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Generalized q-Srivastava-Attiya operator on multivalent functions

Rizwan Salim Badar and Khalida Inayat Noor

Abstract. In this article, we define a generalized q-integral operator on multivalent functions. It generalizes many known linear operators in Geometric Function Theory (GFT). Inclusions results, convolution properties and q-Bernardi integral preservation of the subclasses of analytic functions are discussed.

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1. Introduction

The study of q-extension of classical calculus has been point of focus for various researchers due to its applications. In Physics, q-calculus is amicably used in theories of quantum fields, Newton quantum gravity, special relativity and many other notable fields. In Mathematics, various branches has been established due to its applications in basic hypergeometric functions, combinatorics, calculus of variations, optimal control problems, q-transform analysis. It dates back to great mathematicians of 17th century L. Euler and C. Jacobi. F.H. Jackson formally defined q-difference operator and q-integral operator in [8, 9]. For comprehensive details of concepts of q-calculus, see [5]. The concepts of GFT has been studied in context of q-calculus and q-analogues of various subclasses of analytic functions are defined by the researchers, see [20, 7, 1, 14, 15, 12, 11, 4, 21, 10]. Using the convolution of normalized analytic functions, several q-operators are defined by the researchers, see details in [19]. In this paper we define a generalized q-Srivastava Attiya operator and study its application on multivalent functions.

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Let $A(p)$ $(p \in \mathbb{N} = \{0, 1, 2, ..\})$ denote the set of multivalent functions say f given as

$$f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \tag{1.1}$$

analytic in the open unit disc $E = \{z : |z| < 1\}$.

Let $f(z)$ be given by (1.1) and $g(z)$ defined as

$$f(z) = z^p + \sum_{k=2}^{\infty} b_{k+p-1} z^{k+p-1}.$$

Then Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} b_{k+p-1} z^{k+p-1}.$$

Let $f, g \in A$. Then f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, $z \in E$, if there exists a Schwartz function $\omega(z)$ analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in E$ such that $f(z) = g(\omega(z))$.

A subset $D \subset \mathbb{C}$ is called q-geometric if $zq \in D$ whenever $z \in D$ and it contains all the geometric sequences $\left\{zq^k\right\}_0^{\infty}$. In GFT, the q-derivative of $f(z)$ is defined as;

$$d_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ q \in (0, 1), \quad (z \in D \setminus \{0\}), \tag{1.2}$$

and $d_q f(0) = f'(0)$. For a function $g(z) = z^k$, the q-derivative is

$$d_q g(z) = [k] z^{k-1},$$

where $[k] = \frac{1-q^k}{1-q} = 1 + q + q^2 + ... + q^{k-1}$.

From (1.1) and (1.2) we easily get that

$$d_q f(z) = [p]\, z^{p-1} + \sum_{k=2}^{\infty} [k+p-1]\ a_{k+p-1} z^{k+p-2}.$$

Let $f(z)$ and $g(z)$ be defined on a q-geometric set $D \subset \mathbb{C}$ such that q-derivatives of $f(z)$ and $g(z)$ exist $\forall\ z \in \mathbb{C}$. Then for complex numbers b, c we have:

$$d_q(bf(z) \pm cg(z)) = bd_q f(z) \pm cd_q g(z).$$

$$d_q(f(z)g(z)) = f(qz)d_q g(z) + g(z)d_q f(z).$$

$$d_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\ d_q f(z) - f(z)\ d_q g(z)}{g(z)g(qz)}, \ g(z)g(qz) \neq 0.$$

$$d_q\left(\log f(z)\right) = \frac{\ln q^{-1}}{1-q} \frac{d_q f(z)}{f(z)}.$$

Jackson [8] introduced the q-integral of a function f is given by

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(q^k z),$$

provided that series converges.

Consider a q-analogue of Lerch-Hurwitz function

$$\Phi_q(s,b;z)=\sum_{k=0}^{\infty}\frac{z^k}{[k+b]^s},\, z\in E,$$

($b\in\mathbb{C}\setminus\mathbb{Z}_0^-$; $s\in\mathbb{C}$ when $|z|<1$; $\mathrm{Re}(s)>1$ when $|z|=1$), which is a convergent series of radius 1. Now we define the generalized q-Srivastava Attiya operator $J_{q,b}^{s,p}: A(p)\to A(p)$ as

$$J_{q,b}^{s,p}f(z)=\Psi_q(s,b;z)*f(z), \tag{1.3}$$

where,

$$\Psi_q(s,b;z)=[1+b]^s\left\{\Phi_q(s,b;z)-[b]^{-s}\right\}. \tag{1.4}$$

From (1.1), (1.3) and (1.4), we have

$$J_{q,b}^{s,p}f(z)=z^p+\sum_{k=2}^{\infty}\left(\frac{[1+b]}{[k+b]}\right)^s a_{k+p-1}z^{k+p-1}. \tag{1.5}$$

We observe that $J_{q,b}^{0,p}f(z)=f(z)$. The operator $J_{q,b}^{s,p}$ reduces to known linear operators for different values of parameters p,b and s as:

(i) For $p=1$, $s=1, b=0$ and $q\to 1^-$, $J_{q,b}^{s,p}$ reduces to Alexander operator [2].

(ii) If $p=1$, it is q-Srivastava Attiya operator discussed in [3].

(iii) For $p=1$, $s=1, b>0$ it reduces to q-Choi-Saigo-Srivastava operator discussed in [22].

(iv) For $s=\alpha$ ($\alpha>0$), $b=p$ and $q\to 1^-$, it is operator discussed in [16].

For any complex number s, the operator $I_{q,s}^{b,p}: A(p)\to A(p)$ is defined as;

$$I_{q,b}^{s,p}f(z)=z^p+\sum_{k=2}^{\infty}\left(\frac{[k+b]}{[1+b]}\right)^s a_{k+p-1}z^{k+p-1}. \tag{1.6}$$

The operator $I_{q,b}^{s,p}$ also reduces to known linear operators as:

(i) For $p=1$, $s\in\mathbb{N}_0$, $b=0$, it is q-Sãlãgean differential operator [6].

(ii) For $p=1$, $s=-1$ and $q\to 1^-$, it reduces to Owa-Srivastava Integral Operator [13].

The following identities holds for the two operators $J_{q,b}^{s,p}(z)$and $I_{q,b}^{s,p}(z)$,

$$zd_q(J_{q,b}^{s+1,p}f(z))=q^{p-1}\left(1+\frac{[b]}{q^b}\right)J_{q,b}^{s,p}f(z)+\left(\frac{[p-1]-[b]}{q^b}\right)J_{q,b}^{s+1,p}f(z). \tag{1.7}$$

$$zd_q(I_{q,s}^{b,p}f(z))=q^{p-1}\left(1+\frac{[b]}{q^b}\right)I_{q,s+1}^{b,p}f(z)+\left(\frac{[p-1]-[b]}{q^b}\right)I_{q,s}^{b,p}f(z). \tag{1.8}$$

Here we prove the identity (1.7) as;

$$zd_q(J_{q,b}^{s+1,p}f(z))=[p]\,z^p+\sum_{k=2}^{\infty}\left(\frac{[1+b]}{[k+b]}\right)^{s+1}a_{k+p-1}\,[k+p-1]\,z^{k+p-1},$$

or equivalently,

$$zd_q(J_{q,b}^{s+1,p} f(z)) = [p]\, z^p + \sum_{k=2}^{\infty} \left(\tfrac{[1+b]}{[k+b]}\right)^{s+1} \cdot a_{k+p-1} \left[(k+b) + (p-b-1)\right] z^{k+p-1}.$$

Using the property $[a+b] = q^b[a] + [b]$ we have:

$$zd_q(J_{q,b}^{s+1,p} f(z)) = [p]z^p + \sum_{k=2}^{\infty} \left(\frac{[1+b]}{[k+b]}\right)^{s+1} a_{k+p-1} \left\{q^{p-b-1}[k+b] + [p-b-1]\right\} z^{k+p-1}.$$

By adding and subtracting the terms $q^{p-b-1}[1+b]z^p$ and $[p-b-1]z^p$, using the property $[a+b] = q^b[a] + [b]$ and rearranging the terms: we get

$$zd_q(J_{q,b}^{s+1,p} f(z)) = q^{p-1}\left(1 + \frac{[b]}{q^b}\right) J_{q,b}^{s,p} f(z) + \left(\frac{[p-1]-[b]}{q^b}\right) J_{q,b}^{s+1,p} f(z).$$

On same lines, we can prove the identity (1.8) as well.

Definition 1.1. A function $f \in A(p)$ is said to be in the class $ST_q^p(\varphi)$ if and only if

$$\frac{zd_q f(z)}{[p]\, f(z)} \prec \varphi(z); \tag{1.9}$$

where $\varphi \in \Omega$, the class of analytic and convex multivalent functions in E.

Definition 1.2. A function $f \in A(p)$ is said to be in the class $CV_q^p(\varphi)$ if and only if

$$\frac{d_q(zd_q f(z))}{[p]\, d_q f(z)} \prec \varphi(z);$$

where $\varphi \in \Omega$, the class of analytic and convex multivalent functions in E.

By using operators given by (1.5) and (1.6), we define the classes

$$ST_{q,b}^{s,p}(\varphi) = \left\{f \in A(p) : J_{q,b}^{s,p} f(z) \in ST_q^p(\varphi)\right\}$$

and

$$\widetilde{ST}_{q,s}^{b,p}(\varphi) = \left\{f \in A(p) : I_{q,s}^{b,p} f(z) \in ST_q^p(\varphi)\right\}.$$

Similarly

$$CV_{q,b}^{s,p}(\varphi) = \left\{f \in A(p) : J_{q,b}^{s,p} f(z) \in CV_q^p(\varphi)\right\}$$

and

$$\widetilde{CV}_{q,s}^{b,p}(\varphi) = \left\{f \in A(p) : I_{q,s}^{b,p} f(z) \in CV_q^p(\varphi)\right\}.$$

It is noted

$$f \in CV_{q,b}^{s,p}(\varphi) \Leftrightarrow \frac{zd_q f(z)}{[p]} \in ST_{q,b}^{s,p}(\varphi)$$

and

$$f \in \widetilde{CV}_{q,s}^{b,p}(\varphi) \Leftrightarrow \frac{zd_q f(z)}{[p]} \in \widetilde{ST}_{q,s}^{b,p}(\varphi).$$

We need the following Lemma to obtain our results.

Lemma 1.3. [17] *Let β and γ be complex numbers with $\beta \neq 0$, and let $h(z)$ be a regular in E with $h(0) = 1$ and $\mathrm{Re}[\beta h(z) + \gamma] > 0$. If $p(z) = 1 + p_1 z + p_2 z^2 + ...$is analytic in E, then $p(z) + \frac{z d_q p(z)}{\beta p(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z)$.*

2. Main results

2.1. Inclusions results

In this section, we proved the inclusions results of the classes with respect to parameter s.

Theorem 2.1. *Let $\varphi(z)$ be analytic and convex multivalent function with $\varphi(0) = 1$ and $\mathrm{Re}(\varphi(z)) > 0$ for $z \in E$. Then $ST_{q,b}^{s,p}(\varphi) \subset ST_{q,b}^{s+1,p}(\varphi)$ if $\mathrm{Re}(q^{b-p+1}) > 1$.*

Proof. Let $f \in ST_{q,b}^{s,p}(\varphi)$ and we set

$$\frac{z d_q (J_{q,b}^{s+1,p} f(z))}{[p] J_{q,b}^{s+1,p} f(z)} = Q(z), \ z \in E, \tag{2.1}$$

where $Q(z)$ is analytic in E with $Q(0) = 1$.

Using identity (1.7), we get

$$\frac{z d_q (J_{q,b}^{s+1,p} f(z))}{J_{q,b}^{s+1,p} f(z)} = M_q \frac{J_{q,b}^{s,p} f(z)}{J_{q,b}^{s+1,p} f(z)} - \gamma_q, \tag{2.2}$$

where $M_q = q^{p-1}\left(1 + \frac{[b]}{q^b}\right)$ and $\gamma_q = \frac{[p-1]-[b]}{q^b}$.

From (2.1) and (2.2), we have

$$[p]Q(z) + \gamma_q = M_q \frac{J_{q,b}^{s,p} f(z)}{J_{q,b}^{s+1,p} f(z)}.$$

Applying logarithmic q-differentiation,

$$\frac{z d_q (J_{q,b}^{s,p} f(z))}{J_{q,b}^{s,p} f(z)} = [p]Q(z) + \frac{[p] z d_q Q(z)}{Q(z) + \gamma_q}. \tag{2.3}$$

As $f \in ST_{q,b}^{s,p}(\varphi)$ so,

$$\frac{z d_q (J_{q,b}^{s,p} f(z))}{J_{q,b}^{s,p} f(z)} \prec \varphi(z). \tag{2.4}$$

From (2.3) and (2.4), we have

$$[p]Q(z) + \frac{[p] z d_q Q(z)}{Q(z) + \gamma_q} \prec \varphi(z).$$

As $\mathrm{Re}(q^{b-p+1}) > 1$ and $\mathrm{Re}(\varphi) > 0$ then by Lemma 1.3, we have $Q(z) \prec \varphi(z)$ which implies $f \in ST_{q,b}^{s+1,p}(\varphi)$. So $ST_{q,b}^{s,p}(\varphi) \subset ST_{q,b}^{s+1,p}(\varphi)$. □

Theorem 2.2. *Let $\varphi(z)$ be same as in Theorem 2.1. Then $CV_{q,b}^{s,p}(\varphi) \subset CV_{q,b}^{s+1,p}(\varphi)$ if $\mathrm{Re}(q^{b-p+1}) > 1$.*

Proof. It is evident from the fact $f \in CV_{q,b}^{s,p}(\varphi) \Leftrightarrow \frac{zd_q f(z)}{[p]} \in ST_{q,b}^{s,p}(\varphi)$. □

We can easily prove the following result by using Lemma 1.3 and identity relation (1.8).

Theorem 2.3. *Let $\varphi(z)$ be analytic and convex multivalent function with $\varphi(0) = 1$ and* $\mathrm{Re}(\varphi(z)) > 0$ *for $z \in E$. Then*

$$\widetilde{ST}_{q,s}^{b,p}(\varphi) \subset \widetilde{ST}_{q,s+1}^{b,p}(\varphi)$$

and

$$\widetilde{CV}_{q,s+1}^{b,p}(\varphi) \subset \widetilde{CV}_{q,s}^{b,p}(\varphi).$$

2.2. Integral preservation under generalized q-Bernardi integral operator

In [18], the q-Bernardi integral operator $L_{c,p}f(z)$ for multivalent functions is defined as:

$$L_{c,p}f(z) = \frac{[p+c]}{z^c}\int_0^z t^{c-1}f(t)d_qt, \tag{2.5}$$

where $f \in A(p)$ given by (1.1) with $c > -p$.

Theorem 2.4. *If $f \in ST_{q,b}^{s,p}(\varphi)$ then $L_{c,p}f(z) \in ST_{q,b}^{s,p}(\varphi)$.*

Proof. Let $f(z) \in ST_{q,b}^{s,p}(\varphi)$. Consider

$$\frac{zd_q(J_{q,b}^{s,p}(L_{c,p}f(z)))}{[p]\, J_{q,b}^{s,p}(L_{c,p}f(z))} = Q(z), \tag{2.6}$$

where $Q(z)$ is analytic in E with $Q(0) = 1$. From (2.5), after some calculations, we can write

$$zd_q(L_{b,p}f(z)) = [p+c]\, f(z) - [c]\, L_{c,p}f(z).$$

Now applying the operator $J_{q,b}^{s,p}$ on both sides, we have

$$\frac{zd_q(J_{q,b}^{s,p}(L_{c,p}f(z)))}{J_{q,b}^{s,p}(L_{c,p}f(z))} = [p+c]\,\frac{J_{q,b}^{s,p}f(z)}{J_{q,b}^{s,p}(L_{c,p}f(z))} - [c]. \tag{2.7}$$

Now applying logarithmic q-differentiation on both sides of (2.7), after some calculations and using (2.6), we get

$$\frac{zd_q(J_{q,b}^{s,p}f(z))}{[p]\, J_{q,b}^{s,p}f(z)} = Q(z) + \frac{z[d_qQ(z)]}{[p]\, Q(z) + [c]}. \tag{2.8}$$

As $f \in ST_{q,b}^{s,p}(\varphi)$, so from (2.7) and (2.8), we have

$$Q(z) + \frac{z[d_qQ(z)]}{[p]\, Q(z) + [b]} \prec \varphi(z).$$

As $\mathrm{Re}([p]\varphi(z) + [c]) > 0$ so by Lemma 1.3, we have $Q(z) \prec \varphi(z)$ which implies $L_{c,p}f(z) \in ST_{q,b}^{s,p}(\varphi)$. □

Theorem 2.5. *If $f(z) \in CV_{q,b}^{s,p}(\varphi)$ then $L_{c,p}f(z) \in CV_{q,b}^{s,p}(\varphi)$.*

Proof. It is immediate consequence of the fact $f \in CV_{q,b}^{s,p}(\varphi) \Leftrightarrow \frac{zd_q f(z)}{[p]} \in ST_{q,b}^{s,p}(\varphi)$. □

2.3. Convolution property of $ST_{q,b}^{s,p}(\varphi)$

We now obtain convolution property for the class $ST_{q,b}^{s,p}(\varphi)$.

Theorem 2.6. *Let $f \in ST_{q,b}^{s,p}(\varphi)$. Then*

$$f(z) = z^{[p]}.\exp\left(\frac{\ln q^{-1}}{1-q}[p]\int_0^z \frac{\varphi(\omega(\varsigma))-1}{\varsigma}d_q\varsigma\right) * \left(z^p + \sum_{k=2}^{\infty}\left(\frac{[k+b]}{[1+b]}\right)^s a_{k+p-1}z^{k+p-1}\right), \quad (2.9)$$

where ω is Schwartz function.

Proof. Suppose that $f \in ST_{q,b}^{s,p}(\varphi)$. The subordination condition (1.9) can be written as:

$$\frac{zd_q(J_{q,b}^{s,p}f(z))}{J_{q,b}^{s,p}f(z)} = [p]\,\varphi(\omega(z)), \quad (2.10)$$

where ω is Schwartz function.

From (2.10), after q-integrating we get

$$\log\left(\frac{J_{q,b}^{s,p}f(z)}{z^{[p]}}\right) = \frac{\ln q^{-1}}{1-q}[p]\int_0^z \frac{\varphi(\omega(\varsigma))-1}{\varsigma}d_q\varsigma. \quad (2.11)$$

It follows from (2.11) that

$$J_{q,b}^{s,p}f(z) = z^{[p]}.\exp\left(\frac{\ln q^{-1}}{1-q}[p]\int_0^z \frac{\varphi(\omega(\varsigma))-1}{\varsigma}d_q\varsigma\right). \quad (2.12)$$

The assertion can be obtained easily from (1.5) and (2.12). □

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Rizwan Salim Badar
Allama Iqbal Open University,
Department of Mathematics, Sector H-8,
Islamabad, Pakistan
e-mail: `rizwan.salim@aiou.edu.pk`

Khalida Inayat Noor
Comsats University Islamabad,
Department of Mathematics, Park Road,
Islamabad, Pakistan
e-mail: `khalidainayat@comsats.edu.pk`