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A note on Bloch functions

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. We construct a natural linear isomorphism between the little Bloch space $\mathcal{B}_0$ and the Banach space c_0 of complex null sequences. This paper is written for the special issue of Studia Universitatis Babeş-Bolyai Mathematica in memory of Professor Gabriela Kohr.

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the complex open unit disc. A holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{C}$ satisfying

$$|f|_{\mathcal{B}} := \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty$$

is known as a *Bloch function*, where $|\cdot|_{\mathcal{B}}$ is called the *Bloch semi-norm*. Obviously, bounded holomorphic functions on $\mathbb{D}$, complex polynomials in particular, are Bloch functions, but also, unbounded Bloch functions abound. With the usual addition and scalar multiplication, the Bloch functions on $\mathbb{D}$ form a Banach space $\mathcal{B}$, called the *Bloch space*, in the *Bloch norm* $\|\cdot\|_{\mathcal{B}}$ defined by

$$\|f\|_{\mathcal{B}} = |f(0)| + |f|_{\mathcal{B}} \qquad (f \in \mathcal{B}).$$

The following subspace

$$\mathcal{B}_0 := \{f \in \mathcal{B} : \lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0\}$$

of $\mathcal{B}$ is often called the *little Bloch space*.

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It is well-known that $\mathcal{B}_0$ is linearly isomorphic to the Banach space c_0 of complex null sequences, and a common recourse of its proof is a result in [7, Theorem 7] asserting that c_0 is linearly isomorphic to the Banach space

$$h_0 = \{h : h \text{ is complex harmonic on } \mathbb{D}, \sup_{|z|<1}(1-|z|^2)|h(z)| < \infty, \lim_{|z|\to 1}(1-|z|^2)|h(z)| = 0\}$$

which is equipped with the norm

$$\|h\| = \sup_{|z|<1}(1-|z|^2)|h(z)| \qquad (h \in h_0).$$

This result implies that $\mathcal{B}_0$ is linearly isomorphic to c_0 since $\mathcal{B}_0$ is linearly isomorphic to a complemented subspace of h_0 and by [6], every infinite dimensional complemented subspace of c_0 is linearly isomorphic to c_0.

However, besides invoking [6] in this proof, the isomorphism between h_0 and c_0 in [7] is obtained from a composition of mappings on various Banach spaces, involving a series of non-trivial lemmas. In this note, we show directly that $\mathcal{B}_0$ is linearly isomorphic to c_0 by exhibiting an explicit linear isomorphism between them.

2. Isomorphism of Bloch space

We first explain why $\mathcal{B}_0$ is linearly isomorphic to a complemented subspace of h_0. A function $h : \mathbb{D} \longrightarrow \mathbb{C}$ is called *complex harmonic* if its real and imaginary parts are both real harmonic functions. Such a function can be written as $h = f + \overline{g}$, where f and g are holomorphic functions, and the symbol '$-$' denotes the complex conjugation. Plainly, holomorphic functions are complex harmonic. Let

$$\mathcal{A}_0 = \{h : h \text{ is holomorphic on } \mathbb{D}, \sup_{|z|<1}(1-|z|^2)|h(z)| < \infty, \lim_{|z|\to 1}(1-|z|^2)|h(z)| = 0\}$$

which forms a complex Banach space with the norm

$$\|h\|_{\mathcal{A}_0} = \sup_{|z|<1}(1-|z|^2)|h(z)| \qquad (h \in \mathcal{A}_0)$$

and the preceding remark implies that $\mathcal{A}_0$ is a complemented subspace of h_0. On the other hand, the map

$$f \in \mathcal{B}_0 \mapsto f' \in \mathcal{A}_0 \tag{2.1}$$

is a linearly isometry and therefore $\mathcal{B}_0$ is isomorphic to a complemented subspace of h_0.

In view of (2.1), to construct a linear isomorphism between $\mathcal{B}_0$ and c_0, it suffices to build one between $\mathcal{A}_0$ and c_0.

We will denote the elements in c_0 by bold letters such as

$$\boldsymbol{a} = (a_0, a_1, a_2, \ldots) \in c_0$$

and make use of the fact that $\mathcal{B}_0$ is the $\|\cdot\|_{\mathcal{B}}$-closure of polynomials in $\mathcal{B}$. Further, if $f(x) = \sum_{k=0}^{\infty} b_k z^k$ belongs to $\mathcal{B}_0$, then $\lim_{k\to\infty} b_k = 0$, by a remark following [1, Lemma 3.1].

Given a sequence (f_n) in $\mathcal{B}_0$ converging to $f \in \mathcal{B}$ (in the Bloch norm), we have

$$\text{(i) } (f_n) \text{ converges to } f \text{ locally uniformly on } \mathbb{D}, \tag{2.2}$$

$$\text{(ii) } \lim_{|z|\to 1}(1-|z|^2)|f_n'(z)| = 0, \text{ uniformly in } n \tag{2.3}$$

(cf. [1, p.14]).

The norm of each $\boldsymbol{a} = (a_k) \in c_0$ is given by $\|\boldsymbol{a}\|_{c_0} = \sup_k |a_k|$. Let c_{00} be the subspace of c_0, consisting of elements $\boldsymbol{a} = (a_k)$ with $a_k = 0$ except a finite number of indices k.

Lemma 2.1. *The linear map* $\varphi : c_{00} \longrightarrow \mathcal{A}_0$ *defined by*

$$\varphi(\boldsymbol{a})(z) = \sum_k a_k z^k \qquad (z \in \mathbb{D}, \boldsymbol{a} = (a_k) \in c_{00})$$

is continuous.

Proof. We have

$$\|\varphi(\boldsymbol{a})\|_{\mathcal{A}_0} = \sup\left\{(1-|z|^2)\left|\sum_k a_k z^k\right| : |z| < 1\right\}$$

where

$$\left|\sum_k a_k z^k\right| \le (\sup_k |a_k|)(1+|z|+|z|^2+\cdots) = \frac{\|\boldsymbol{a}\|_{c_0}}{1-|z|}$$

and hence

$$\|\varphi(\boldsymbol{a})\|_{\mathcal{A}_0} \le \sup\{(1+|z|)\|\boldsymbol{a}\|_{c_0} : |z| < 1\} \le 2\|\boldsymbol{a}\|_{c_0}. \qquad \square$$

Since c_{00} is dense in c_0, the map φ in Lemma 2.1 extends to a continuous linear map, *still denoted by* φ, from c_0 to $\mathcal{A}_0$. We show that this map is actually a linear isomorphism.

Theorem 2.2. *The extension* $\varphi : c_0 \longrightarrow \mathcal{A}_0$ *of the map in Lemma* 2.1 *is a linear homeomorphism.*

Proof. We begin by showing that φ is injective. Let $\boldsymbol{a} \in c_0$ and $\varphi(\boldsymbol{a}) = 0$. We show $\boldsymbol{a} = 0$. By definition of the map φ, there is a sequence $(\boldsymbol{a}_n)$ in c_{00} norm converging to $\boldsymbol{a}$ such that $\lim_n \varphi(\boldsymbol{a}_n) = 0$ in $\mathcal{A}_0$, where

$$\boldsymbol{a}_n = (a_{nk}) = (a_{n0}, a_{n1}, \ldots, a_{nk}, \ldots).$$

By virtue of (2.1) and (2.2), the sequence $\varphi(\boldsymbol{a}_n)$ of functions converges to 0 locally uniformly on $\mathbb{D}$, where

$$\varphi(\boldsymbol{a}_n)(z) = \sum_k a_{nk} z^k \qquad (z \in \mathbb{D}).$$

For $k = 0, 1, 2, \ldots$, the k-th derivative $\varphi(\boldsymbol{a}_n)^{(k)}$ converges to 0 locally uniformly, as $n \to \infty$. It follows that

$$k!|a_{nk}| = |\varphi(\boldsymbol{a}_n)^{(k)}(0)| \le \sup\{|\varphi(\boldsymbol{a}_n)^{(k)}(z)| : |z| \le 1/2\} \to 0$$

as $n \to \infty$. Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies

$$\sup\{|\varphi(\boldsymbol{a}_n)^{(k)}(z)| : |z| \le 1/2\} < k!\varepsilon$$

and hence $|a_{nk}| < \varepsilon$ for $n \geq n_0$. Therefore we have

(i) $\lim_n a_{nk} = 0$ for each k,
(ii) $\lim_n \lim_k a_{nk} = 0$ since $\boldsymbol{a}_n \in c_{00}$.

By [5, IV.13.10], the sequence $(\boldsymbol{a}_n)$ converges weakly to 0 in c_0 and hence $\boldsymbol{a} = 0$.

Finally, we show that φ is surjective. Let $f \in \mathcal{A}_0$. By (2.1), there is a sequence (p_n) of polynomials such that $\|p_n - f\|_{\mathcal{A}_0} \to 0$ as $n \to \infty$. Write

$$p_n(z) = \sum_k a_{nk} z^k.$$

Then $p_n = \varphi(\boldsymbol{a}_n)$ where $\boldsymbol{a}_n = (a_{nk}) \in c_{00}$.

As before, $(p_n - f)$ converges locally uniformly to 0 on $\mathbb{D}$, and we have

$$\sup\{|(p_m - p_n)'(z)| : |z| \leq 1/2\} \leq 2 \sup\{|(p_m - p_n)(z)| : |z| \leq 1/2\}$$

from the Cauchy formula. Iterating this inequality yields

$$\begin{aligned} k!|a_{mk} - a_{nk}| &= |(p_m - p_n)^{(k)}(0)| \leq \sup\{|(p_m - p_n)^{(k)}(z)| : |z| \leq 1/2\} \\ &\leq 2^k \sup\{|(p_m - p_n)(z)| : |z| \leq 1/2\} \to 0 \quad (k = 0, 1, 2, \ldots) \end{aligned}$$

as $m, n \to \infty$. It follows that the sequence $(a_{nk})_{n=1}^{\infty}$ converges to some $a_k \in \mathbb{C}$ for each k, and for some $m_0 \in \mathbb{N}$ and for all k, we have

$$|a_{m_0 k} - a_{nk}| \leq \frac{2^k}{k!} \quad \text{whenever} \quad n \geq m_0.$$

Since $(a_{m_0 k}) \in c_{00}$, there is some k_0 such that $a_{m_0 k} = 0$ for $k \geq k_0$, which gives

$$|a_{nk}| \leq 2^k/k!$$

for $n \geq m_0$ and $k \geq k_0$, Hence $|a_k| \leq 2^k/k!$ for $k \geq k_0$ and $\lim_k a_k = 0$.

By [5, IV.13.10] again, the following properties

(i) $\lim_n a_{nk} = a_k$ for each k,
(ii) $\lim_n \lim_k a_{nk} = 0 = \lim_k a_k$

imply that $(\boldsymbol{a}_n)$ converges weakly to $\boldsymbol{a} = (a_k)$ in c_0. Since φ is weakly continuous, the sequence $\varphi(\boldsymbol{a}_n)$ converges weakly to $\varphi(\boldsymbol{a})$ in $\mathcal{A}_0$. On the other hand, $\varphi(\boldsymbol{a}_n)$ norm converges f in $\mathcal{A}_0$ and hence $\varphi(\boldsymbol{a}) = f$. This proves surjectivity of φ.

By the open mapping theorem, the map $\varphi : c_0 \longrightarrow \mathcal{A}_0$ is a linear homeomorphism which completes the proof. □

It has been shown in [1] that the second dual space $\mathcal{B}_0^{**}$ is linearly isomorphic to $\mathcal{B}$. It follows that $\mathcal{B}$ is linearly isomorphic to the Banach space ℓ_∞ of bounded complex sequences.

3. Bloch functions of several complex variables

The concept of a Bloch function has been extended to higher and infinite dimensions by several authors. We refer to [2] for references of these extensions. The various definitions of Bloch functions on bounded symmetric domains in these references are all equivalent to the one given in [3] and below.

We recall that a *bounded symmetric domain* is a bounded domain D in a complex Banach space V such that each point $p \in D$ admits a (unique) symmetry $s_p : D \longrightarrow D$ which, by definition, is a biholomorphic map such that p is an isolated fixed-point of s_p and $s_p \circ s_p$ is the identity map on D. Further details of infinite dimensional bounded symmetric domains including their realisation as the open unit ball of a complex Banach space with a Jordan structure, alias JB*-triple, can be found in [2].

Definition 3.1. Let D be a bounded symmetric domain realised as the open unit ball of a JB*-triple V and let $\operatorname{Aut} D$ be the automorphism group of D, consisting of biholomorphisms of D. The *Bloch semi-norm* of a holomorphic map $f : D \longrightarrow \mathbb{C}^d$ is defined by

$$|f|_{\mathcal{B}} = \sup\{\|(f \circ g)'(0)\| : g \in \operatorname{Aut} D\}$$

where $d \in \mathbb{N}$ and $\mathbb{C}^d$ is equipped with the Euclidean norm. We call f a *Bloch map* if $|f|_{\mathcal{B}} < \infty$. A Bloch map $f : D \longrightarrow \mathbb{C}$ is often called a *Bloch function*.

We note that on the unit disc $\mathbb{D}$, the two definitions of the Bloch semi-norm $|\cdot|_{\mathcal{B}}$ given previously coincide, that is,

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| = \sup\{|(f \circ g)'(0)| : g \in \operatorname{Aut} \mathbb{D}\}.$$

On higher dimensional domains D, however, they are not equal, even on the bidisc, although we always have

$$\sup_{z \in D} (1 - \|z\|^2)\|f'(z)\| \leq \sup\{\|(f \circ g)'(0)\| : g \in \operatorname{Aut} D\}.$$

The following example has been given in [4].

Example 3.2. Let $f : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$ be defined by

$$f(z_1, z_2) = (1 - z_2) \log \frac{1}{1 - z_1}, \qquad (z_1, z_2) \in \mathbb{D} \times \mathbb{D}.$$

Then we have

$$\sup_{(z_1, z_2) \in \mathbb{D} \times \mathbb{D}} (1 - \|(z_1, z_2)\|^2)\|f'(z_1, z_2)\| < \infty$$

where $\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}$, but in contrast

$$\sup\{\|(f \circ g)'(0)\| : g \in \operatorname{Aut}(\mathbb{D} \times \mathbb{D})\} = \infty.$$

As in the one dimensional case, the Bloch functions on D form a Banach space $\mathcal{B}(D)$ in the following Bloch norm:

$$\|f\|_{\mathcal{B}} = \|f(0)\| + |f|_{\mathcal{B}} \qquad (f \in \mathcal{B}(D)).$$

One can also define the *little Bloch space* $\mathcal{B}_0(D)$ as the closure of the polynomials in $\mathcal{B}(D)$ and likewise, we have

$$\mathcal{B}_0(D) = \{f \in \mathcal{B}(D) : \lim_{\|z\| \to 1} (1 - \|z\|^2)\|f'(z)\| = 0\}$$

if D is the open unit ball of a Hilbert space V (cf. [2, Theorem 4.3.11]). While it is known that the little Bloch space $\mathcal{B}_0(B_d)$ of a d-dimensional Euclidean ball $B_d \subset \mathbb{C}^d$ is linearly isomorphic to c_0, as in the case of $\mathbb{D}$ by similar arguments, the little Bloch space $\mathcal{B}_0(B)$ of the open unit ball B of a non-separable Hilbert space is not linearly isomorphic to c_0 since $\mathcal{B}_0(B)$ is not separable.

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