

A numerical method for two-dimensional Hammerstein integral equations

Sanda Micula

Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. In this paper we investigate a collocation method for the approximate solution of Hammerstein integral equations in two dimensions. As in [8], collocation is applied to a reformulation of the equation in a new unknown, thus reducing the computational cost and simplifying the implementation. We start with a special type of piecewise linear interpolation over triangles for a reformulation of the equation. This leads to a numerical integration scheme that can then be extended to any bounded domain in \mathbb{R}^2 , which is used in collocation. We analyze and prove the convergence of the method and give error estimates. As the quadrature formula has a higher degree of precision than expected with linear interpolation, the resulting collocation method is superconvergent, thus requiring fewer iterations for a desired accuracy. We show the applicability of the proposed scheme on numerical examples and discuss future research ideas in this area.

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1. Introduction

Integral equations are a special topic in Applied Mathematics, as they are an important tool for modeling many applications in fields ranging from engineering, computer graphics to astrophysics, chemistry, quantum mechanics and more (see [13]). They also arise in reformulations of initial and boundary value problems for ordinary and partial differential equations.

Having such a wide variety of applications, they have been studied extensively, especially from the approximation perspective. Numerical solutions have been obtained using moving least squares [5], Adomian decomposition, [4], kernel methods [6], collocation [8, 3, 10, 7], Galerkin and Nyström methods [9]. Also, good results were

obtained using wavelets [11, 12] and other iterative methods [1]. For more details on approximating methods for integral equations, see [2].

In this paper, we consider the following integral equation of Hammerstein type

$$u(x) = \int_D k(x, y)g(y, u(y)) dy + f(x), \quad x \in D \subset \mathbb{R}^2, \quad (1.1)$$

with a smooth kernel k and $g : D \times \mathbb{R} \rightarrow \mathbb{R}$ a continuous nonlinear function. Later, other assumptions will be made on k, g and f .

As in [8], collocation is applied to a reformulation of the equation in a new unknown, thus reducing the computational cost and simplifying the implementation. For the new integral equation, we define a collocation scheme based on linear interpolation and show that at the collocation nodes, it converges faster than over the entire domain, thus requiring fewer iterations for a given accuracy. These two aspects make this method much more efficient from the computational point of view.

In operator form, we write (1.1) as

$$u = Ku + f, \quad (1.2)$$

where

$$(Ku)(x) = \int_D k(x, y)g(y, u(y)) dy. \quad (1.3)$$

Following the ideas in [8], we reformulate (1.1) for the new unknown

$$v(y) := g(y, u(y)).$$

By (1.1), v must satisfy the equation

$$v(x) = g \left(x, \int_D k(x, y)v(y)dy + f(x) \right), \quad x \in D \quad (1.4)$$

and u is given by

$$u(x) = \int_D k(x, y)v(y)dy + f(x), \quad x \in D. \quad (1.5)$$

We define a collocation scheme for v in equation (1.4), which will be then used to find an approximate solution of (1.5).

We are interested in finding a numerical solution of equation (1.1), which approximates the exact solution, assumed to exist. To this end, we work under the following assumptions:

- (A1) The equation (1.3) has an isolated solution u^* with non-zero index, which is assumed to be smooth enough;
- (A2) The integral operator $K : C(D) \rightarrow C(D)$ defined by (1.3) is completely continuous;
- (A3) The derivative $g_u(y, u)$ exists and is continuous on $D \times \mathbb{R}$;
- (A4) The function $f \in C(D)$.

The ideas and results described in this paper work for any closed bounded domain $D \subset \mathbb{R}^2$ that can be triangulated in a smooth way. For simplicity, we restrict the discussion to the case of a rectangular region $D = [a, b] \times [c, d]$.

The rest of the paper is organized as follows: in Section 2, we define a collocation scheme for equation (1.4), based on a special type of linear interpolation on a triangular region. We prove the convergence and give error estimates (for both u and v) in Section 3. Section 4 shows the applicability of the proposed method to numerical examples, where the theoretical error bounds are confirmed by the numerical results. We draw some important conclusions and discuss future research ideas in Section 5.

2. Numerical method

2.1. Preliminaries for collocation

Let us recall the collocation method in the general framework of projection methods. Consider a set of nodes $\{x_1, \dots, x_n\} \subset D$ and let $\{l_1, \dots, l_n\}$ be a set of functions defined on D such that

$$l_j(x_i) = \delta_{ij}, 1 \leq i, j \leq n.$$

Denote by $D_n = \text{span}\{l_1, \dots, l_n\}$ and define the interpolatory projection operator $P_n : D \rightarrow D_n$ by

$$(P_n u)(x) = \sum_{j=1}^n u(x_j) l_j(x), x \in D. \tag{2.1}$$

Then $P_n : C(D) \rightarrow C(D)$ is a linear operator (see e.g. [2]) and its norm is given by

$$\|P_n\| = \sup_{x \in D} \sum_{j=1}^n |l_j(x)|.$$

We will assume that $\|P_n\| < \infty$ and that

$$\lim_{n \rightarrow \infty} \|u - P_n u\| = 0, \text{ for all } u \in C(D). \tag{2.2}$$

Let v^* be the solution of (1.4) corresponding to u^* . Using P_n , we define an approximation of v^* by

$$v_n(x) = P_n v(x) = \sum_{j=1}^n v_n(x_j) l_j(x). \tag{2.3}$$

The values $\{v_n(x_j)\}_{j=1}^n$ are determined by forcing equation (1.4) to be true at the collocation points. This leads to the system

$$v_n(x_i) = g \left(x_i, \sum_{j=1}^n v_n(x_j) \int_D k(x_i, y) l_j(y) dy + f(x_i) \right), i = 1, \dots, m, \tag{2.4}$$

or

$$\sum_{j=1}^n v_n(x_j) l_j(x_i) = g \left(x_i, \sum_{j=1}^n v_n(x_j) \int_D k(x_i, y) l_j(y) dy + f(x_i) \right). \tag{2.5}$$

It is worth mentioning that the integrals on the right hand side only have to be evaluated once, not at every iteration, since they are dependent only on the basis functions, *not* on v_n . This reduces the computational cost of the method and simplifies the implementation.

From (2.5), the approximate solution of (1.5) is found by

$$\begin{aligned} u_n(x) &= \int_D k(x, y)v_n(y)dy + f(x) \\ &= \sum_{j=1}^n v_n(x_j) \int_D k(x, y)l_j(y)dy + f(x). \end{aligned} \tag{2.6}$$

For the two approximate solutions, the following result holds:

Theorem 2.1. ([8, Theorem 2]) *Assume conditions (A1) – (A4) hold and that the operator P_n defined in (2.1) satisfies (2.2). Then*

$$\|v_n - v^*\| \rightarrow 0, \quad \|u_n - u^*\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Moreover, there exists an $n_0 \in \mathbb{N}$ and a constant c , independent of n , such that for all $n \geq n_0$,

$$\|u_n - u^*\| \leq c \inf_{\phi \in D_n} \|\phi - v^*\|.$$

This means that u_n converges to u^* at least as fast as v_n converges to v^* .

2.2. Interpolation-based collocation

To define the projection operator P_n , we start with piecewise linear interpolation of an unknown function on a triangle. First, we consider the unit simplex $\sigma = \{(s, t) \mid 0 \leq s, t, s + t \leq 1\}$. Let h be a continuous function on σ and denote by $w = 1 - s - t$. To approximate h , we use linear interpolation

$$h(s, t) \approx \sum_{i=1}^3 h(\tau_i)l_i(s, t), \tag{2.7}$$

where the nodes

$$\tau_1 = \left(\frac{1}{2}, \frac{1}{2}\right), \tau_2 = \left(\frac{1}{2}, 0\right), \tau_3 = \left(0, \frac{1}{2}\right) \tag{2.8}$$

are the midpoints of the three sides of σ and

$$l_1(s, t) = 1 - 2w, \quad l_2(s, t) = 1 - 2t, \quad l_3(s, t) = 1 - 2s \tag{2.9}$$

are the corresponding Lagrange interpolation basis functions. Obviously, the approximation formula (2.7) is exact for all polynomials of degree less than or equal to 1.

This formula can be extended to any triangle Δ , using an affine mapping $T : \sigma \xrightarrow[onto]{} \Delta$ given by

$$x = (\xi, \eta) = T(s, t) = wz_1 + tz_2 + sz_3, \tag{2.10}$$

where z_1, z_2, z_3 are the vertices of Δ . This mapping transforms a polynomial of degree r in (s, t) into a polynomial of the same degree in (ξ, η) (and its inverse acts the same

way on polynomials in (ξ, η) .

Denote by

$$q_i = T(\tau_i), i = 1, 2, 3. \tag{2.11}$$

For a given $h \in C(\Delta)$, define $P_n h$ by

$$\begin{aligned} P_n h(x) &= P_n h(T(s, t)) \\ &= \sum_{i=1}^3 h(q_i) l_i(s, t), \quad (s, t) \in \sigma. \end{aligned} \tag{2.12}$$

Then the approximation formula

$$h(x) \approx P_n h(x) = \sum_{i=1}^3 h(q_i) l_i(s, t), \tag{2.13}$$

is still exact for polynomials of degree $r \leq 1$.

From general interpolation theory, we have the following error bound for this approximation (see e.g. [2]).

Lemma 2.2. *Let Δ be a planar triangle and assume $h \in C^2(\Delta)$. Then, the following holds*

$$\|h - P_n h\|_\infty \leq c \delta^2 \|D^2 h\|_\infty, \tag{2.14}$$

where $\delta = \text{diameter}(\Delta)$ and $D^2 h = \max_{0 \leq i \leq 2} \left| \frac{\partial^2 h(\xi, \eta)}{\partial \xi^i \partial \eta^{2-i}} \right|$. The constant c is independent of both h and Δ .

Now, to define our collocation method, let $\mathcal{T}_n = \{\Delta_1, \dots, \Delta_n\}$ be a triangulation of D with grid size δ_n and $T_k : \sigma \rightarrow \Delta_k$ be defined as in (2.10), for every $k = 1, \dots, n$. At each iteration, the triangulation is refined by splitting each triangle into four triangles, obtained by connecting the midpoints of the three sides. The new triangulation, \mathcal{T}_{4n} , has four times as many triangles and grid size $\delta_{4n} = \frac{1}{2} \delta_n$.

For a function $h \in C(D)$, restrict it to some $\Delta \in \mathcal{T}_n$ and use (2.13) to approximate it on Δ .

Integrating (2.7), we obtain the quadrature formula

$$\int_\sigma h(s, t) d\sigma \approx \frac{1}{6} \left[h\left(\frac{1}{2}, \frac{1}{2}\right) + h\left(\frac{1}{2}, 0\right) + h\left(0, \frac{1}{2}\right) \right]. \tag{2.15}$$

Using the same affine mapping (2.10), this leads to a quadrature formula for integrals on Δ

$$\begin{aligned} \int_\Delta h(y) dy &\approx \int_\Delta P_n h(y) dy \\ &= \sum_{i=1}^3 h(q_i) \int_\sigma l_i(s, t) J_T(s, t) d\sigma, \end{aligned} \tag{2.16}$$

where J_T is the Jacobian of the transformation given in (2.10).

Thus, we have the approximation

$$\int_D h(y)dy = \sum_{k=1}^n \int_{\Delta_k} h(y)dy \tag{2.17}$$

$$\approx \sum_{k=1}^n \sum_{i=1}^3 h(q_{k,i}) \int_{\sigma} l_i(s, t) J_{T_k}(s, t) d\sigma, \tag{2.18}$$

where $q_{k,i} = T_k(\tau_i), i = 1, 2, 3$.

Then the collocation method is given by

$$v_n(q_i) = g \left(q_i, \sum_{j=1}^3 v_n(q_j) \int_D k(q_i, y) l_j(y) dy + f(q_i) \right), \tag{2.19}$$

which leads to the system

$$v_n(q_i) = g \left(q_i, \sum_{k=1}^n \sum_{j=1}^3 v_n(q_{k,j}) \int_{\sigma} k(q_i, T_k(s, t)) l_j(s, t) J_{T_k}(s, t) d\sigma + f(q_i) \right), \tag{2.20}$$

for all $i = 1, \dots, 3n$. Once all the unknowns $v_n(q_i)$ are found from this system, we have, for each $x = T_k(s, t) \in \Delta_k$,

$$\begin{aligned} v_n(x) &= \sum_{i=1}^3 v_n(q_i) l_i(s, t), \\ u_n(x) &= \sum_{i=1}^3 v_n(q_i) \int_{\sigma} k(x, T_k(s, t)) l_i(s, t) J_{T_k}(s, t) d\sigma + f(x). \end{aligned} \tag{2.21}$$

3. Convergence and error analysis

We write the system (2.20) in operator form as

$$(I - P_n \mathcal{K})v_n = 0, \tag{3.1}$$

with

$$\mathcal{K}(v)(x) = g \left(x, \int_D k(x, y)v(y)dy + f(x) \right) \tag{3.2}$$

and P_n defined as in (2.12).

Since we are using piecewise linear interpolation in our collocation method, we have the following well known general result (see, e.g. [2, p. 177]).

Theorem 3.1. *Assuming A1 – A4 hold, the operators $I - P_n \mathcal{K}$ are invertible on $C(D)$ and have uniformly bounded inverses, for all sufficiently large n , say $n \geq n_0$. In addition, the following error bounds hold:*

$$\|v^* - v_n\|_{\infty} \leq \|(I - P_n \mathcal{K})^{-1}\| \cdot \|v^* - P_n v^*\|, \quad n \geq n_0$$

and

$$\|v^* - v_n\|_\infty \leq O(\delta^2), \quad n \geq n_0, \tag{3.3}$$

where $\delta = \delta_n$ denotes the mesh size of the triangulation \mathcal{T}_n .

Now, this result holds in general, when using linear spline approximation. However, because of our particular choice of collocation (and interpolation) nodes, the approximation has higher order than $O(\delta^2)$. Notice that the quadrature formula (2.15) has degree of precision $d = 2$, *higher* than expected with interpolation of degree 1. Then formula (2.16) also has degree of precision $d = 2$. This will lead to a higher rate of convergence at the collocation nodes than $O(\delta^2)$, i. e., we get *superconvergence*.

Theorem 3.2. *Assume the conditions A1 – A4 hold, that $k \in C^2(D \times D)$ and that $v^* \in C^3(D)$. Then*

$$\max_{1 \leq i \leq n} |v^*(q_i) - v_n(q_i)|, \quad \max_{1 \leq i \leq n} |u^*(q_i) - u_n(q_i)| \leq O(\delta^3). \tag{3.4}$$

Proof. The proof is computational and follows the same ideas as the ones given for similar results e.g. in [3, 10].

Since v^* is the exact solution of (1.4), $v^* = \mathcal{K}v^*$. By the interpolation formula (2.13), $v_n = P_n \mathcal{K}v_n$. Then

$$\begin{aligned} (I - P_n \mathcal{K})(v^* - v_n) &= v^* - v_n - P_n \mathcal{K}v^* + P_n \mathcal{K}v_n \\ &= \mathcal{K}v^* - P_n \mathcal{K}v^* = (I - P_n) \mathcal{K}v^* \end{aligned}$$

and, thus, under our assumptions, for each $i = 1, \dots, n$,

$$\begin{aligned} |(I - P_n \mathcal{K})(v^* - v_n)(q_i)| &= \left| g \left(q_i, \int_D k(q_i, y) v^*(y) dy + f(q_i) \right) \right. \\ &\quad \left. - g \left(q_i, \int_D k(q_i, y) P_n v^*(y) dy + f(q_i) \right) \right| \\ &\leq c \left| \sum_{k=1}^n \int_{\Delta_k} k(q_i, y) (I - P_n) v^*(y) dy \right|. \end{aligned}$$

Now, on each triangle Δ_k , let p_j denote Taylor polynomial expansions of v^* around a suitable point in Δ_k , for $j = 1, 2$. Then

$$\begin{aligned} \|v^* - p_j\|_\infty &\leq c \delta^{j+1}, \\ \|p_2 - p_1\|_\infty &\leq c \delta^2. \end{aligned} \tag{3.5}$$

Also, since $k \in C^2(D \times D)$, there exists a constant k_0 such that, for all $y \in \Delta_k$,

$$|k(q_i, y) - k_0| \leq c \delta. \tag{3.6}$$

Since the interpolation formula (2.13) has degree of precision 1, we have

$$k(q_i, y) (I - P_n) p_1(y) = 0$$

and because the quadrature formula (2.15) has degree of precision 2, it follows that

$$k_0 \int_{\Delta_k} (I - P_n)p_2(y)dy = 0.$$

Then, we can write

$$\begin{aligned} \left| \int_{\Delta_k} k(q_i, y)(I - P_n)v^*(y)dy \right| &\leq c \left| \int_{\Delta_k} k(q_i, y)(I - P_n)(v^* - p_2)(y)dy \right. \\ &\quad + \int_{\Delta_k} (k(q_i, y) - k_0)(I - P_n)(p_2 - p_1)(y)dy \\ &\quad \left. - \int_{\Delta_k} k_0(I - P_n)p_1(y)dy \right|. \end{aligned}$$

Now, using the bounds (3.5), (3.6), we get

$$\begin{aligned} \max_{1 \leq i \leq n} |v^*(q_i) - v_n(q_i)| &\leq \max_{1 \leq i \leq n} |(I - P_n)(\mathcal{K}v^*)(q_i)| \\ &\leq c\delta^3 \sum_{k=1}^n \int_{\Delta_k} dy \\ &= O(\delta^3) \cdot n \cdot \text{Area}(\Delta_k) \\ &= O(\delta^3) \cdot O(\delta^{-2}) \cdot O(\delta^2) = O(\delta^3). \end{aligned}$$

Then by Theorem 2.1, we also have

$$\max_{1 \leq i \leq n} |u^*(q_i) - u_n(q_i)| \leq O(\delta^3).$$

□

4. Numerical experiments

We will use the notation $x = (x_1, x_2), y = (y_1, y_2) \in D = [a, b] \times [c, d]$.

Example 4.1. Let us consider the integral equation

$$u(x_1, x_2) = \frac{1}{2} \int_0^1 \int_0^1 \frac{x_1^2 + 1}{y_1^2 + 2y_2^2} u^2(y_1, y_2) dy_1 dy_2 + 2x_2^2, \tag{4.1}$$

for $(x_1, x_2) \in [0, 1] \times [0, 1]$, with exact solution $u^*(x_1, x_2) = x_1^2 + 2x_2^2 + 1$.

Here, we have

$$\begin{aligned} k(x, y) &= k(x_1, x_2, y_1, y_2) = \frac{x_1^2 + 1}{y_1^2 + 2y_2^2}, \\ g(y, u(y)) &= \frac{1}{2}u^2(y), \\ f(x) &= f(x_1, x_2) = 2x_2^2 \end{aligned}$$

and $D = [0, 1] \times [0, 1]$.

We start with $n = 2$ triangles that cover D and then refine the triangulation as described earlier. We compute the errors

$$e_n(v) = \max_{1 \leq i \leq n} |v^*(q_i) - v_n(q_i)| \text{ and}$$

$$e_n(u) = \max_{1 \leq i \leq n} |u^*(q_i) - u_n(q_i)|$$

Also, we look at the ratios

$$r_1 = \frac{e_n(v)}{e_{4n}(v)}, \quad r_2 = \frac{e_n(u)}{e_{4n}(u)}$$

from one iteration to the next. If indeed the numerical method has order of convergence $O(\delta^d)$, then these ratios should equal approximately 2^d .

We approximate the integrals needed for the coefficients of the nonlinear system (2.20) using tiled adaptive quadratures (function *integral2* in Matlab).

In Table 1, we give the errors $e_n(v)$ and $e_n(u)$, as well as the values $\log_2 r_1$ and $\log_2 r_2$, for each iteration.

n	$e_n(v)$	$\log_2 r_1$	$e_n(u)$	$\log_2 r_2$
2	$1.121e - 1$		$2.382e - 2$	
8	$1.874e - 2$	2.58	$3.397e - 3$	2.81
32	$2.582e - 3$	2.86	$4.519e - 4$	2.91
128	$3.296e - 4$	2.97	$5.653e - 5$	2.99

TABLE 1. Errors in Example 4.1

The table shows that both r_1 and r_2 approach the value 2^3 , which is consistent with the superconvergence $O(\delta^3)$ proved in Theorem 3.2.

Example 4.2. Next, we consider the equation

$$u(x_1, x_2) = \frac{1}{8} \int_0^2 \int_0^2 e^{2x_1+x_2} (y_1^2 - y_2) \ln(u(y_1, y_2)) dy_1 dy_2, \quad (4.2)$$

for $(x_1, x_2) \in [0, 2] \times [0, 2]$, whose exact solution is $u^*(x_1, x_2) = e^{2x_1+x_2}$.

In this case, $D = [0, 2] \times [0, 2]$ and we can take

$$k(x, y) = \frac{1}{8} e^{2x_1+x_2},$$

$$g(y, u(y)) = \ln(u(y)),$$

$$f(x) = 0.$$

Again, we start with $n = 2$ triangles and proceed as before.

The errors are shown in Table 2. The results show, again, an $O(\delta^3)$ rate of convergence.

n	$e_n(v)$	$\log_2 r_1$	$e_n(u)$	$\log_2 r_2$
2	$2.143e - 1$		$3.628e - 2$	
8	$3.344e - 2$	2.68	$5.356e - 3$	2.76
32	$4.735e - 3$	2.82	$7.428e - 4$	2.85
128	$6.086e - 4$	2.96	$9.415e - 5$	2.98

TABLE 2. Errors in Example 4.2

5. Conclusions

We have developed a collocation method for the approximate solution of non-linear Hammerstein integral equations over bounded domains of \mathbb{R}^2 . The numerical scheme is based on a special choice of linear spline interpolation over triangles. Having a higher degree of precision of the quadrature formula than expected, the resulting collocation method is superconvergent at the collocation nodes, converging faster than over the entire domain. This is one major advantage of the proposed numerical scheme over other projection-type collocation methods. Another important aspect is the fact that by applying collocation to a newly reformulated integral equation, the integrals needed for the coefficients of the linear system only have to be evaluated once, not at every iteration, which reduces the computational cost of the method and simplifies the implementation. Thus, this is a very efficient numerical method.

These ideas can be taken further, considering more complicated domains, higher dimensions or other types of interpolation.

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Sanda Micula
Babeş-Bolyai University,
Faculty of Mathematics and Computer Sciences,
1, Kogălniceanu Street,
400084 Cluj-Napoca, Romania
e-mail: smicula@math.ubbcluj.ro