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# On singular $\phi-$Laplacian BVPs of nonlinear fractional differential equation

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**Abstract.** This paper investigates the existence of multiple positive solutions for a class of $\phi-$Laplacian boundary value problem with a nonlinear fractional differential equation and fractional boundary conditions. Multiple solutions are proved under slight conditions on a possibly degenerating source term. Approximation techniques together with the fixed point index theory a on cone of a Banach space are employed. Some illustrating examples of are also supplied.

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**Keywords:** Fractional differential equation, $\phi-$Laplacian, fixed points index, cone, positive solution, multiple solutions.

## 1. Introduction

This paper deals with the existence of multiple positive solutions to the following nonlinear fractional differential equation with a $\phi$-Laplacian operator and Riemann-Liouville derivatives:

$$\begin{cases} -D_{0^+}^{\alpha}(\phi(-D_{0^+}^{\beta}x(t)) = q(t)f(t,x(t),D_{0^+}^{\gamma}x(t)), \quad 0<t<1, \\ x(0)=x'(0)=D_{0^+}^{\beta-1}x(1)=D_{0^+}^{\beta}x(0)=[D_{0^+}^{\alpha-1}(\phi(-D_{0^+}^{\beta}x(t))]_{t=1}=0, \end{cases} \tag{1.1}$$

where $\gamma > 0$, $\alpha \in (1,2]$, $\beta \in (2,3]$, $\beta-\gamma-2 \geq 0$, and $D_{0^+}^{\alpha}, D_{0^+}^{\beta}, D_{0^+}^{\gamma}$ are the standard Riemann-Liouville derivatives. The nonlinear term $f = f(t,x,y)$ : $[0,1]\times[0,+\infty)\times[0,+\infty) \longrightarrow \mathbb{R}^+$ is continuous but may be singular at $x=0$ and/or at $y=0$ in a sense to be made precise. The function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing

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homeomorphism such that $\phi(0) = 0$. The sets $\mathbb{R}^+$ and $I = (0, +\infty)$ will stand for the nonnegative real numbers and the positive real numbers, respectively.

In the last couple of years, fractional boundary value problems (BVPs for short) have been the subject of intensive research works, see, e.g., [2, 12, 11, 15] and reference therein. They can thought of as extension of BVPs with ordinary differential equations (see [5, 6]). For the $p-$Laplacian $\varphi_p(s) = |s|^{p-2}s\ \ p > 1$, the authors of [13] discussed the BVP

$$\begin{cases} D^{\beta}_{0+}(\varphi_p(D^{\alpha}_{0+}y(x))) = f(x, y(x)), 0 < x < 1, \\ y(0) = y'(0) = y(1) = D^{\alpha}_{0+}y(0) = 0, D^{\alpha}_{0+}y(1) = \lambda D^{\alpha}_{0+}y(\varepsilon), \end{cases}$$

where $\alpha, \beta \in \mathbb{R},\ \alpha \in (2.3],\ \beta \in (1, 2],\ \varepsilon \in (0, 1),\ \lambda \in [0, +\infty),\ D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the standard Riemann-Liouville derivatives, and $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$. They proved the existence of positive solutions by means of the Guo-Krasnosel'skii fixed point theorem. In [8], Lu *et al.* considered the BVP

$$\begin{cases} D^{\alpha}_{0+}(\varphi_p(D^{\beta}_{0+}u(t))) = f(t, u(t)), 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0,\ D^{\beta}_{0+}u(0) = D^{\beta}_{0+}u(1) = 0, \end{cases}$$

where $\alpha \in (1, 2], \beta \in (2, 3]$, $D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the standard Riemann-Liouville derivatives, and $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$. Existence results are proved by combination of the Guo-Krasnosel'skii fixed point theorem, the Leggett-Williams fixed point theorem, and the method of upper and lower solutions. In [14], the authors considered the BVP

$$\begin{cases} -D^{\alpha_1}_{0+}(\varphi_p(D^{\beta_1}_{0+}u(t)) = f(t, u(t)), 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0,\ D^{\beta_1}_{0+}u(0) = 0, D^{\beta_1}_{0+}u(1) = bD^{\beta_1}_{0+}u(\eta), \end{cases} \tag{1.2}$$

where $\alpha \in (1, 2], \beta \in (3, 4],\ \eta \in (0, 1), b \in (0, \eta^{\frac{1-\alpha}{p-1}})$, and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$. They established the existence of positive solutions by the upper and lower solutions method combined with the Schauder fixed point theorem. More recently, the existence of positive solutions is proved in [15] by fixed point theory. In [3], A. Boucenna and T. Moussaoui have used the Krasnoselskii fixed point theorem to establish the existence of positive solution on the half-line for the BVP:

$$\begin{cases} -D^{\alpha}_{0+}u(t) = a(t)g(u(t), D^{\beta}_{0+}u(t)), t > 0, \\ u(0) = D^{\alpha-1}_{0+}u(\infty) = 0, \end{cases} \tag{1.3}$$

where $\alpha \in (1, 2], \beta > 0$, and $\alpha - \beta \geq 1$ and the nonlinear function $g$ satisfies some growth assumptions.

This work discusses the existence and the multiplicity of positive solutions to Problem (1.1) where the function $f$ depends on $x$ and on the standard Rieman-Liouville derivative $D^{\gamma}_{0+}x$. The nonlinear term $f$ may be singular point at $x = 0$ and/or $D^{\gamma}_{0+}x = 0$. $\phi$ is a homeomorphism. We will make use of the fixed point index theory on a suitable cone in some Banach space. Each existence result is illustrated by an example. In this section, we also recall some preliminary results we need in this paper. The first reminders concern the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivation. For more details, we refer to [7, 10, 9].

**Definition 1.1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, 1) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

provided that the right-hand side is pointwise defined on $(0, 1)$. $\Gamma(\alpha)$ is the Euler gamma function $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$.

**Definition 1.2.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, 1) \to \mathbb{R}$ is defined by

$$D_{0^+}^{\alpha} u(t) = \frac{d^n}{dt^n} I_{0^+}^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where $n$ is the smallest integer greater than or equal to $\alpha$, provided the right-hand side is pointwise defined on $(0, 1)$.

**Lemma 1.3.** *Let $\alpha > 0$. Then for $u \in L(0, 1)$ and $D_{0^+}^{\alpha} u(t) \in L(0, 1)$, we have*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},$$

*where $c_1, c_2, \ldots, c_n \in (-\infty, +\infty), n - 1 < \alpha \leq n$.*

For the theory and the computation of the fixed point index on cones in Banach spaces, we refer to [1, 2, 4]. An operator $A : E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets. A nonempty subset $\mathcal{P}$ of a Banach space $E$ is called a cone if it is convex, closed and satisfies $\alpha x \in \mathcal{P}$ for all $x \in \mathcal{P}$ and $\alpha \geq 0$ and $x, -x \in \mathcal{P}$ implies that $x = 0$.

**Lemma 1.4.** *Let $\Omega$ be a bounded open set in a real Banach space $E$, $\mathcal{P}$ a cone of $E$ and $A : \overline{\Omega} \cap \mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose that $\lambda Ax \neq x, \forall\, x \in \partial\Omega \cap \mathcal{P}, \lambda \in (0, 1]$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 1$.*

**Lemma 1.5.** *Let $\Omega$ be a bounded open set in a real Banach space $E$, $\mathcal{P}$ a cone of $E$ and $A : \overline{\Omega} \cap \mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose that $Ax \not\leq x, \forall\, x \in \partial\Omega \cap \mathcal{P}$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 0$.*

The basic space to study Problem (1.1) is

$$E = \{x \in C([0, 1], \mathbb{R}) : D_{0^+}^{\gamma} x \in C([0, 1], \mathbb{R})\}.$$

$E$ is a Banach space with the norm $\|x\| = \|x\|_1 + \|x\|_2$, where $\|x\|_1 = \sup_{t\in[0,1]} |x(t)|$ and $\|x\|_2 = \sup_{t\in[0,1]} |D_{0^+}^{\gamma} x(t)|$. The following lemma is the fractional version of Ascoli-Arzéla Theorem.

**Lemma 1.6.** *Let $M \subseteq E$, then $M$ is relatively compact in $E$ if the following conditions hold:*

*(a) $M$ is bounded in $E$,*

*(b) the functions belonging to $\{x, x \in M\}$ and $\{z : z(t) = D_{0^+}^{\gamma} x(t), x \in M\}$ are equicotinuous, i.e., $\forall\, \varepsilon > 0, \exists\, \delta > 0, \forall\, t_1, t_2 \in [0, 1]$, and for all $x \in M$,*

$$|t_1 - t_2| < \delta \Rightarrow |x(t_1) - x(t_2)| < \varepsilon \text{ and } |D_{0^+}^{\gamma} x(t_1) - D_{0^+}^{\gamma} x(t_2)| < \varepsilon.$$

The cases where $f$ is either regular or singular are discussed separately in Section 3 and Section 4, respectively. Some technical lemmas are collected in the following section.

## 2. Fixed point setting

Consider the boundary value problem

$$\begin{cases} -D^{\alpha}_{0+}u(t) = v(t), \quad 0 < t < 1, \\ u(0) = D^{\alpha-1}_{0+}u(1) = 0. \end{cases} \tag{2.1}$$

It is easy to verify

**Lemma 2.1.** *If $v \in C([0,1])$, then Problem* (2.1) *has the unique solution*

$$u(t) = \int_0^1 H(t,s)v(s)ds,$$

*where*

$$H(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

If we set $\phi(-D^{\beta}_{0+}x(t)) = u(t)$, then $-D^{\beta}_{0+}x(t) = \phi^{-1}(u)(t)$. Thus the BVP

$$\begin{cases} -D^{\alpha}_{0+}(\phi(-D^{\beta}_{0+}x(t)) = v(t), \quad 0 < t < 1 \\ x(0) = x'(0) = D^{\beta-1}_{0+}x(1) = D^{\beta}_{0+}x(0) = [D^{\alpha-1}(\phi(-D^{\beta}_{0+}x(t)]_{t=1} = 0 \end{cases} \tag{2.2}$$

is equivalent to

$$\begin{cases} -D^{\beta}_{0+}x(t) = \phi^{-1}(\int_0^1 H(t,s)v(s))ds), \quad t \in (0,1) \\ \\ x(0) = x'(0) = D^{\beta-1}_{0+}x(1) = 0. \end{cases}$$

**Lemma 2.2.** *Given $v \in C[0,1]$, Problem* (2.2) *has the unique solution*

$$x(t) = \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)v(\tau)d\tau\right)ds,$$

*where*

$$G(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ t^{\beta-1}, & 0 \le t \le s \le 1. \end{cases}$$

A direct computation yields

$$D^{\gamma}_{0+}G(t,s) = \frac{1}{\Gamma(\beta-\gamma)} \begin{cases} t^{\beta-\gamma-1} - (t-s)^{\beta-\gamma-1}, & 0 \le s \le t \le 1, \\ t^{\beta-\gamma-1}, & 0 \le t \le s \le 1. \end{cases}$$

*Proof.* By Lemma 1.3,

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = -I^{\alpha}_{0+}v(t).$$

Then

$$u(t) = -I^{\alpha}_{0+}v(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2}, \quad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions $u(0) = D_{0+}^{\alpha-1}u(1) = 0$ imply $c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 v(s)ds$ and $c_2 = 0$. Hence the solution $u$ of Problem (2.1) is

$$\begin{aligned} u(t) &= - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 v(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1}-(t-s)^{\alpha-1}]v(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}v(s)ds \\ &= \int_0^1 H(t,s)v(s)ds. \end{aligned}$$

Also,

$$I_{0+}^{\beta}D_{0+}^{\beta}x(t) = I_{0+}^{\beta}\phi^{-1}\left(\int_0^1 H(t,s)v(s))ds\right).$$

Then

$$x(t) = -I_{0+}^{\beta}\phi^{-1}\left(\int_0^1 H(t,s)v(s))ds\right) + c_1t^{\beta-1} + c_2t^{\beta-2} + c_3t^{\beta-3},$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. By the boundary conditions $x(0) = x'(0) = D_{0+}^{\beta-1}x(1) = 0$, we have

$$c_1 = \frac{1}{\Gamma(\beta)}\int_0^1 \phi^{-1}\left(\int_0^1 H(s,\tau)v(\tau)d\tau\right)ds, \quad c_2 = c_3 = 0.$$

Finally, the explicit solution $x$ of Problem (2.2) is

$$\begin{aligned} x(t) &= - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi^{-1}(\int_0^1 H(s,\tau)v(\tau)d\tau)ds \\ &\quad + \frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^1 \phi^{-1}(\int_0^1 H(s,\tau)v(\tau)d\tau)ds \\ &= \frac{1}{\Gamma(\beta)}\int_0^t [t^{\beta-1}-(t-s)^{\beta-1}]\phi^{-1}(\int_0^1 (H(s,\tau)v(\tau)d\tau)ds \\ &\quad + \frac{1}{\Gamma(\beta)}\int_t^1 t^{\beta-1}\phi^{-1}(\int_0^1 H(s,\tau)v(\tau)d\tau)ds \\ &= \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)v(\tau)d\tau\right)ds. \end{aligned}$$

□

**Lemma 2.3.** *The function $H(t,s), G(t,s), D_{0+}^{\gamma}G(t,s)$ enjoys the properties*

($a_1$) $H, G, D_{0+}^{\gamma}G$ *are continuous on* $[0,1]\times[0,1]$,

($a_2$) $H(t,s) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \leq \frac{1}{\Gamma(\alpha)}$, $\forall\,(t,s)\in[0,1]\times[0,1]$,

($a_3$) $G(t,s) \leq \frac{t^{\beta-1}}{\Gamma(\beta)} \leq \frac{1}{\Gamma(\beta)}$, $\forall\,(t,s)\in[0,1]\times[0,1]$,

($a_4$) $D_{0+}^{\gamma}G(t,s) \leq \frac{t^{\beta-1}}{\Gamma(\beta-\gamma)} \leq \frac{1}{\Gamma(\beta-\gamma)}$, $\forall\,(t,s)\in(0,1)\times(0,1)$,

($a_5$) $\frac{(\beta-1)\rho(t)s}{\Gamma(\beta)} \leq G(t,s) \leq \frac{(\beta-1)s}{\Gamma(\beta)}$, $\forall\,(t,s)\in[0,1]\times[0,1]$,

($a_6$) $G(t,s) \geq \rho(t)\sup\limits_{t\in[0,1]} G(t,s)$, $\forall\,(t,s)\in[0,1]\times[0,1]$,

($a_7$) $\frac{(\beta-1)\rho(t)s}{\Gamma(\beta-\gamma)} \leq D_{0+}^{\gamma}G(t,s) \leq \frac{(\beta-1)s}{\Gamma(\beta-\gamma)}$, $\forall\,(t,s)\in(0,1)\times(0,1)$

($a_8$) $D_{0+}^{\gamma}G(t,s) \geq \rho(t)\sup\limits_{t\in[0,1]} D_{0+}^{\gamma}G(t,s)$, $\forall\,(t,s)\in(0,1)\times(0,1)$,

($a_9$) $G(t,s) \geq \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\sup\limits_{t\in[0,1]} D_{0+}^{\gamma}G(t,s)$, $\forall\,(t,s)\in(0,1)\times(0,1)$,

($a_{10}$) $D_{0+}^{\gamma}G(t,s) \geq \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\sup\limits_{t\in[0,1]} G(t,s)$, $\forall\,(t,s)\in(0,1)\times(0,1)$,

*where* $\rho(t) = \frac{1}{\beta-1}t^{\beta-1}(1-t)^{\beta-1}$.

*Proof.* Let $(t,s)\in(0,1)\times(0,1)$.
$(a_5)$ If $s\le t$, then

$$\begin{array}{rcl} t^{\beta-1}-(t-s)^{\beta-1} & = & (\beta-1)\int_{t-s}^{t} z^{\beta-2}dz \\ & \le & (\beta-1)(t-t+s)=(\beta-1)s \end{array}$$

and

$$\begin{array}{rcl} t^{\beta-1}-(t-s)^{\beta-1} & \ge & t^{\beta-1}(1-s)^{\beta-1}-(t-s)^{\beta-1} \\ & = & t^{\beta-2}(1-s)^{\beta-2}(t-ts)-(t-s)^{\beta-2}(t-s) \\ & \ge & t^{\beta-2}(1-s)^{\beta-2}(t-ts)-(t-ts)^{\beta-2}(t-s) \\ & \ge & t^{\beta-2}(1-s)^{\beta-2}[(t-ts)-(t-s)] \\ & \ge & t^{\beta-2}(1-s)^{\beta-2}s(1-t) \\ & \ge & t^{\beta-1}(1-t)^{\beta-1}s. \end{array}$$

If $t\le s$, then

$$t^{\beta-1}=(\beta-1)\int_0^t z^{\beta-2}dz\le(\beta-1)s$$

and

$$t^{\beta-1}\ge t^{\beta-1}(1-t)^{\beta-1}s.$$

Hence

$$\frac{t^{\beta-1}(1-t)^{\beta-1}s}{\Gamma(\beta)}\le G(t,s)\le\frac{(\beta-1)s}{\Gamma(\beta)},\quad \forall\,(t,s)\in[0,1]\times[0,1].$$

$(a_7)$ For $s\le t$

$$\begin{array}{rcl} t^{\beta-\gamma-1}-(t-s)^{\beta-\gamma-1} & = & (\beta-\gamma-1)\int_{t-s}^{t} z^{\beta-\gamma-2}dz \\ & \le & (\beta-\gamma-1)(t-t+s) \\ & = & (\beta-\gamma-1)s\le(\beta-1)s \end{array}$$

and

$$\begin{array}{rl} t^{\beta-\gamma-1}-(t-s)^{\beta-\gamma-1} & \ge t^{\beta-\gamma-1}(1-s)^{\beta-\gamma-1}-(t-s)^{\beta-\gamma-1} \\ & = t^{\beta-\gamma-2}(1-s)^{\beta-\gamma-2}(t-ts)-(t-s)^{\beta-\gamma-2}(t-s) \\ & \ge t^{\beta-\gamma-2}(1-s)^{\beta-\gamma-2}(t-ts)-(t-ts)^{\beta-\gamma-2}(t-s) \\ & \ge t^{\beta-\gamma-2}(1-s)^{\beta-\gamma-2}[(t-ts)-(t-s)] \\ & \ge t^{\beta-\gamma-2}(1-s)^{\beta-\gamma-2}s(1-t) \\ & \ge t^{\beta-1}(1-t)^{\beta-1}s. \end{array}$$

If $t\le s$, then

$$t^{\beta-\gamma-1}=(\beta-\gamma-1)\int_0^t z^{\beta-2}dz\le(\beta-1)s$$

and

$$t^{\beta-\gamma-1}\ge t^{\beta-1}(1-t)^{\beta-1}s.$$

Hence

$$\frac{t^{\beta-1}(1-t)^{\beta-1}s}{\Gamma(\beta-\gamma)}\le D_{0^+}^{\gamma}G(t,s)\le\frac{(\beta-1)s}{\Gamma(\beta-\gamma)}.$$

$(a_9)$ By $(a_5)$ and $(a_7)$,

$$\begin{aligned} G(t,s) &\geq \tfrac{t^{\beta-1}(1-t)^{\beta-1}s}{\Gamma(\beta)} \\ &= \tfrac{\Gamma(\beta-\gamma)t^{\beta-1}(1-t)^{\beta-1}}{(\beta-1)\Gamma(\beta)} \tfrac{(\beta-1)s}{\Gamma(\beta-\gamma)} \\ &\geq \tfrac{\Gamma(\beta-\gamma)t^{\beta-1}(1-t)^{\beta-1}}{(\beta-1)\Gamma(\beta)} \sup_{t\in[0,1]} D^{\gamma}_{0+}G(t,s). \end{aligned}$$

$(a_{10})$ By $(a_5)$ and $(a_7)$

$$\begin{aligned} D^{\gamma}_{0+}G(t,s) &\geq \tfrac{t^{\beta-1}(1-t)^{\beta-1}s}{\Gamma(\beta-\gamma)} \\ &= \tfrac{\Gamma(\beta)t^{\beta-1}(1-t)^{\beta-1}}{(\beta-1)\Gamma(\beta)} \tfrac{(\beta-1)s}{\Gamma(\beta)} \\ &\geq \tfrac{\Gamma(\beta)t^{\beta-1}(1-t)^{\beta-1}}{(\beta-1)\Gamma(\beta-\gamma)} \sup_{t\in[0,1]} G(t,s). \end{aligned}$$

□

Define the cone $\mathcal{P}$

$$P = \{x \in E : \ \ x(t) \geq \lambda_1\rho(t)\|x\|,\ D^{\gamma}_{0+}x(t) \geq \lambda_2\rho(t)\|x\|,\ \ \forall\, t \in [0,1]\},$$

where

$$\lambda_1 = \frac{1}{2(\beta-1)} \max\left\{\frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}, 1\right\}$$

and

$$\lambda_2 = \frac{1}{2(\beta-1)} \max\left\{\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}, 1\right\}.$$

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ with $0 < \alpha_1 < \alpha_2$ be such that

$$t^{\alpha_2}\phi(x) \leq \phi(tx) \leq t^{\alpha_1}\phi(x); \qquad \forall\, t \in [0,1],\ \forall\, x \geq 0. \tag{2.3}$$

Then

$$t^{\frac{1}{\alpha_1}}\phi^{-1}(x) \leq \phi^{-1}(tx) \leq t^{\frac{1}{\alpha_2}}\phi^{-1}(x); \quad \forall\, t \in [0,1],\ \forall\, x \geq 0. \tag{2.4}$$

Let

$$\rho_1(x) = \begin{cases} x^{\frac{1}{\alpha_1}}, & x \leq 1 \\ x^{\frac{1}{\alpha_2}}, & x \geq 1 \end{cases} \tag{2.5}$$

$$\rho_2(x) = \begin{cases} x^{\frac{1}{\alpha_2}}, & x \leq 1 \\ x^{\frac{1}{\alpha_1}}, & x \geq 1. \end{cases} \tag{2.6}$$

From Equation (2.4), we get

$$\rho_1(t)\phi^{-1}(x) \leq \phi^{-1}(tx) \leq \rho_2(t)\phi^{-1}(x); \quad \forall\, t \geq 0, \forall\, x \geq 0. \tag{2.7}$$

By Lemma 2.1 and Lemma 2.2, Problem 1.1 is equivalent to the nonlinear integral equation

$$x(t) = \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau, x(\tau), D^{\gamma}_{0+}x(\tau))d\tau\right) ds. \tag{2.8}$$

Thus the fixed point operator is the operator $A : E \longrightarrow C([0,1])$ given by

$$A(x)(t) = \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau, x(\tau), D^{\gamma}_{0+}x(\tau))d\tau\right) ds,$$

where $x \in \mathcal{P}$. The fractional derivative is

$$D^{\gamma}_{0+}A(x)(t) = \int_0^1 D^{\gamma}_{0+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds.$$

Lemma 2.3 will help in investigating the properties of the fixed point operator. Existence of fixed points will be investigated in the next two sections.

## 3. Regular nonlinear term

Suppose that $f:[0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous such that $f(t_0,0,0)\neq 0$ for some $t_0\in(0,1]$. Let the hypotheses

$(\mathcal{H}_1)$ There exist $m\in C([0,1],\mathbb{R}^+)$ and a nondecreasing function in each argument $g\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ such that

$$f(t,x,y)\le m(t)g(x,y),\quad \forall\, t\in[0,1],\forall\, x,y\in\mathbb{R}^+.$$

$(\mathcal{H}_2)$

$$\sup_{c>0}\frac{c}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c,c)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)}>1.$$

$(\mathcal{H}_3)$ There exist $a,b$ $(0<a<b<1)$ such that

$$\lim_{x\to+\infty}\frac{f(t,x,y)}{\phi(x)}=+\infty,\qquad \text{uniformly in } t\in[a,b] \text{ and } y\in\mathbb{R}^+.$$

**Proposition 3.1.** *Suppose $(\mathcal{H}_1)$. Then the operator $A$ maps $\mathcal{P}$ into $\mathcal{P}$ and it is completely continuous.*

*Proof.*

**(1)** $A(\mathcal{P})\subset\mathcal{P}$. $A(x)(t)\ge 0, D^{\gamma}_{0+}Ax(t)\ge 0\ \ \forall\ t\in[0,1]$ and by Lemma 2.3($a_6$)

$$\begin{aligned}
A(x)(t) &= \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds\\
&\ge \rho(t)\\
&\quad \sup_{t\in[0,1]}\int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds\\
&\ge \rho(t)\sup_{t\in[0,1]}Ax(t)\\
&\ge \rho(t)\|Ax\|_1.
\end{aligned}$$

By Lemma 2.3($a_9$),

$$\begin{aligned}
A(x)(t) &= \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds\\
&\ge \tfrac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\\
&\quad \sup_{t\in[0,1])}\int_0^1 D^{\gamma}_{0+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds\\
&\ge \tfrac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\sup_{t\in[0,1]}D^{\gamma}_{0+}Ax(t)\\
&\ge \tfrac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\|Ax\|_2.
\end{aligned}$$

Hence

$$\begin{array}{rcl} A(x) & = & \frac{1}{2}(A(x)+A(x)) \\ & \geq & \frac{1}{2}(\rho(t)\|Ax\|_1 + \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\|Ax\|_2) \\ & \geq & \lambda_1\rho(t)\|Ax\|. \end{array}$$

Also by Lemma 2.3($a_{10}$),

$$\begin{array}{rcl} D^{\gamma}_{0^+}A(x)(t) & = & \int_0^1 D^{\gamma}_{0^+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\int_0^1 \sup\limits_{t\in[0,1]} G(t,s) \\ & & \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\sup\limits_{t\in[0,1]}\int_0^1 G(t,s) \\ & & \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\sup\limits_{t\in[0,1]} Ax(t) \\ & \geq & \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\|Ax\|_1 \end{array}$$

and Lemma 2.3($a_8$) implies

$$\begin{array}{rcl} D^{\gamma}_{0^+}A(x)(t) & = & \int_0^1 D^{\gamma}_{0^+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \rho(t)\sup\limits_{t\in[0,1]}\int_0^1 D^{\gamma}_{0^+}G(t,s) \\ & & \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \rho(t)\sup\limits_{t\in[0,1]} D^{\gamma}_{0^+}Ax(t) \\ & \geq & \rho(t)\|Ax\|_2. \end{array}$$

Hence

$$\begin{array}{rcl} D^{\gamma}_{0^+}A(x) & \geq & \frac{1}{2}(\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\|Ax\|_1 + \rho(t)\|Ax\|_2) \\ & \geq & \lambda_2\rho(t)\|Ax\|, \end{array}$$

proving the claim.

**(2)** Let $D\subset E$ be a bounded set. Then there exists $r>0$ such that $\forall\, x\in D, \|x\|\leq r$. By $(\mathcal{H}_1)$ and the properties $(a_2),(a_3),(a_4)$ of Lemma 2.3, we have the estimates

$$\begin{array}{rcl} \|A(x)\|_1 & = & \|\int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0^+}(\tau))d\tau\right)ds\|_1 \\ & \leq & \frac{1}{\Gamma(\beta)}\int_0^1 \phi^{-1}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)f(\tau,x(\tau),D^{\gamma}_{0^+}x(\tau))d\tau\right)ds \\ & \leq & \frac{1}{\Gamma(\beta)}\phi^{-1}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)g(x(\tau),D^{\gamma}_{0^+}x(\tau))d\tau\right) \\ & \leq & \frac{1}{\Gamma(\beta)}\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)(m(\tau)d\tau\right) < +\infty \end{array}$$

and

$$\begin{array}{rcl} \|A(x)\|_2 & = & \|\int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0^+}x(\tau))d\tau\right)ds\|_2 \\ & \leq & \frac{1}{\Gamma(\beta-\gamma)}\phi^{-1}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)g(x(\tau),D^{\gamma}_{0^+}x(\tau))d\tau\right) \\ & \leq & \frac{1}{\Gamma(\beta-\gamma)}\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right) < +\infty. \end{array}$$

Hence $\|A(x)\| < [\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)(m(\tau)d\tau\right)$, that is $A(D)$ is uniformly bounded.

(3) $A(D)$ is equicontinuous. For $t,t'\in[0,1]$ $(t<t')$, we have

$$\begin{aligned} & |A(x)(t)-A(x)(t')| \\ \leq\ & \int_0^1 |G(t,s)-G(t',s)|\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds \\ \leq\ & \int_0^1 |G(t,s)-G(t',s)|\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)ds \end{aligned}$$

and

$$\begin{aligned} & |D^{\gamma}_{0+}A(x)(t)-D^{\gamma}_{0+}A(x)(t')| \\ \leq\ & \int_0^1 |D^{\gamma}_{0+}G(t,s)-D^{\gamma}_{0+}G(t',s)|\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)ds. \end{aligned}$$

Then for any $\varepsilon>0$, there exists $\delta>0$ such that

$$|Ax(t)-A(x')|<\varepsilon \text{ and } |D^{\gamma}_{0+}A(x)(t)-D^{\gamma}_{0+}A(x)(t')|<\varepsilon,$$

for all $t,t'\in[0,1]$ and $|t-t'|<\delta$, proving that $A(D)$ is relatively compact.

**(4)** $A$ is continuous: Let some sequence $\{x_n\}_{n\geq 0}\subset\mathcal{P}$ be such that $\lim\limits_{n\to+\infty} x_n = x_0$. Then there exists $r>0$ such that $\|x_n\|\leq r$, $\forall n\geq 0$. By $(\mathcal{H}_1)$, for all $t\in[0,1]$, we have

$$\begin{aligned} & |Ax_n(t)-Ax_0(t)| \\ =\ & |\int_0^1 G(t,s)[\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_n(\tau),D^{\gamma}_{0+}x_n(\tau))d\tau)ds \\ & -\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^{\gamma}_{0+}x_0(\tau))d\tau)]ds| \\ \leq\ & \frac{2}{\Gamma(\beta)}\phi^{-1}(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau) \end{aligned}$$

and

$$|D^{\gamma}_{0+}Ax_n(t)-D^{\gamma}_{0+}Ax_0(t)|\leq\frac{2}{\Gamma(\beta-\gamma)}\phi^{-1}(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau).$$

With the Lebegue Dominated convergence theorem, we conclude that

$$\lim_{n\to+\infty}\|Ax_n-Ax_0\|=0,$$

i.e., $A$ is continuous. □

We state and prove our first existence result

**Theorem 3.2.** *Under Assumptions* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold, BVP* (1.1) *has at least one positive solution.*

*Proof.* From Condition $(\mathcal{H}_2)$, there exists $R>0$ such that

$$\frac{R}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(R,R)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)}>1. \tag{3.1}$$

Let $\Omega_1=\{x\in E;\|x\|\leq R\}$. To prove that $x\neq\lambda Ax$ for all $x\in\partial\Omega_1\cap\mathcal{P}$ and $\lambda\in(0,1]$, suppose by contradiction that there exist $x_0\in\partial\Omega_1\cap\mathcal{P}$ and $\lambda_0\in(0,1]$ such that $x_0 = \lambda_0 A x_0$. By $(\mathcal{H}_1)$ and the properties $(a_2), (a_3)$ and $(a_4)$ of Lemma 2.3, we have

$$\begin{array}{rcl} R & = & \|x_0\| \\ & = & \|\lambda_0 A x_0\| \\ & \leq & \|Ax_0\|_1 + \|Ax_0\|_2 \\ & \leq & \sup\limits_{t\in[0,1]} \int_0^1 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^{\gamma}_{0+}x_0(\tau))d\tau)ds \\ & & + \sup\limits_{t\in[0,1]} \int_0^1 D^{\gamma}_0 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^{\gamma}_{0+}x_0(\tau))d\tau)ds \\ & \leq & \frac{1}{\Gamma(\beta)}\phi^{-1}(\frac{g(R,R)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau) + \frac{1}{\Gamma(\beta-\gamma)}\phi^{-1}(\frac{g(R,R)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau) \\ & \leq & [\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}(\frac{g(R,R)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau), \end{array}$$

which contradicts (3.1). Lemma 1.4 implies that

$$i(A, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1.$$

Then there exists $x_0 \in \Omega_1 \cap \mathcal{P}$ such that $Ax_0 = x_0$. Since

$$f(t_0, 0, 0) \neq 0 \text{ and } x_0(t) \geq \lambda_1 \rho(t)\|x_0\|,$$

$x_0$ is a positive solution of Problem (1.1). □

**Example 3.3.** Consider the BVP

$$\begin{cases} -D^{\frac{3}{2}}_{0+}\left(-D^{\frac{5}{2}}_{0+}x(t)\right)^{\frac{5}{3}} = \frac{3}{25}t^{\frac{1}{4}}(1+\cos(\frac{\pi}{4}t^{\frac{5}{4}}))(x + D^{\frac{1}{6}}_{0+}x + 1)^{\frac{5}{3}}, t \in (0,1) \\ x(0) = x'(0) = D^{\frac{3}{2}}_{0+}x(1) = D^{\frac{5}{2}}_{0+}x(0) = [D^{\frac{1}{2}}_{0+}(\phi(-D^{\frac{5}{2}}_{0+}x(t))]_{t=1} = 0, \end{cases} \tag{3.2}$$

where

$$f(t,x,y) = (1+\cos(\frac{\pi}{4}t^{\frac{5}{4}}))(x+y+1)^{\frac{5}{3}}, \; q(t) = \frac{3}{25}t^{\frac{1}{4}} \text{ and } \phi(t) = t^{\frac{5}{3}}.$$

Then $\phi$ is an increasing homeomorphism such that $\phi(0) = 0$. For

$$g(x,y) = \frac{3}{25}(x+y+1)^{\frac{5}{3}} \text{ and } m(t) = 1+\cos(\frac{\pi}{4}t^{\frac{5}{4}}),$$

Assumption $(\mathcal{H}_2)$

$$\sup_{c>0} \frac{c}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c,c)}{\Gamma(\alpha)}, \int_0^1 q(\tau)m(\tau)d\tau\right)} \geq 1.013 > 1$$

is satisfied and then all conditions of Theorem 3.2 hold. Therefore Problem (3.2) has at least one positive solution.

The existence of positive solutions is given by

**Theorem 3.4.** *Assume that* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold and suppose that there exist* $\alpha_1, \alpha_2$, $0 < \alpha_1 < \alpha_2$, *such that*

$$t^{\alpha_2}\phi(x) \leq \phi(tx) \leq t^{\alpha_1}\phi(x), \; \forall\, t \in [0,1], \; \forall\, x \geq 0.$$

*Then Problem* (1.1) *has at least two positive solutions.*

*Proof.* Choose $R$ as in the proof of Theorem 3.2. Then

$$i(A,\Omega_1\cap\mathcal{P},\mathcal{P})=1 \tag{3.3}$$

and there exist $x_0\in\Omega_1$ solution of Problem (1.1). Let $0<a<b<1$ be as in $(\mathcal{H}_3)$ and

$$a_0=\min_{(t,s)\in[a,b]^2}G(t,s)>0,\ b_0=\min_{(t,s)\in[a,b]^2}H(t,s)>0,$$

$$c=\lambda_1\min_{t\in[a,b]}\rho(t)>0,\ N>1+\frac{1}{c^{\alpha_2}(b-a)^{\alpha_2}a_0^{\alpha_2}b_0\int_a^b q(\tau)d\tau}.$$

By $(\mathcal{H}_3)$, there exists $R'>\lambda_1 R$ such that

$$f(t,x,y)>N\phi(x),\ \forall\, t\in[a,b],\ \ \forall\, x>R',\ \ \forall\, y\in\mathbb{R}^+.$$

Define the open ball $\Omega_2=\left\{x\in E:\|x\|\le\frac{R'}{c}\right\}$.
To show that $Ax\nleq x$ for all $x\in\partial\Omega_2\cap\mathcal{P}$, suppose on the contrary that there exists $x_0\in\partial\Omega_2\cap\mathcal{P}$ such that $Ax_0\le x_0$. Since $x_0\in\mathcal{P}$, then

$$\begin{array}{rcl} x_0(t) & \geq & Ax_0(t)\\ & = & \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^{\gamma}_{0+}x_0(\tau))d\tau\right)ds\\ & \geq & \int_a^b G(t,s)\phi^{-1}\left(\int_a^b H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^{\gamma}_{0+}x_0(\tau))d\tau\right)ds\\ & = & (b-a)a_0\phi^{-1}\left(b_0\int_a^b q(\tau)N\phi(x_0(\tau))d\tau\right)\\ & = & (b-a)a_0\phi^{-1}\left(b_0N\phi(R')\int_a^b q(\tau)d\tau\right)\\ & = & (b-a)a_0\phi^{-1}\left([b_0N\int_a^b q(\tau)d\tau]\phi(R')\right)\\ & \geq & (b-a)a_0\rho_1\left(b_0N\int_a^b q(\tau)d\tau\right)R'\\ & \geq & (b-a)a_0b_0^{\frac{1}{\alpha_2}}N^{\frac{1}{\alpha_2}}\left(\int_a^b q(\tau)d\tau\right)^{\frac{1}{\alpha_2}}R'\\ & > & \frac{R'}{c}, \end{array}$$

contradicting $\|x_0\|=\frac{R'}{c}$. By Lemma 1.5, we conclude that

$$i(A,\Omega_2\cap\mathcal{P},\mathcal{P})=0. \tag{3.4}$$

(3.3) and (3.4) imply

$$i(A,(\Omega_2\setminus\Omega_1)\cap\mathcal{P},\mathcal{P})=-1. \tag{3.5}$$

Then $A$ has a second fixed point $y_0\in(\Omega_2\setminus\Omega_1)\cap\mathcal{P}$. Moreover $y_0\ge\lambda_1\rho(t)R$ and $R\le\|y_0\|<\frac{R'}{c}$. Then $x_0$ and $y_0$ are two positive solutions of Problem (1.1). ☐

**Example 3.5.** Consider the BVP

$$\begin{cases} -D^{\frac{3}{2}}_{0+}\left(-D^{\frac{11}{4}}_{0+}x(t)\right)^p=(2\delta t)(x+(D^{\frac{1}{4}}_{0+}x)+1),\ t\in(0,1),\\ x(0)=x'(0)=D^{\frac{9}{4}}_{0+}x(1)=D^{\frac{11}{4}}_{0+}x(0)=[D^{\frac{1}{2}}_{0+}(\phi(-D^{\frac{11}{4}}_{0+}x(t))]_{t=1}=0, \end{cases} \tag{3.6}$$

where $f(t,x,y)=(2\delta\sqrt{t})(x+y+1), q(t)=\sqrt{t},\ \delta>0$, and $\phi(t)=t^p, (p=\frac{a}{b}$ are such that $0<a<b$ and $(b-a)$ is an even number. $\phi$ is an increasing homeomorphism such that $\phi(0)=0$ and there exist $\alpha_1=p^2, \alpha_2=p$

$$t^p\phi(x)\le\phi(tx)\le t^{p^2}\phi(x),\quad \forall\, t\in[0,1], \forall\, x\ge 0.$$

For $g(x,y)=x+y+1$ and $m(t)=2\delta\sqrt{t}$, Assumption $(\mathcal{H}_2)$

$$\sup_{c>0}\frac{c}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c,c)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)}\ge\sup_{c>0}\frac{0.72(\frac{\sqrt{\pi}}{2})^{\frac{1}{p}}c}{(\delta(2c+1))^{\frac{1}{p}}}$$

and $(\mathcal{H}_3)$

$$\begin{aligned}\lim_{x\to+\infty}\frac{f(t,x,y)}{\phi(x)} &= \lim_{x\to+\infty}\frac{(2\delta\sqrt{t})(x+y+1)}{x^p}\\ &\ge \lim_{x\to+\infty}2\delta\sqrt{a}x^{1-p}=+\infty,\quad \forall\, t\in[a,b],\ \forall\ y\ge 0\end{aligned}$$

are satisfied for $\delta<\left(\sup_{c>0}\frac{0.72(\frac{\sqrt{\pi}}{2})^{\frac{1}{p}}c}{(\delta(2c+1))^{\frac{1}{p}}}\right)^p$. Finally all hypotheses of Theorem 3.2 are fulfilled. Hence Problem (3.6) has at least two positive solutions.

## 4. Degenerating nonlinear term

First suppose that $f$ may have a singular point at $x=0$ only. More precisely $f:[0,1]\times I\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous. Assume that
$(\mathcal{H}_1')$ There exist $m\in C([0,1],\mathbb{R}^+), \psi\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $g,h,\in C(I,I)$ such that $h$ is a decreasing function and $\psi$, $\frac{g}{h}$ are increasing functions with

$$f(t,x,y)\le m(t)g(x)\psi(y),\quad \forall\ t\in[0,1], \forall\, x\in I,\ \forall y\in\mathbb{R}^+$$

and for each $c>0$,

$$\int_0^1 q(\tau)m(\tau)h(c\rho(\tau))d\tau<+\infty,$$

$(\mathcal{H}_2')$

$$\sup_{c>0}\frac{c}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c)\psi(c)}{\Gamma(\alpha)h(c)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)c)d\tau\right)}>1.$$

$(\mathcal{H}_3')$ There exist $a,b$ $(0<a<b<1)$ such that

$$\lim_{x\to+\infty}\frac{f(t,x,y)}{\phi(x)}=+\infty,\quad \text{uniformly in }\ t\in[a,b]\text{ and }y\in\mathbb{R}^+.$$

$(\mathcal{H}_4')$ For any $c>0$, there exist $\psi_c\in C([0,1],\mathbb{R}^+)$ and an interval $J\subset[0,1]$ such that $\psi_c(t)>0$ in $J$ and

$$f(t,x,y)\ge\psi_c(t),\quad \forall\ t\in[0,1],\ \forall\, x\in(0,c], \forall\, y\in[0,c].$$

Given $f\in C([0,1]\times I\times\mathbb{R}^+,\mathbb{R}^+)$, define the sequence of functions $\{f_n\}_{n\ge1}$

$$f_n(t,x,y)=f(t,\max\{\tfrac{1}{n},x\},y),\quad n\in\{1,2,\ldots\},$$

and for $x \in \mathcal{P}$, define the sequences of operators

$$A_n(x)(t) = \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x(\tau), D^\gamma_{0+}x(\tau)\right) d\tau)ds.$$

Then

$$D^\gamma_{0+}A_n(x)(t) = \int_0^1 D^\gamma_{0+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x(\tau, D^\gamma_{0+}x(\tau))d\tau\right) ds.$$

The proof of the following result is the same as that of the operator $A$ in Proposition 3.1. We omit it.

**Proposition 4.1.** *Suppose $(\mathcal{H}'_1)$ holds.Then for each $n \geq 1$, the operator $A_n$ maps $\mathcal{P}$ into $\mathcal{P}$ and it is completely continuous.*

As in the regular case, we prove two theorems: one of the existence of a single solution and one of a pair of solutions.

**Theorem 4.2.** *Suppose $(\mathcal{H}'_1), (\mathcal{H}'_2), (\mathcal{H}'_4)$ hold. Then Problem (1.1) has at least one positive solution.*

*Proof.* **(1) Construction of a sequence $(x_n)_n$ of approximating fixed points.**
By condition $(\mathcal{H}'_2)$, there exists $R > 0$ such that

$$\frac{R}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(R)\psi(R)}{\Gamma(\alpha)h(R)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)R)d\tau\right)} > 1. \tag{4.1}$$

Let $\Omega_1 = \{x \in E : \|x\| < R\}$. Then $x \neq \lambda A_n(x)$ for any $x \in \partial\Omega_1 \cap \mathcal{P}, \lambda \in (0,1]$ and $n \geq n_0 \geq \frac{1}{R}$. Otherwise there exist $n_1 \geq n_0$, $x_1 \in \partial\Omega_1 \cap \mathcal{P}$ and $\lambda_0 \in (0,1]$ such that $x_1 = \lambda_0 A_{n_1} x_1$. Since $x_1 \in \partial\Omega_1 \cap \mathcal{P}$, we have $x_1(t) \geq \lambda_1\rho(t)\|x_1\| = \lambda_1\rho(t)R$, then

$$\begin{aligned}
R &= \|x_1\| \\
&= \|\lambda_0 A_{n_1} x_1\| \\
&\leq \|A_{n_1}x_1\|_1 + \|Ax_1\|_2 \\
&\leq \sup_{t\in[0,1]} \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_{n_1}(\tau, x_1(\tau), D^\gamma_{0+}x_1(\tau))d\tau\right) ds \\
&+ \sup_{t\in[0,1]} \int_0^1 D^\gamma_{0+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_{n_1}(\tau, x_1(\tau), D^\gamma_{0+}x_1(\tau))d\tau\right) ds \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\tfrac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)g(\max\{\tfrac{1}{n_1}, x_1(\tau)\})\psi(D^\gamma_{0+}x_1(\tau))d\tau\right) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}(\tfrac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)h(\max\{\tfrac{1}{n_1}, x_1(\tau)\})\tfrac{g(\max\{\frac{1}{n_1}, x_1(\tau)\})}{h(\max\{\frac{1}{n_1}, x_1(\tau)\})} \\
&\psi(D^\gamma_{0+}x_1(\tau))d\tau) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\tfrac{g(R)\psi(R)}{h(R)\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)R)d\tau\right)
\end{aligned}$$

which is a contradiction to (4.1). By Lemma 1.4, we deduce that

$$i(A_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \text{ for all } n \in \{n_0, n_0+1, \ldots\}. \tag{4.2}$$

Hence there exists an $x_n \in \Omega_1 \cap \mathcal{P}$ such that $A_n x_n = x_n$, $\forall\, n \geq n_0$.

**(2) The sequence $(x_n)_n$ is relatively compact.**

(a) Since $\|x_n\| < R$, by $(\mathcal{H}_4')$ there exists $\psi_R \in C([0,1], \mathbb{R}^+)$ such that

$$f_n(t, x_n(t), D_{0^+}^{\gamma} x_n(t)) \geq \psi_R(t), \quad \forall\, t \in [0,1].$$

Then, by Lemma 2.3($a_5$),

$$\begin{array}{rcl} x_n(t) & = & A_n x_n(t) \\ & = & \int_0^1 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x_n(\tau), D_{0^+}^{\gamma} x_n(\tau))d\tau)ds \\ & \geq & \int_0^1 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)\psi_R(\tau)d\tau)ds \\ & \geq & \frac{\rho(t)(\beta-1)}{\Gamma(\beta)} \int_0^1 s\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)\psi_R(\tau)d\tau\right)ds. \end{array}$$

Let

$$c^* = \frac{(\beta-1)}{\Gamma(\beta)} \int_0^1 s\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)\psi_R(\tau)d\tau\right)ds > 0.$$

Then

$$x_n(t) \geq c^*\rho(t), \qquad \forall\, t \in [0,1], \forall\, n \geq n_0.$$

(b) For any $t, t' \in [0,1]$ $(t > t')$,

$$\begin{array}{l} |x_n(t) - x_n(t')| \\ \leq \int_0^1 \left|G(t,s) - G(t',s)\right| \\ \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x_n(\tau), D_{0^+}^{\gamma} x_n(\tau))d\tau\right)ds \\ \leq \int_0^1 \left|G(t,s) - G(t',s)\right| \phi^{-1}\left(\frac{g(R)\psi(R)}{\Gamma(\alpha)h(R)} \int_0^1 q(\tau)m(\tau)h(c^*\rho(\tau))d\tau\right). \end{array}$$

Also

$$\begin{array}{l} \left|D_{0^+}^{\gamma} x_n(t) - D_{0^+}^{\gamma} x_n(t')\right| \\ \leq \int_0^1 \left|D_{0^+}^{\gamma} G(t,s) - D_{0^+}^{\gamma} G(t',s)\right| \\ \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x_n(\tau), D_{0^+}^{\gamma} x_n(\tau))d\tau\right)ds \\ \leq \int_0^1 \left|D_{0^+}^{\gamma} G(t,s) - D_{0^+}^{\gamma} G(t',s)\right| \phi^{-1}\left(\frac{g(R)\psi(R)}{\Gamma(\alpha)h(R)} \int_0^1 q(\tau)m(\tau)h(c^*\rho(\tau))d\tau\right)ds. \end{array}$$

Since $G$ and $D_{0^+}^{\gamma} G$ are continuous, by Lemma 1.6 $(x_n)_n$ is relatively compact in $E$. Then there exists a subsequence $(x_{n_k})_{k\geq 1}$ such that $\lim\limits_{k\to+\infty} x_{n_k} = x_0$. Since $x_{n_k}(t) \geq c^*\rho(t)\ \forall\, k \geq 1, \forall\, t \in [0,1]$, we have $x_0(t) \geq c^*\rho(t), \quad \forall\, t \in [0,1]$. Since $f$ is continuous, by the Lebesgue dominated convergence theorem,

$$\begin{array}{rl} & x_0(t) \\ = & \lim\limits_{k\to+\infty} x_{n_k}(t) \\ = & \lim\limits_{k\to+\infty} \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_{n_k}(\tau, x_{n_k}(\tau), D_{0^+}^{\gamma} x_{n_k}(\tau))d\tau\right)ds \\ = & \lim\limits_{k\to+\infty} \int_0^1 G(t,s) \\ & \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau, \max\{\frac{1}{n_k}, x_{n_k}(\tau)\}, D_{0^+}^{\gamma} x_{n_k}(\tau))d\tau\right)ds \\ = & \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau, \max\{0, x_0(\tau)\}, D_{0^+}^{\gamma} x_0(\tau))d\tau\right)ds \\ = & \int_0^1 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau, x_0(\tau), D_{0^+}^{\gamma} x_0(\tau))d\tau)ds. \end{array}$$

Therefore $x_0$ is a positive solution of Problem (1.1). □

**Example 4.3.** Consider the BVP

$$\begin{cases} -D_{0^+}^{\frac{7}{4}}\left(-D_{0^+}^{\frac{9}{4}}x(t)\right)^{\frac{1}{3}} = \delta t^{\frac{5}{4}}\frac{e^{2x}}{x}Ch(D_{0^+}^{\frac{1}{6}}x),\ t\in(0,1) \\ x(0)=x'(0)=D_{0^+}^{\frac{5}{4}}x(1)=D_{0^+}^{\frac{9}{4}}x(0)=[D_{0^+}^{\frac{3}{4}}(\phi(-D_{0^+}^{\frac{9}{4}}x(t))]_{t=1}=0, \end{cases} \tag{4.3}$$

where

$$f(t,x,y)=\delta\frac{e^{2x+t}}{x}Ch(y),\ (\delta>0),\ q(t)=t^{\frac{5}{4}}e^{-t} \text{ and } \phi(t)=t^{\frac{1}{3}}.$$

Hence $\phi$ is an increasing homeomorphism and $\phi(0)=0$ We check the conditions of Theorem 4.2.

($\mathcal{H}_1'$) Let $m(t)=\delta e^t,\ g(x)=\frac{e^x}{x},\ \psi(y)=Ch(y),\ h(y)=\frac{1}{y}$. Then $\frac{g(x)}{h(x)}=e^x$ and $\psi$ are increasing,

$$f(t,x,y)\le m(t)g(x)\psi(y),\ \forall t\in[0,1],\ \forall x\in I,\ \forall y\in\mathbb{R}^+,$$

and for any $c>0$

$$\int_0^1 m(\tau)q(\tau)h(c\rho(\tau))d\tau=\frac{16}{45c}<+\infty.$$

($\mathcal{H}_2'$)

$$\sup_{c>0}\frac{c}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c)\psi(c)}{\Gamma(\alpha)h(c)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)c)d\tau\right)}\ge\sup_{c>0}\frac{81c^4}{(\delta e^cCh(c))^3}.$$

($\mathcal{H}_4'$) For every $c>0$, there exists $\psi_c=\frac{\delta e^t}{c}$ such that

$$f(t,x,y)\ge\psi_c(t),\quad \forall\ t\in[0,1],\forall\ \ \forall x\in(0,c],y\in[0,c].$$

Let $0<\delta\le\left(\sup\limits_{c>0}\frac{81c^4}{(e^cCh(c))^3}\right)^{\frac{1}{3}}$. Then Problem (4.3) has at least one positive solution.

The existence of two positive solutions is given by

**Theorem 4.4.** *Let* ($\mathcal{H}_1'$) $-$ ($\mathcal{H}_4'$) *and suppose that there exist* $\alpha_1,\alpha_2$ *with* $0<\alpha_1<\alpha_2$ *such that*

$$t^{\alpha_2}\phi(x)\le\phi(tx)\le t^{\alpha_1}\phi(x),\ \forall t\in[0,1],\ \forall x\ge 0.$$

*Then Problem* (1.1) *has at least two positive solutions.*

*Proof.* With $R$ the same as in the proof of Theorem 4.2, we get

$$i(A_n,\Omega_1\cap\mathcal{P},\mathcal{P})=1,\quad\text{for all } n\in\{n_0,n_1,\dots\}. \tag{4.4}$$

Then for every $n\in\{n_0,n_1,\dots\}$, there exists a solution $x_n$ of Problem (1.1) in $\Omega_1$. Let $0<a<b<1$ be as in ($\mathcal{H}_3'$) and $a_0,b_0,c$ as in the proof of Theorem 3.4. Choose

$$N>1+\frac{1}{c^{\alpha_2}(b-a)^{\alpha_2}a_0^{\alpha_2}b_0\int_a^b q(\tau)d\tau}.$$

By ($\mathcal{H}_3'$), there exists a positive constant $R'>\max\{1,\lambda_1 R\}$ such that

$$f(t,x,y)>N\phi(x),\ \forall t\in[a,b],\ \ \forall x\ge R',\ \ \forall y\in\mathbb{R}^+.$$

Consider the open ball $\Omega_2=\left\{x\in E:\|x\|\leq\frac{R'}{c}\right\}$. Then $A_nx\nleq x$ for all $x\in\partial\Omega_2\cap\mathcal{P}$ and $n\in\{1,2\ldots\}$. Otherwise there exist $n\in\{1,2,\ldots\}$ and $x_0\in\partial\Omega_2\cap\mathcal{P}$ such that $A_nx_0\leq x_0$. Since $x_0\in\partial\Omega_2\cap\mathcal{P}$,

$$\begin{aligned}
x_0(t)\ &\geq A_nx_0(t)\\
&=\int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau,x_0(\tau),D^{\gamma}_{0+}x_0(\tau))d\tau\right)ds\\
&\geq\int_a^b G(t,s)\phi^{-1}\left(\int_a^b H(s,\tau)q(\tau)f(\tau,\max\{\tfrac{1}{n},x_0(\tau)\},D^{\gamma}_{0+}x_0(\tau))d\tau\right)ds\\
&=(b-a)a_0\phi^{-1}\left(b_0\int_a^b q(\tau)N\phi(x_0(\tau))d\tau\right)\\
&=(b-a)a_0\phi^{-1}\left(b_0N\phi(R')\int_a^b q(\tau)d\tau\right)\\
&=(b-a)a_0\phi^{-1}\left([b_0N\int_a^b q(\tau)d\tau]\phi(R')\right)\\
&\geq(b-a)a_0\rho_1\left(b_0N\int_a^b q(\tau)d\tau\right)R'\\
&\geq(b-a)a_0b_0^{\frac{1}{\alpha_2}}N^{\frac{1}{\alpha_2}}\left(\int_a^b q(\tau)d\tau\right)^{\frac{1}{\alpha_2}}R'\\
&>\tfrac{R'}{c},
\end{aligned}$$

contradicting $\|x_0\|=\frac{R'}{c}$. Finally, Lemma 1.5 entails

$$i(A_n,\Omega_2\cap\mathcal{P},\mathcal{P})=0,\ \forall n\in\mathbb{N}^*\tag{4.5}$$

whereas (4.4) and (4.5) imply

$$i(A_n,(\Omega_2\setminus\Omega_1)\cap\mathcal{P},\mathcal{P})=-1,\ \forall n\geq n_0.\tag{4.6}$$

Then $A_n$ has a second fixed point $y_n\in(\Omega_2\setminus\Omega_1)\cap\mathcal{P}$, $\forall n\geq n_0$.
In addition $y_n(t)\geq\lambda_1\rho(t)R$, $\forall t\in[0,1]$ and $\|y_n\|<\frac{R'}{c}$. As above, we can show that $(y_n)_{n\geq n_0}$ has a subsequence $(y_{n_j})_{j\geq1}$ such that $\lim\limits_{j\to+\infty}y_{n_j}=y_0$ and $y_0$ is a solution of Problem (1.1). Finally $R\leq\|y_0\|<\frac{R'}{c}$, i.e., $x_0$ and $y_0$ are two positives solutions of Problem (1.1). □

**Example 4.5.** Consider the BVP

$$\begin{cases}-D^{\frac{7}{4}}_{0+}\left(-D^{\frac{9}{4}}_{0+}x(t)\right)^{\frac{1}{3}}=\delta t^{\frac{5}{4}}\dfrac{e^{2x}Ch(D^{\frac{1}{6}}_{0+}x)}{x}, & 0<t<1\\ x(0)=x'(0)=D^{\frac{5}{4}}_{0+}x(1)=D^{\frac{9}{4}}_{0+}x(0)=[D^{\frac{3}{4}}_{0+}(\phi(-D^{\frac{9}{4}}_{0+}x(t))]_{t=1}=0,\end{cases}\tag{4.7}$$

where $f(t,x,y)=\delta\frac{e^{2x+t}Ch(y)}{x}$, $(\delta>0)$, $q(t)=t^{\frac{5}{4}}e^{-t}$. $\phi(t)=t^{\frac{1}{3}}$. Hence $\phi$ is an increasing homeomorphism, $\phi(0)=0$, and there exist $\alpha_1=\frac{1}{4},\alpha_2=2$ such that

$$t^2\phi(x)\leq\phi(tx)\leq t^{\frac{1}{4}}\phi(x),\forall t\in[0,1],\ \forall x\geq0.$$

$(\mathcal{H}_3')$

$$\lim_{x\to+\infty}\frac{f(t,x,y)}{\phi(x)}\geq\lim_{x\to+\infty}\frac{\delta e^{2x}}{x^{\frac{4}{3}}}=+\infty,\quad\forall t\geq0,\ \forall y\geq0.$$

Choosing $\delta\leq\sup\limits_{c>0}\left(\frac{81c^4}{(e^cCh(c))^3}\right)^{\frac{1}{3}}$, all conditions of Theorem 4.4 are fulfilled and Problem (4.7) has at least two positive solutions.

In the last part of this work, the nonlinear function $f$ may be degenerating at both $x = 0$ and $y = 0$. More precisely $f : [0,1] \times I \times I \longrightarrow \mathbb{R}^+$ satisfies Assumption $(\mathcal{H}_1'')$, i.e., there exist $m \in C([0,1], \mathbb{R}^+)$ and $g, h, \psi, l \in C(I, I)$ such that $h, l$ are decreasing functions and $\frac{\psi}{l}, \frac{g}{h}$ are increasing functions and satisfies

$$f(t,x,y) \leq m(t)g(x)\psi(y), \quad \forall\, t \in [0,1], \forall\, x, y \in I,$$

and for any $c, c' > 0$,

$$\int_0^1 q(\tau)m(\tau)h(c\rho(\tau))l(c'\rho(t)d\tau < +\infty.$$

Assumption $(\mathcal{H}_2'')$ is

$$\sup_{c>0} \frac{c}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c)\psi(c)}{\Gamma(\alpha)h(c)l(c)} \int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)c)l(\lambda_2\rho(\tau)c)d\tau\right)} > 1.$$

Regarding Assumption $(\mathcal{H}_3'')$, there exist $a, b$ $(0 < a < b < 1)$ such that

$$\lim_{x \to +\infty} \frac{f(t,x,y)}{\phi(x)} = +\infty, \quad \text{uniformly in } t \in [a,b] \text{ and } y > 0.$$

As for Assumption $(\mathcal{H}_4'')$, we have that for any $c > 0$, there exist $\psi_c \in C([0,1], \mathbb{R}^+)$ and an interval $J \subset (0,1]$ such that $\psi_c(t) > 0$, in $J$ and

$$f(t,x,y) \geq \psi_c(t), \quad \forall\, t \in [0,1],\ \forall\, x, y \in (0,c].$$

For $f \in C([0,1] \times I \times I, \mathbb{R}^+)$, define the sequence $(f_n)_{n\geq 1}$ by

$$f_n(t,x,y) = f(t, \max\{\tfrac{1}{n}, x\}, \max\{\tfrac{1}{n}, y\}), \quad n \in \{1, 2, \ldots\}$$

and for $x \in \mathcal{P}$, define the sequence of operators

$$A_n(x)(t) = \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x(\tau), D_{0^+}^{\gamma}x(\tau))d\tau\right) ds.$$

Then

$$D_{0^+}^{\gamma}A_n(x)(t) = \int_0^1 D_{0^+}^{\gamma}G(t,s)\phi^{-1}\left(\int_0^1 H_2(s,\tau)q(\tau)f_n(\tau, x(\tau), D_{0^+}^{\gamma}x(\tau))d\tau\right) ds.$$

As for Proposition 3.1, we can prove

**Proposition 4.6.** *Suppose $(\mathcal{H}_1'')$ holds then, for each $n \geq 1$, the operator $A_n$ sends $\mathcal{P}$ into $\mathcal{P}$ and is completely continuous.*

As in the previous cases, we prove the existence of one solution and then two solutions. The first result is

**Theorem 4.7.** *Assume that $(\mathcal{H}_1''), (\mathcal{H}_2''), (\mathcal{H}_4'')$ hold. Then Problem (1.1) has at least one positive solution.*

*Proof.* From the condition $(\mathcal{H}_2'')$, there exists $R > 0$ such that

$$\frac{R}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(R)\psi(R)}{\Gamma(\alpha)h(R)l(R)} \int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)R)l(\lambda_2\rho(\tau)R)d\tau\right)} > 1. \quad (4.8)$$

Let $\Omega_1 = \{x \in E : \|x\| < R\}$. We claim that $x \neq \lambda A_n(x)$, for any $x \in \partial\Omega_1 \cap \mathcal{P}$, $\lambda \in (0,1]$ and $n \geq n_0 \geq \frac{1}{R}$. On the contrary, there exist $n_1 \geq n_0$, $x_1 \in \partial\Omega_1 \cap \mathcal{P}$ and $\lambda_0 \in (0,1]$ such that $x_1 = \lambda_0 A_{n_1} x_1$. Since $x_1 \in \partial\Omega_1 \cap \mathcal{P}$, then

$$x_1(t) \geq \lambda_1 \rho(t) \|x_1\| = \lambda_1 \rho(t) R, \ \forall\, t \in [0,1]$$

and

$$D^{\gamma}_{0+} x_1(t) \geq \lambda_2 \rho(t) \|x_1\| = \lambda_2 \rho(t) R, \ \forall\, t \in [0,1].$$

Hence

$$\begin{aligned}
R &= \|x_1\| \\
&= \|\lambda_0 A_{n_1} x_1\| \\
&\leq \|A_{n_1} x_1\|_1 + \|A x_1\|_2 \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}] \\
&\phi^{-1}\left(\int_0^1 H(s,\tau) q(\tau) f(\tau, \max\{\tfrac{1}{n}, x_1(\tau)\}, \max\{\tfrac{1}{n}, D^{\gamma}_{0+} x_1(\tau)\} d\tau\right) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}] \\
&\phi^{-1}\left(\tfrac{1}{\Gamma(\alpha)} \int_0^1 q(\tau) m(\tau) g(\max\{\tfrac{1}{n_1}, x_1(\tau)\}) \psi(\max\{\tfrac{1}{n}, D^{\gamma}_{0+} x_1(\tau)\}) d\tau\right) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}] \phi^{-1}(\tfrac{1}{\Gamma(\alpha)} \int_0^1 q(\tau) m(\tau) h(\max\{\tfrac{1}{n_1}, x_1(\tau)\}) \tfrac{g(\max\{\frac{1}{n_1}, x_1(\tau)\})}{h(\max\{\frac{1}{n_1}, x_1(\tau)\})} \\
&\tfrac{\psi(\max\{\frac{1}{n}, D^{\gamma}_{0+} x_1(\tau)\})}{l(\max\{\frac{1}{n}, D^{\gamma}_{0+} x_1(\tau)\})} l(\max\{\tfrac{1}{n}, D^{\gamma}_{0+} x_1(\tau)\}) d\tau) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}] \phi^{-1}\left(\tfrac{g(R)\psi(R)}{h(R) l(R) \Gamma(\alpha)} \int_0^1 q(\tau) m(\tau) h(\lambda_1 \rho(\tau) R) l(\lambda_2 \rho(\tau) R) d\tau\right)
\end{aligned}$$

which is a contraction to (4.8). By Lemma 1.4, we deduce that

$$i(A_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \ \text{ for all } \ n \in \{n_0, n_0+1, \ldots\}. \tag{4.9}$$

Then there exists $x_n \in \Omega_1 \cap \mathcal{P}$ such that $A_n x_n = x_n$; $\forall\ n \geq n_0$. As in the proof of Theorem 4.2(2), $(x_n)$ is proven to be relatively compact in $E$ and thus there exists a subsequence $(x_{n_k})_{k \geq 1}$ such that $\lim\limits_{k \to +\infty} x_{n_k} = x_0$, where $x_0$ is a positive solution of Problem (1.1). □

**Example 4.8.** Consider the BVP

$$\begin{cases} -D^{\frac{11}{6}}_{0+} \phi\left(-D^{\frac{11}{5}}_{0+} x(t)\right) = \delta t^{\frac{12}{5}} (1-t)^{\frac{12}{5}} e^{-t} \dfrac{e^{x + D^{\frac{1}{2}}_{0+} x}}{x D^{\frac{1}{2}}_{0+} x}, \ t \in (0,1) \\ x(0) = x'(0) = D^{\frac{5}{4}}_{0+} x(1) = D^{\frac{11}{5}}_{0+} x(0) = [D^{\frac{5}{6}}_{0+}(\phi(-D^{\frac{11}{5}}_{0+} x(t))]_{t=1} = 0, \end{cases} \tag{4.10}$$

where

$$f(t,x,y) = \delta t^{\frac{5}{4}} e^{-t} \frac{e^{x+y}}{xy}, \ (\delta > 0), \ q(t) = t^{\frac{12}{5}} (1-t)^{\frac{12}{5}} \text{ and } \phi(t) = t^3 + t.$$

Hence $\phi$ is an increasing homeomorphism such that $\phi(0) = 0$.
$(\mathcal{H}_1'')$ Let $m(t) = 1, g(x) = \frac{e^x}{x}, \psi(y) = \frac{e^y}{y}, h(y) = \frac{1}{y}, l(y) = \frac{1}{y}$. Then

$$f(t,x,y) \leq m(t) g(x) \psi(y), \ \ \forall\, t \in [0,1], \forall\, x, y \in I,$$

and for any $c, c' > 0$

$$\int_0^1 m(\tau)q(\tau)h(c\rho(\tau))l(c'\rho(\tau))d\tau = \frac{35}{36cc'} < +\infty.$$

$(\mathcal{H}_2'')$

$$\begin{aligned}&\sup_{c>0} \frac{c}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\big(\frac{g(c)\psi(c)}{\Gamma(\alpha)h(c)l(c)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)c)l(\lambda_2\rho(\tau)c)d\tau\big)}\\ &\geq \sup_{c>0} \frac{0.94c}{\phi^{-1}(\delta\frac{e^{2c}}{c^2})}\\ &\geq \frac{1}{\delta^{\frac{1}{4}}} \sup_{c>0} \frac{0.94c}{\phi^{-1}(\frac{e^{2c}}{c^2})}.\end{aligned}$$

$(\mathcal{H}_4'')$ For any $c > 0$ there exists $\psi_c = \frac{t^{\frac{12}{5}}(1-t)^{\frac{12}{5}}}{c^2}$ such that

$$f(t, x, y) \geq \psi_c(t), \quad \forall\, t \in [0, 1],\ \forall\, x \in (0, c], y \in (0, c].$$

For $\delta \leq \left(\sup_{c>0} \frac{0.94c}{\phi^{-1}(\frac{e^{2c}}{c^2})}\right)^4$, all conditions of Theorem 4.7 hold. Then Problem (4.10) has at least one positive solution.

The last result of this work concerns the existence of two positive solutions. The proof is similar to the proof of Theorem 4.4 and is omitted.

**Theorem 4.9.** *Assume that* $(\mathcal{H}_1'') - (\mathcal{H}_4'')$ *hold and there exist* $\alpha_1, \alpha_2$ *with* $0 < \alpha_1 < \alpha_2$ *such that*

$$t^{\alpha_2}\phi(x) \leq \phi(tx) \leq t^{\alpha_1}\phi(x),\ \forall\, t \in [0, 1],\ \forall\, x \geq 0.$$

*Then Problem* (1.1) *has at least two positive solutions.*

**Example 4.10.** Let the BVP

$$\begin{cases} -D_{0^+}^{\frac{11}{6}}\phi\left(-D_{0^+}^{\frac{11}{5}}x(t)\right) = \delta t^{\frac{12}{5}}(1-t)^{\frac{12}{5}}e^{-t}\dfrac{e^{x+D_{0^+}^{\frac{1}{2}}x}}{xD_{0^+}^{\frac{1}{2}}x},\ t \in (0, 1)\\ x(0) = x'(0) = D_{0^+}^{\frac{5}{4}}x(1) = D_{0^+}^{\frac{11}{5}}x(0) = [D_{0^+}^{\frac{5}{6}}(\phi(-D_{0^+}^{\frac{11}{5}}x(t))]_{t=1} = 0, \end{cases} \tag{4.11}$$

where

$$f(t, x, y) = \delta t^{\frac{5}{4}}e^{-t}\frac{e^{x+y}}{xy},\ (\delta > 0),\ q(t) = t^{\frac{12}{5}}(1-t)^{\frac{12}{5}} \text{ and } \phi(t) = t^3 + t.$$

Hence $\phi$ is an increasing homeomorphism such that $\phi(0) = 0$. Moreover there exist $\alpha_1 = 1, \alpha_2 = 4$ such that

$$t^4\phi(x) \leq \phi(tx) \leq t\phi(x), \forall\, t \in [0, 1],\ \forall\, x \geq 0.$$

Assumption $(\mathcal{H}_3''')$ reads

$$\lim_{x\to+\infty} \frac{f(t, x, y)}{\phi(x)} \geq \lim_{x\to+\infty} \frac{\delta a^{\frac{5}{4}}e^{-b}e^x}{(x + x^3)x} = +\infty, \quad \forall t \in [a, b], \forall y > 0.$$

If we choose $\delta$ such that $\delta \leq \left(\sup_{c>0} \frac{0.94c}{\phi(\frac{e^{2c}}{c^2})^{-1}}\right)^4$, all conditions of Theorem 4.9 hold. Consequently Problem (4.11) has at least two positive solutions.

**Remark 4.11.** The same results can be obtained in case the nonlinear function $f$ has a singular point at $y=0$ but not at $x=0$. The corresponding assumptions are
$(\mathcal{H}_1''')$ There exist $m \in C([0,1],\mathbb{R}^+), \psi \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $g, l, \in C(I,I)$ such that $l$ is a decreasing function and $g, \frac{\psi}{l}$ are increasing functions with

$$f(t,x,y) \leq m(t)g(x)\psi(y), \quad \forall\, t \in [0,1], \forall\, x \in \mathbb{R}^+,\ \forall\, y \in I$$

and for each $c>0$,

$$\int_0^1 q(\tau)m(\tau)l(c\rho(\tau))d\tau < +\infty,$$

$(\mathcal{H}_2''')$

$$\sup_{c>0} \frac{c}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c)\psi(c)}{\Gamma(\alpha)l(c)} \int_0^1 q(\tau)m(\tau)h(\lambda_2\rho(\tau)c)d\tau\right)} > 1.$$

$(\mathcal{H}_3''')$ There exist $a, b$ $(0<a<b<1)$ such that

$$\lim_{x\to+\infty} \frac{f(t,x,y)}{\phi(x)} = +\infty, \quad \text{uniformly in } \ t \in [a,b] \text{ and } y>0.$$

$(\mathcal{H}_4''')$ For any $c>0$ there exists $\psi_c \in C([0,1],\mathbb{R}^+)$ and there exists an interval $J \subset (0,1]$ such that $\psi_c(t)>0$, in $J$ and

$$f(t,x,y) \geq \psi_c(t), \quad \forall\, t \in [0,1], \forall\ \ \forall\, x \in [0,c], y \in (0,c]$$

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