almost common fixed point of the family $\{T(\omega)\}_{\omega\in\Omega}$.

3. Auxiliary results

In this section we present two lemmata which will be used in the proofs of our main results. We start by defining three sequences which we use in the proofs of these lemmata.

Choose $z_0 \in F$ and set $r_0 = 1$. We first define the sequence $\{\alpha_k\}_{k=1}^{\infty}$ of positive numbers by

$$\alpha_k = 2^{-1}\left(1 + 2k\left(2\sup_{z\in F}\|z\| + 1\right)\right)^{-1} \in (0,1)\,. \tag{3.1}$$

Clearly, for each positive integer k,

$$(1-\alpha_k)\left(2\sup_{z\in F}\|z\| + 1\right)^{-1} \in (0,1) \tag{3.2}$$

and

$$(1-\alpha_k)\left(2\sup_{z\in F}\|z\| + 1\right)^{-1} - 2\alpha_k k = 2^{-1}\left(2\sup_{z\in F}\|z\| + 1\right)^{-1} > 0. \tag{3.3}$$

Using (3.3), for each $r \in (0, r_0]$, we choose sequences $\{\gamma_k^r\}_{k=1}^{\infty}$ and $\{N_k^r\}_{k=1}^{\infty}$ of positive numbers such that

$$\gamma_k^r \in \left(2\alpha_k k r, (1-\alpha_k)\, r\left(2\sup_{z\in F}\|z\| + 1\right)^{-1}\right) \tag{3.4}$$

and

$$N_k^r > 2\left(\gamma_k^r k^{-1} - 2\alpha_k r\right)^{-1}\sup_{z\in F}\|z\| + 1 \tag{3.5}$$

for each positive integer k. Evidently, by(3.1), (3.2) and (3.4), $\gamma_k^r \in (0,1)$.

Lemma 3.1. *Assume that k is a positive integer and let $\mathcal{F}_k$ be the set of all sequences $\{T_n\}_{n=1}^{\infty} \in \mathcal{M}_\Omega$ for which there exists a positive integer N such that for each mapping $s : \{1,2,\dots\} \to \{1,2,\dots\}$, we have*

$$\left\|\widetilde{T_{s(N)}}\dots\widetilde{T_{s(1)}}x - \widetilde{T_{s(N)}}\dots\widetilde{T_{s(1)}}y\right\| < k^{-1}$$

for each $x, y \in K$. Then the set $\mathcal{G}_k := \mathcal{M}_\Omega\backslash\mathcal{F}_k$ is a porous subset of $\mathcal{M}_\Omega$.

Proof. Assume that $\{T_n\}_{n=1}^{\infty} \in \mathcal{M}_\Omega$ and $r \in (0, r_0]$. Define a sequence of mappings $\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty}$, $T_n^{\gamma_k^r} : \Omega \to \mathcal{N}$, by

$$T_n^{\gamma_k^r}(\omega)\, x := (1-\gamma_k^r)\, T_n(\omega)\, x + \gamma_k^r z_0,\ n = 1,2,\dots$$

for each $\omega \in \Omega$ and each $x \in K$. Clearly, $\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty} \in \mathcal{M}_\Omega$ and for each $n = 1,2,\dots$,

$$\begin{aligned}\widetilde{T_n^{\gamma_k^r}}x &= \int_\Omega \left((1-\gamma_k^r)\, T_n(\omega)\, x + \gamma_k^r z_0\right) d\mu(\omega) = \gamma_k^r z_0 + (1-\gamma_k^r)\int_\Omega T_n(\omega)\, x d\mu(\omega)\\ &= (1-\gamma_k^r)\,\widetilde{T_n}x + \gamma_k^r z_0\end{aligned}$$

for each $x \in K$. We have

$$\rho_{\mathcal{M}_\Omega}\left(\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty}, \{T_n\}_{n=1}^{\infty}\right) \le 2\gamma_k^r \sup_{z\in F}\|z\|\,, \tag{3.6}$$

as well as, for each positive integer n,

$$\left\| \widetilde{T_n^{\gamma_k^r}} x - \widetilde{T_n^{\gamma_k^r}} y \right\| \leq (1 - \gamma_k^r) \|x - y\| \tag{3.7}$$

for each $x, y \in K$.
Let $\{S_n\}_{n=1}^{\infty} \in \mathcal{M}_\Omega$ satisfy

$$\rho_{\mathcal{M}_\Omega} \left(\left\{ T_n^{\gamma_k^r} \right\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty} \right) < \alpha_k r. \tag{3.8}$$

Assume that $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$ is an arbitrary mapping. We claim that

$$\left\| \widetilde{S_{s\left(N_k^r\right)}} \dots \widetilde{S_{s(1)}} x - \widetilde{S_{s\left(N_k^r\right)}} \dots \widetilde{S_{s(1)}} y \right\| < k^{-1} \tag{3.9}$$

for each $x, y \in K$. Suppose to the contrary that this does not hold. Then there exist points $x_0, y_0 \in K$ such that for each $i = 0 \dots N_k^r$, we have

$$\left\| \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}} y_0 \right\| \geq k^{-1}. \tag{3.10}$$

Using the triangle inequality, (3.8), (3.7) and (3.10), we obtain that for each $i = 1 \dots N_k^r$,

$$\begin{aligned}
&\left\| \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}} y_0 \right\| \\
&\leq \left\| \widetilde{S_{s(i)}} \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{T_{s(i)}^{\gamma_k^r}} \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1}} x_0 \right\| \\
&+ \left\| \widetilde{T_{s(i)}^{\gamma_k^r}} \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{T_{s(i)}^{\gamma_k^r}} \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} y_0 \right\| \\
&+ \left\| \widetilde{T_{s(i)}^{\gamma_k^r}} \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} y_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}} y_0 \right\| \\
&< 2\alpha_k r + (1 - \gamma_k^r) \left\| \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} y_0 \right\| \\
&\leq \left\| \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} y_0 \right\| + 2\alpha_k r - \gamma_k^r k^{-1}.
\end{aligned}$$

Hence by (3.4), we have

$$\begin{aligned}
\left\| \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} y_0 \right\| - \left\| \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}} y_0 \right\| \\
> \gamma_k^r k^{-1} - 2\alpha_k r > 0
\end{aligned}$$

for each $i = 1 \dots N_k^r$. Therefore

$$\begin{aligned}
2 \sup_{z \in F} \|z\| &\geq \left\| \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s(1)}} y_0 \right\| - \left\| \widetilde{S_{s\left(N_k^r\right)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s\left(N_k^r\right)}} \dots \widetilde{S_{s(1)}} y_0 \right\| \\
&= \Sigma_{i=2}^{N_k^r} \left(\left\| \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}} y_0 \right\| \right. \\
&\left. - \left\| \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}} x_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}} y_0 \right\| \right) > (N_k^r - 1) \left(\gamma_k^r k^{-1} - 2\alpha_k r \right).
\end{aligned}$$

As a result,

$$N_k^r < 2 \left(\gamma_k^r k^{-1} - 2\alpha_k r \right)^{-1} \sup_{z \in F} \|z\| + 1.$$

This, however, contradicts (3.5). Thus (3.9) does hold. Next, using the triangle inequality, we see by (3.6), (3.8) and (3.4) that

$$\begin{aligned}\rho_{\mathcal{M}_\Omega}\left(\{T_n\}_{n=1}^{\infty},\{S_n\}_{n=1}^{\infty}\right) &\leq \rho_{\mathcal{M}_\Omega}\left(\{T_n\}_{n=1}^{\infty},\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty}\right)\\ &+\rho_{\mathcal{M}_\Omega}\left(\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty},\{S_n\}_{n=1}^{\infty}\right)\\ &< 2\gamma_k^r \sup_{z\in F}\|z\| + \alpha_k r < (1-\alpha_k)\, r + \alpha_k r = r. \end{aligned} \tag{3.11}$$

From (3.9) and (3.11) it now follows that

$$B_{\rho_{\mathcal{M}_\Omega}}\left(\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty},\alpha_k r\right) \subset B_{\rho_{\mathcal{M}_\Omega}}\left(\{T_n\}_{n=1}^{\infty},r\right)\cap \mathcal{F}_k = B_{\rho_{\mathcal{M}_\Omega}}\left(\{T_n\}_{n=1}^{\infty},r\right)\backslash \mathcal{G}_k.$$

Hence $\mathcal{G}_k$ is indeed a porous subset of $\mathcal{M}_\Omega$, as asserted. □

Lemma 3.2. *Assume that k is a positive integer and let $\mathcal{F}_k$ be the set of all mappings $T \in \mathcal{N}_\Omega$ for which there exists a positive integer N such that*

$$\left\|\widetilde{T}^N x - \widetilde{T}^N y\right\| < k^{-1}$$

for each $x, y \in K$. Then the set $\mathcal{G}_k := \mathcal{N}_\Omega \backslash \mathcal{F}_k$ is a porous subset of $\mathcal{N}_\Omega$.

Proof. Assume that $T \in \mathcal{N}_\Omega$ and $r \in (0, r_0]$. Define a mapping $T_{\gamma_k^r}$, $T_{\gamma_k^r} : \Omega \to \mathcal{N}$, by

$$T_{\gamma_k^r}(\omega)\, x := (1-\gamma_k^r)\, T(\omega)\, x + \gamma_k^r z_0$$

for each $\omega \in \Omega$ and each $x \in K$. Clearly, $T_{\gamma_k^r} \in \mathcal{N}_\Omega$ and

$$\begin{aligned}\widetilde{T_{\gamma_k^r}} x &= \int_\Omega \left((1-\gamma_k^r)\, T(\omega)\, x + \gamma_k^r z_0\right) d\mu(\omega) = \gamma_k^r z_0 + (1-\gamma_k^r)\int_\Omega T(\omega)\, x d\mu(\omega)\\ &= (1-\gamma_k^r)\, \widetilde{T} x + \gamma_k^r z_0\end{aligned}$$

for each $x \in K$. We have

$$\rho_{\mathcal{N}_\Omega}\left(T_{\gamma_k^r}, T\right) \leq 2\gamma_k^r \sup_{z\in F}\|z\|, \tag{3.12}$$

as well as

$$\left\|\widetilde{T_{\gamma_k^r}} x - \widetilde{T_{\gamma_k^r}} y\right\| \leq (1-\gamma_k^r)\, \|x-y\| \tag{3.13}$$

for each $x, y \in K$.
Let $S \in \mathcal{N}_\Omega$ satisfy

$$\rho_{\mathcal{N}_\Omega}\left(T_{\gamma_k^r}, S\right) < \alpha_k r. \tag{3.14}$$

We claim that

$$\left\|\widetilde{S}^{N_k^r} x - \widetilde{S}^{N_k^r} y\right\| < k^{-1} \tag{3.15}$$

for each $x, y \in K$. Suppose to the contrary that this does not hold. Then there exist points $x_0, y_0 \in K$ such that for each $i = 0 \dots N_k^r$, we have

$$\left\|\widetilde{S}^i x_0 - \widetilde{S}^i y_0\right\| \geq k^{-1}. \tag{3.16}$$

Using the triangle inequality, (3.14), (3.13) and (3.16), we see that for each $i = 1 \dots N_k^r$,

$$\begin{aligned}\left\|\widetilde{S}^i x_0 - \widetilde{S}^i y_0\right\| &\leq \left\|\widetilde{S}\widetilde{S}^{i-1} x_0 - \widetilde{T_{\gamma_k^r}}\widetilde{S}^{i-1} x_0\right\| \\ &+ \left\|\widetilde{T_{\gamma_k^r}}\widetilde{S}^{i-1} x_0 - \widetilde{T_{\gamma_k^r}}\widetilde{S}^{i-1} y_0\right\| + \left\|\widetilde{T_{\gamma_k^r}}\widetilde{S}^{i-1} y_0 - \widetilde{S}\widetilde{S}^{i-1} y_0\right\| \\ &< 2\alpha_k r + (1-\gamma_k^r)\left\|\widetilde{S}^{i-1} x_0 - \widetilde{S}^{i-1} y_0\right\| \\ &\leq \left\|\widetilde{S}^{i-1} x_0 - \widetilde{S}^{i-1} y_0\right\| + 2\alpha_k r - \gamma_k^r k^{-1}.\end{aligned}$$

Hence by (3.4),

$$\left\|\widetilde{S}^{i-1} x_0 - \widetilde{S}^{i-1} y_0\right\| - \left\|\widetilde{S}^i x_0 - \widetilde{S}^i y_0\right\| > \gamma_k^r k^{-1} - 2\alpha_k r > 0,$$

for each $i = 1 \dots N_k^r$. Therefore

$$\begin{aligned}2\sup_{z\in F}\|z\| &\geq \left\|\widetilde{S} x_0 - \widetilde{S} y_0\right\| - \left\|\widetilde{S}^{N_k^r} x_0 - \widetilde{S}^{N_k^r} y_0\right\| \\ &= \Sigma_{i=2}^{N_k^r}\left(\left\|\widetilde{S}^{i-1} x_0 - \widetilde{S}^{i-1} y_0\right\| - \left\|\widetilde{S}^i x_0 - \widetilde{S}^i y_0\right\|\right) \\ &> (N_k^r - 1)\left(\gamma_k^r k^{-1} - 2\alpha_k r\right).\end{aligned}$$

As a result,

$$N_k^r < 2\left(\gamma_k^r k^{-1} - 2\alpha_k r\right)^{-1}\sup_{z\in F}\|z\| + 1.$$

This, however, contradicts (3.5). Thus (3.15) does hold. Next, using the triangle inequality, we see by (3.12), (3.14) and (3.4) that

$$\begin{aligned}\rho_{\mathcal{N}_\Omega}(T,S) &\leq \rho_{\mathcal{N}_\Omega}\left(T, T_{\gamma_k^r}\right) + \rho_{\mathcal{N}_\Omega}\left(T_{\gamma_k^r}, S\right) \\ &< 2\gamma_k^r \sup_{z\in F}\|z\| + \alpha_k r < (1-\alpha_k) r + \alpha_k r = r.\end{aligned} \tag{3.17}$$

From (3.15) and (3.17) it now follows that

$$B_{\rho_{\mathcal{N}_\Omega}}\left(T_{\gamma_k^r}, \alpha_k r\right) \subset B_{\rho_{\mathcal{N}_\Omega}}(T,r) \cap \mathcal{F}_k = B_{\rho_{\mathcal{N}_\Omega}}(T,r) \backslash \mathcal{G}_k.$$

Hence $\mathcal{G}_k$ is indeed a porous subset of $\mathcal{N}_\Omega$, as asserted. □

4. Proofs of the main results

Proof of Theorem 2.1. By Lemma 3.1, there is a sequence of subsets $\{\mathcal{F}_n\}_{n=1}^\infty$ of $\mathcal{M}_\Omega$ such that for each positive integer n, the set $\mathcal{G}_n := \mathcal{M}_\Omega \backslash \mathcal{F}_n$ is a porous subset of $\mathcal{M}_\Omega$ and $\mathcal{F}_n$ is the set of all sequences $\{T_n\}_{n=1}^\infty \in \mathcal{M}_\Omega$ for which there exists a positive integer N such that for each mapping $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$, we have

$$\left\|\widetilde{T_{s(N)}} \dots \widetilde{T_{s(1)}} x - \widetilde{T_{s(N)}} \dots \widetilde{T_{s(1)}} y\right\| < n^{-1} \tag{4.1}$$

for each $x, y \in K$. Set $\mathcal{F} := \cap_{n=1}^\infty \mathcal{F}_n$. Then $\mathcal{M}_\Omega \backslash \mathcal{F} = \cup_{n=1}^\infty \mathcal{G}_n$ is a σ-porous subset of $\mathcal{M}_\Omega$.

Let $\{T_n\}_{n=1}^{\infty} \in \mathcal{F}$ and let $\varepsilon > 0$. Choose a positive integer n_0 such that $n_0^{-1} < \varepsilon$. Since $\{T_n\}_{n=1}^{\infty} \in \mathcal{F}_{n_0}$, we infer from (4.1) that there exists a positive integer N such that for each integer $n \geq N$ and each mapping $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$,

$$\left\|\widetilde{T_{s(n)}} \dots \widetilde{T_{s(1)}}x - \widetilde{T_{s(n)}} \dots \widetilde{T_{s(1)}}y\right\| \leq \left\|\widetilde{T_{s(N)}} \dots \widetilde{T_{s(1)}}x - \widetilde{T_{s(N)}} \dots \widetilde{T_{s(1)}}y\right\| < n_0^{-1} < \varepsilon \tag{4.2}$$

for each $x, y \in K$. This completes the proof. □

Proof of Theorem 2.2. By Lemma 3.2, there is a sequence of subsets $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of $\mathcal{N}_\Omega$ such that for each positive integer n, the set $\mathcal{G}_n := \mathcal{N}_\Omega \backslash \mathcal{F}_n$ is a porous subset of $\mathcal{N}_\Omega$ and $\mathcal{F}_n$ is the set of all mappings $T \in \mathcal{N}_\Omega$ for which there exists a positive integer N satisfying

$$\left\|\widetilde{T}^N x - \widetilde{T}^N y\right\| < n^{-1} \tag{4.3}$$

for each $x, y \in K$. Set $\mathcal{F} := \cap_{n=1}^{\infty} \mathcal{F}_n$. Then $\mathcal{N}_\Omega \backslash \mathcal{F} = \cup_{n=1}^{\infty} \mathcal{G}_n$ is a σ-porous subset of $\mathcal{N}_\Omega$.

Let $T \in \mathcal{F}$ and let $\varepsilon > 0$ be arbitrary. Choose a positive integer n_0 such that $n_0^{-1} < \varepsilon$. Since $T \in \mathcal{F}_{n_0}$, we infer from (4.3) that there exists a positive integer N such that for each integer $n \geq N$,

$$\left\|\widetilde{T}^n x - \widetilde{T}^n y\right\| < \left\|\widetilde{T}^N x - \widetilde{T}^N y\right\| < n_0^{-1} < \varepsilon \tag{4.4}$$

for each $x, y \in K$. Clearly, for all integers $n, m \geq N$, we have

$$\left\|\widetilde{T}^n x - \widetilde{T}^m x\right\| < \varepsilon \tag{4.5}$$

for each $x \in K$. Since ε is an arbitrary positive number, inequality (4.5) and the completeness of the subspace F of $(X, \|\cdot\|)$ imply that the sequence $\left\{\widetilde{T}^n\right\}_{n=1}^{\infty}$ converges to an operator $P : K \to F$, uniformly on K. By taking the limit in (4.4), we see that P is constant on K, that is, there exists a point $x_T \in K$ such that the sequence $\left\{\widetilde{T}^n x\right\}_{n=1}^{\infty} \to x_T$ as $n \to \infty$, uniformly on K. Pick an arbitrary point $x_0 \in K$. Since the operator $\widetilde{T}$ is continuous, it follows that

$$\widetilde{T}x_T = \widetilde{T} \lim_{n\to\infty} \widetilde{T}^k x_0 = \lim_{k\to\infty} \widetilde{T}^{k+1} x_0 = x_T.$$

Hence $x_T \in K$ is the unique fixed point of the operator $\widetilde{T}$, as asserted. □

Remark 4.1. We take this opportunity to correct two misprints in [1].

- Page 332, second paragraph: The sentence "Note that this mapping is onto K." should be replaced by the sentence "Note that the mapping defined on $\mathcal{N}_\Omega$ by $T \mapsto \widetilde{T}$ is onto $\mathcal{N}$."
- Page 347: The formula

  $$\widetilde{R_n} x_R = \widetilde{R_n} \lim_{k\to\infty} \widetilde{R_n}^k x = \lim_{k\to\infty} \widetilde{R_n}^{k+1} x_R = x_R$$

  should be replaced by the formula

  $$\widetilde{R_n} x_R = \widetilde{R_n} \lim_{k\to\infty} \widetilde{R_n}^k x = \lim_{k\to\infty} \widetilde{R_n}^{k+1} x = x_R.$$

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