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Starlikeness and close-to-convexity involving certain differential inequalities

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Abstract. In the present paper, we study certain differential inequalities involving meromorphic functions in the open unit disk and obtain certain sufficient conditions for starlikeness and close-to-convexity of meromorphic functions. In particular, we obtain:
1. If $f(z) \in \Sigma_p$ satisfies the differential inequality $\left|1+\frac{zf''(z)}{f'(z)}+p\right|<\frac{1}{2},\ z\in\mathbb{E}$, then $f(z)$ is meromorphic close-to-convex function.
2. If $f(z) \in \Sigma$ satisfies the differential inequality

$$\left|\frac{zf'(z)}{f(z)}+1\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|^{\gamma}<\frac{1-\alpha}{(1+|1-2\alpha|)^{\gamma}},\ \gamma\geq 0,\ z\in\mathbb{E},$$

then $f(z)$ is meromorphic starlike function of order α.

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1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}a_k z^{k-p}\ (p\in\mathbb{N}=\{1,2,3,\ldots\}),$$

which are analytic and p-valent in the punctured unit disc $\mathbb{E}_0=\mathbb{E}\setminus\{0\}$, where $\mathbb{E}=\{z\in\mathbb{C}:|z|<1\}$. A function $f\in\Sigma_p$ is said to be meromorphic p-valent starlike

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of order α if $f(z) \neq 0$ for $z \in \mathbb{E}$ and

$$-\Re\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}). \tag{1.1}$$

The class of all such meromorphic p-valent starlike functions is denoted by $\mathcal{MS}_p^*(\alpha)$. A function $f \in \Sigma_p$ is called meromorphic p-valent close-to-convex of order α if there exists a function $g \in \mathcal{MS}_p^*$ such that and

$$-\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}).$$

The class of all such meromorphic p-valent close-to-convex functions defined above is denoted by $\mathcal{MC}_p(\alpha)$.
Since $g(z) = z^{-p} \in \mathcal{MS}_p^*$, it follows that a function $f \in \Sigma_p$ satisfying

$$-\Re(z^{p+1}f'(z)) > 0,\ z \in \mathbb{E},$$

or

$$\left|z^{p+1}f'(z) + p\right| < p,\ z \in \mathbb{E}, \tag{1.2}$$

is a member of the class $\mathcal{MC}_p$.
Let $\Sigma = \Sigma_1$, $\mathcal{MS}^*(\alpha) = \mathcal{MS}_1^*(\alpha)$, $\mathcal{MS}^* = \mathcal{MS}_1^*(0)$, $\mathcal{MC}(\alpha) = \mathcal{MC}_1(\alpha)$ and $\mathcal{MC} = \mathcal{MC}_1(0)$.
In the literature of meromorphic functions, many authors obtained the conditions for meromorphic close-to-convex functions and meromorphic starlike functions. Some of the results from literature are given below:

Goyal and Prajapat [1] proved the following results:

Theorem 1.1. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - z^2f'(z) + 1\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha} \qquad (0 \leq \alpha < 1),$$

then $f \in \mathcal{MC}(\alpha)$.

Theorem 1.2. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - z^2f'(z) + 1\right| < \frac{3}{2},$$

then $f \in \mathcal{MC}$.

Theorem 1.3. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| < \frac{1}{2},$$

then $f \in \mathcal{MC}$.

Theorem 1.4. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)}\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha},\ (0 \leq \alpha < 1),$$

then $f \in \mathcal{MS}^*(\alpha)$.

Theorem 1.5. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1\right| < \frac{1}{2},$$

then $f \in \mathcal{MS}^*$.

Xu and Yang [4] proved the following results:

Theorem 1.6. *If $f \in \Sigma_n$ satisfies $f'(z) \neq 0$ in $\mathbb{E}_0$ and*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < a,$$

for some a $(0 < a \leq n)$, then $f \in \mathcal{MS}_n^*(e^{-a/n})$ and the order $e^{-a/n}$ is sharp.

Z-G Wang et al. [3] proved the following results:

Theorem 1.7. *If $f(z) \in \Sigma_p$ satisfies the following inequality*

$$\left|\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \mu \ \left(0 < \mu < \frac{1}{p}\right),$$

then $f \in \mathcal{MS}_p^*\left(\dfrac{p}{1+p\mu}\right)$.

Theorem 1.8. *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$\left|\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} - 1\right| < \delta \ (0 < \delta < 1),$$

then $f \in \mathcal{MS}_p^*(p(1-\delta))$.

2. Preliminaries

We shall use the following lemma of Jack [2] to prove our result.

Lemma 2.1. *Suppose w is a nonconstant analytic function in $\mathbb{E}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in \mathbb{E}$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = m w(z_0)$, where $m \geq 1$, is some real number.*

Theorem 2.2. *Let $f(z) \in \Sigma_p$ and suppose that it satisfies, for $\gamma \geq 0$, the inequality*

$$\left|z^{p+1}f'(z) + p\right|^{1-\gamma} \left|z^{p+2}f''(z) + (p+1)z^{p+1}f'(z)\right|^{\gamma} < p, \qquad z \in \mathbb{E}. \tag{2.1}$$

Then $|z^{p+1}f'(z) + p| < p$, i.e. $f(z) \in \mathcal{MC}_p$ and is a bounded function in $\mathbb{E}$.

Proof. For a function $f \in \Sigma_p$ satisfying the assumption (2.1), we define a function w by

$$w(z) = \frac{1}{p}\left(z^{p+1}f'(z) + p\right) = b_k z^k + \ldots, \qquad z \in \mathbb{E}. \tag{2.2}$$

Then w is analytic in $\mathbb{E}$ with $w(0) = 0$. To prove our conclusion we will show that $|w(z)| < 1, z \in \mathbb{E}$. Differentiating (2.2), we have

$$z^{p+2}f''(z) + (p+1)z^{p+1}f'(z) = pzw'(z) \tag{2.3}$$

From (2.2) and (2.3) we obtain that

$$\begin{aligned}\left|z^{p+1}f'(z)+p\right|^{1-\gamma}\left|z^{p+2}f''(z)+(p+1)z^{p+1}f'(z)\right|^{\gamma}\\ =|pw(z)|^{1-\gamma}|pzw'(z)|^{\gamma}\\ =p|w(z)|\left|\frac{zw'(z)}{w(z)}\right|^{\gamma},\ z\in\mathbb{E}.\end{aligned}\tag{2.4}$$

Supposing that there exists a point $z_0\in\mathbb{E}$ such that $\max\limits_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$. Then by Lemma 2.1, we have $z_0w'(z_0)=kw(z_0), k\geq 1$.
Hence, from (2.4) we obtain

$$\begin{aligned}\left|z_0^{p+1}f'(z_0)+p\right|^{1-\gamma}\left|z_0^{p+2}f''(z_0)+(p+1)z_0^{p+1}f'(z_0)\right|^{\gamma}&=|pw(z_0)|^{1-\gamma}|pzw'(z_0)|^{\gamma}\\ &=p|k|^{\gamma}\geq p,\end{aligned}$$

which contradicts (2.1).Therefore, $|w(z)|<1$ for all $z\in\mathbb{E}$, and the conclusion has been proved.
Finally, from (1.2) it follows that $|f'(z)|\leq 2p|z|^{-(p+1)}<2p,\ z\in\mathbb{E}$, hence

$$|f(z)|=\left|\int_0^z f'(t)dt\right|\leq\int_0^r|f'(\rho e^{\iota\theta})|d\rho\leq 2pr<2p,$$

$$z=re^{\iota\theta}\in\mathbb{E},\ \theta\in[0,2\pi).$$

Consequently, f is bounded in $\mathbb{E}$. □

Setting $\gamma=1$ in Theorem 2.2 reduces to the next result.

Corollary 2.3. *If $f\in\Sigma_p$ satisfies*

$$|z^{p+2}f''(z)+(p+1)z^{p+1}f'(z)|<p,\ z\in\mathbb{E},$$

then the inequality (1.2) holds, i.e., $f\in\mathcal{MC}_p$ and it is bounded function in $\mathbb{E}$.

Theorem 2.4. *Let $f(z)\in\Sigma_p$ and suppose that it satisfies, for $\gamma\geq 0$, the inequality*

$$\left|\frac{z^{p+1}f'(z)}{p}+1\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}+p\right|^{\gamma}<\left(\frac{1}{2}\right)^{\gamma},\qquad z\in\mathbb{E}.\tag{2.5}$$

Then $|z^{p+1}f'(z)+p|<p$, i.e. $f(z)\in\mathcal{MC}_p$ and is a bounded function in $\mathbb{E}$.

Proof. For a function $f\in\Sigma_p$ satisfying the assumption (2.5), we define a function w by (2.2). Then w is analytic in $\mathbb{E}$ with $w(0)=0$ and differentiating (2.2), we have

$$1+\frac{zf''(z)}{f'(z)}+p=\frac{zw'(z)}{w(z)-1}.\tag{2.6}$$

From the assumption (2.5), it follows that the left-hand side of (2.6) is an analytic function in $\mathbb{E}$, hence $w(z)\neq 1$ for all $z\in\mathbb{E}$. From (2.2) and (2.6) we have

$$\left|\frac{z^{p+1}f'(z)}{p}+1\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}+p\right|^{\gamma}=|w(z)|^{1-\gamma}\left|\frac{zw'(z)}{w(z)-1}\right|^{\gamma},\ z\in\mathbb{E}.\tag{2.7}$$

If we suppose that there exists a point $z_0 \in \mathbb{E}$ such that $\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$.
Then by Lemma 2.1, we have $z_0 w'(z_0) = kw(z_0), k \geq 1$.
Hence, from (2.7) we have

$$\left|\frac{z_0^{p+1} f'(z_0)}{p} + 1\right|^{1-\gamma} \left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} + p\right|^{\gamma} = |w(z_0)|^{1-\gamma} \left|\frac{z_0 w'(z_0)}{w(z_0) - 1}\right|^{\gamma}$$
$$= |w(z_0)| \left|\frac{k}{w(z_0) - 1}\right|^{\gamma}$$
$$\geq \left(\frac{1}{2}\right)^{\gamma},$$

which contradicts (2.5). Therefore, $|w(z)| < 1$ for all $z \in \mathbb{E}$ and our conclusion (1.2) has been proved.
Since under the assumption (2.5) the inequality holds, as in the proof of the previous theorem it follows that f is bounded in $\mathbb{E}$. □

Selecting $\gamma = 1$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.5. *If $f \in \Sigma_p$ satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)} + p\right| < \frac{1}{2}, \ z \in \mathbb{E},$$

then $|z^{p+1} f'(z) + p| < p$, i.e. $f(z) \in \mathcal{MC}_p$ and is a bounded function in $\mathbb{E}$.

Putting p=1 in the above corollary, we have the following result.

Corollary 2.6. *If $f \in \Sigma$ satisfies*

$$\left|2 + \frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}, \ z \in \mathbb{E},$$

then $|z^2 f'(z) + 1| < 1$, i.e. $f(z) \in \mathcal{MC}$ and is a bounded function in $\mathbb{E}$.

Remark 2.7. From above corollary, we obtained the result of Goyal and Prajapat [1, Corollary 3].

Theorem 2.8. *Let $f(z) \in \Sigma_p$ and suppose that it satisfies, for $\gamma \geq 0$, the inequality*

$$\left|\frac{zf'(z)}{f(z)} + p\right|^{1-\gamma} \left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right|^{\gamma} < \frac{p - \alpha}{(p + |p - 2\alpha|)^{\gamma}}, \qquad z \in \mathbb{E}, \tag{2.8}$$

then assume that for $f(z) \neq 0$, $f(z) \in \mathcal{MS}_p^*(\alpha)$.

Proof. For a function $f \in \Sigma_p$ satisfying the assumption (2.8), we define a function w by

$$\frac{-zf'(z)}{f(z)} = \frac{p + (p - 2\alpha)w(z)}{1 - w(z)}, \ z \in \mathbb{E}, \qquad (0 \leq \alpha < p). \tag{2.9}$$

Since $w(z) = b_k z^k + \ldots$ is analytic in $\mathbb{E}$ with $w(0) = 0$ and from assumption (2.8) it follows that the left hand side of (2.9) is an analytic function in $\mathbb{E}$, hence $w(z) \neq 1$

for all $z \in \mathbb{E}$.
Differentiating (2.9), we have

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{2(p-\alpha)zw'(z)}{(p+(p-2\alpha)w(z))(1-w(z))},\ z\in\mathbb{E}. \tag{2.10}$$

From (2.9) and (2.10), we get

$$\left|\frac{zf'(z)}{f(z)}+p\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|^{\gamma}$$
$$=2(p-\alpha)\left|\frac{w(z)}{1-w(z)}\right|\left|\frac{\frac{zw'(z)}{w(z)}}{p+(p-2\alpha)w(z)}\right|^{\gamma},\ z\in\mathbb{E}. \tag{2.11}$$

If we suppose that there exists a point $z_0 \in \mathbb{E}$ such that $\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1$.
Then by Lemma 2.1, we have $z_0w'(z_0)=kw(z_0)$, $k\ge 1$.

$$\left|\frac{z_0f'(z_0)}{f(z_0)}+p\right|^{1-\gamma}\left|1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right|^{\gamma}$$
$$=2(p-\alpha)\left|\frac{w(z_0)}{1-w(z_0)}\right|\left|\frac{\frac{z_0w'(z_0)}{w(z_0)}}{p+(p-2\alpha)w(z_0)}\right|^{\gamma},\ z\in\mathbb{E}, \tag{2.12}$$
$$\ge(p-\alpha)\left|\frac{k}{p+(p-2\alpha)w(z_0)}\right|^{\gamma},$$
$$\left|\frac{z_0f'(z_0)}{f(z_0)}+p\right|^{1-\gamma}\left|1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right|^{\gamma}\ge\frac{p-\alpha}{(p+|p-2\alpha|)^{\gamma}},$$

which contradicts (2.8).
This proves that $|w(z)<1$ for all $z\in\mathbb{E}$ and hence $f(z)\in\mathcal{MS}_p^*(\alpha)$. □

Putting $p=1$ in Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let $f\in\Sigma$ and suppose that f satisfies, for $\gamma\ge 0$, the inequality*

$$\left|\frac{zf'(z)}{f(z)}+1\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|^{\gamma}<\frac{1-\alpha}{(1+|1-2\alpha|)^{\gamma}},\ z\in\mathbb{E},$$

then $f\in\mathcal{MS}^*(\alpha)$.

If we take $\alpha=0$ in Theorem 2.8, then we obtain the next corollary.

Corollary 2.10. *Let $f\in\Sigma_p$ and suppose that f satisfies, for $\gamma\ge 0$, the inequality*

$$\left|\frac{zf'(z)}{f(z)}+p\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|^{\gamma}<\frac{p}{(2p)^{\gamma}},\ z\in\mathbb{E},$$

then $f\in\mathcal{MS}_p^*$.

For $p=1$ and $\gamma=1$, above corollary reduces to

Corollary 2.11. *Let $f \in \Sigma$ satisfies the inequality*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|<\frac{1}{2},\ z\in\mathbb{E},$$

and $f(z)\neq 0$ for all $z\in\mathbb{E}_0$ then $f\in\mathcal{MS}^*$.

Remark 2.12. From above corollary, we obtained another result of Goyal and Prajapat [1, Corollary 7].

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