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Certain geometric properties of generalized Bessel-Maitland function

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Abstract. In the present study, we first introduce Generalized Bessel-Maitland function $\mathbb{J}^{\xi}_{\zeta,a}(z)$ and then derive sufficient conditions under which the Generalized Bessel-Maitland function $\mathbb{J}^{\xi}_{\zeta,a}(z)$ have geometric properties like univalency, starlikeness and convexity in the open unit disk $\mathscr{D}$.

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1. Introduction and preliminaries

Let $\mathscr{H}$ denote the class of all functions analytic in the open unit disk

$$\mathscr{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and $\mathscr{A}$ be the class of all functions $f \in \mathscr{H}$ which are normalized by $f(0) = 0$ and $f'(0) = 1$. Each $f(z) \in \mathscr{A}$ has a Maclaurin series expansion of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... \tag{1.1}$$

Let $g, h \in \mathscr{H}$, we say that g is subordinated to h in $\mathscr{D}$, and write $g(z) \prec h(z)$, if there exists a function $\omega \in \mathscr{H}$ with $|\omega(z)| < |z|,\ z \in \mathscr{D}$, such that $g(z) = h(\omega(z))$ in $\mathscr{D}$. In particular, if h is univalent in $\mathscr{D}$, then we have:

$$g(z) \prec h(z) \iff g(0) = h(0) \text{ and } g(\mathscr{D}) \subset h(\mathscr{D}).$$

For a given $0 \leq \beta < 1$, a function $g \in \mathscr{A}$ is called starlike function of order β, if $\Re\left(zg'(z)/g(z)\right) > \beta,\ z \in \mathscr{D}$ class of such functions denoted by $\mathscr{S}^*(\beta)$. Similarly, for $0 \leq \beta < 1$, a function $g \in \mathscr{A}$ is called convex function of order β if

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$\Re\left(1+zg''(z)/g'(z)\right) > \beta$, $z \in \mathscr{D}$, class of such function denoted by $\mathscr{K}(\beta)$. It is customary that $\mathscr{S}^*(0) = \mathscr{S}^*$ and $\mathscr{K}(0) = \mathscr{K}$. Moreover, a function $g \in \mathscr{A}$ is said to be close-to-convex with respect to a fixed starlike function h, denoted by $\mathcal{C}_h$, if $\Re\left(zg'(z)/h(z)\right) > 0$, $z \in \mathscr{D}$. For more details one can refer [6].

In the present perusal, we study some geometric properties of Generalized Bessel-Maitland function (see, e.g., [9], Eq.(8.3)), $J^{\xi}_{\zeta}(z)$. This function is defined by the following series representation:

$$J^{\xi}_{\zeta}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(\xi n+\zeta+1)} \ (\Re(\xi) \geq 0,\ \Re(\zeta) \geq -1 \text{ and } z \in \mathscr{D}). \tag{1.2}$$

It has many application in various research fields of Science and Engineering. For a comprehensive description of applications of Bessel functions and its generalization, the reader may be referred to [20]. Here in the present paper, we define a new (probably) generalization of Bessel-Maitland function called generalized Bessel-Maitland function $J^{\xi}_{\zeta,c}(z)$, given by:

$$J^{\xi}_{\zeta,a}(z) = \sum_{n=0}^{\infty} \frac{(-a)^n z^n}{n!\Gamma(\xi n+\zeta+1)} \ (a \in \mathbb{C}-\{0\}, \xi > 0, \zeta > -1 \text{ and } z \in \mathscr{D}). \tag{1.3}$$

It can be easily seen that

$$J^{\xi}_{\zeta,-1}(z) = W_{\xi,\zeta+1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\xi n+\zeta+1)} \tag{1.4}$$

where $W_{\xi,\zeta+1}(z)$ is called Wright function and

$$J^{\xi}_{\zeta,1}(z) = J^{\xi}_{\zeta}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(\xi n+\zeta+1)}. \tag{1.5}$$

Observe that the Generalized Bessel-Maitland function $J^{\xi}_{\zeta,a}(z) \notin \mathscr{A}$. We can consider the following two types of normalization of the Generalized Bessel-Maitland function:

$$\mathbb{J}^{\xi}_{\zeta,a}(z) = z\Gamma(\zeta+1)J^{\xi}_{\zeta,a}(z) = z + \sum_{n=1}^{\infty} \frac{(-a)^n\Gamma(\zeta+1)z^{n+1}}{n!\Gamma(\xi n+\zeta+1)} \tag{1.6}$$

and

$$\mathcal{J}^{\xi}_{\zeta,a}(z) = \frac{\Gamma(\xi+\zeta+1)}{(-a)}\left(J^{\xi}_{\zeta,a}(z) - \frac{1}{\Gamma(\zeta+1)}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(-a)^n\Gamma(\xi+\zeta+1)z^{n+1}}{(n+1)!\Gamma(\xi n+\xi+\zeta+1)}. \tag{1.7}$$
$$(\xi > 0,\ \xi+\zeta > -1,\ a \in \mathbb{C}-\{0\},\ z \in \mathscr{D})$$

Also note that

$$\mathbb{J}^{1}_{\zeta,1}(z) = \mathbb{J}_{\zeta}(z) = \Gamma(\zeta+1)z^{1-\zeta/2}J_{\zeta}(2\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\zeta+1)z^{n+1}}{n!\Gamma(n+\zeta+1)}. \tag{1.8}$$

where $J_\zeta(z)$ is well known Bessel function of order ζ and $\mathbb{J}_\zeta(z)$ is the normalized Bessel function, studied recently for the various geometric properties (see [14]-[18]). Conversely, it can be easily seen that

$$J_\zeta(z) = \frac{1}{\Gamma(\zeta+1)}\left(\frac{z}{2}\right)^{\zeta-2}\mathbb{J}^1_{\zeta,1}\left(\frac{z^2}{4}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n+\zeta}}{n!\Gamma(n+\zeta+1)}.$$

Additionally, we observe that

$$\mathbb{V}^\xi_{\zeta,a}(z) = \frac{\mathbb{J}^\xi_{\zeta,a}(z)}{z} = \frac{1}{z}\left[z + \sum_{n=1}^{\infty}\frac{(-a)^n\Gamma(\zeta+1)z^{n+1}}{n!\Gamma(\xi n+\zeta+1)}\right]$$
$$= 1 + \sum_{n=1}^{\infty}\frac{(-a)^n\Gamma(\zeta+1)z^n}{n!\Gamma(\xi n+\zeta+1)}$$

and

$$z(\mathbb{V}^\xi_{\zeta,a}(z))' = \sum_{n=1}^{\infty}\frac{(-a)^n\Gamma(\zeta+1)nz^n}{n!\Gamma(\xi n+\zeta+1)}.$$

The following identity relations can be easily established:

$$\xi z(\mathbb{J}^\xi_{\zeta,a}(z))' = (\zeta+1)\mathbb{J}^\xi_{\zeta,a}(z) + (\xi-\zeta-1)\mathbb{J}^\xi_{\zeta+1,a}(z) \tag{1.9}$$

$$z(\mathcal{J}^\xi_{\zeta,a}(z))' = \mathbb{J}^\xi_{\xi+\zeta,a}(z) \tag{1.10}$$

and

$$\left(\mathbb{V}^\xi_{\zeta,a}(z)\right)' = \frac{(-a)\Gamma(\zeta+1)}{\Gamma(\xi+\zeta+1)}\mathbb{V}^\xi_{\xi+\zeta,a}(z). \tag{1.11}$$

Lately, several researchers have studied innumerable special functions belonging to class $\mathscr{A}$ and found sufficient conditions such that the special functions belonging to class $\mathscr{A}$ have certain properties like univalency, starlikeness or convexity in $\mathscr{D}$. For the generalized hypergeometric functions one can refer [10, 13, 16], Bessel functions [3, 1, 2, 4] and Wright function [15]. In the present paper, we derive sufficient conditions for the same geometric properties for the functions $\mathbb{J}^\xi_{\zeta,a}(z)$ and $\mathcal{J}^\xi_{\zeta,a}(z)$.

2. Lemmas

To prove main results, we requisite the following results:

Lemma 2.1. (see [7]). *Let $g \in \mathscr{A}$ satisfy the inequality*

$$|(g(z)/z) - 1| < 1 \ (z \in \mathscr{D})$$

then g is starlike in the disk $\mathscr{D}_{1/2} = \{z : |z| < 1/2\}$.

Lemma 2.2. (see [8]). *Let $g \in \mathscr{A}$ satisfy the inequality*

$$|g'(z) - 1| < 1 \ (z \in \mathscr{D})$$

then g is convex in the disk $\mathscr{D}_{1/2}$.

Lemma 2.3. (see [11]). *Let $g \in \mathscr{A}$ satisfy*

$$|g'(z) - 1| < 2/\sqrt{5}\ (z \in \mathscr{D})$$

then g is starlike in the disk $\mathscr{D}$.

Lemma 2.4. (see [21]). *Let $g \in \mathscr{A}$ satisfy the inequality*

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| < L, \quad z \in \mathscr{D},$$

where L is solution of the equation $\cos L = L$, then $\Re(g'(z)) > 0$.

Lemma 2.5. (see [12]). *Let $\delta \in \mathbb{C}$ with $\Re(\delta) > 0, d \in \mathbb{C}$ with $|d| \leq 1,\ d \neq -1$. If $h \in \mathscr{A}$ satisfies*

$$\left| d|z|^{2\delta} + (1 - |z|^{2\delta}) \frac{zh''(z)}{\delta h'(z)} \right| \leq 1, \quad z \in \mathscr{D}$$

then the integral operator

$$\mathcal{C}_\delta(z) = \left\{ \delta \int_0^z t^{\delta - 1} h'(t) dt \right\}^{1/\delta}, \ z \in \mathscr{D}$$

is analytic and univalent in $\mathscr{D}$.

For $\delta = 1$ and $d = 0$, Lemma 2.5 is equivalent to Becker's criterion for univalency [5], which shows that, if $f \in \mathscr{A}$ satisfy the inequality $(1 - |z|^2)|zf''(z)/f'(z)| \leq 1$ for each $z \in \mathscr{D}$, then f is one-to-one (univalent) in $\mathscr{D}$.

3. Main results

Theorem 3.1. (i)*Let $\xi \geq 1$ and $\zeta \geq \dfrac{3(|a| - 1) + \sqrt{9|a|^2 + 2|a| + 1}}{2}$, then $\mathbb{J}^{\xi}_{\zeta,a}$ is starlike in $\mathscr{D}$.*

(ii) *Let $\xi \geq 1$ and $\xi + \zeta \geq \dfrac{3(|a| - 1) + \sqrt{9|a|^2 + 2|a| + 1}}{2}$, then $\mathbb{V}^{\xi}_{\zeta,a}$ is convex in $\mathscr{D}$.*

(iii) *Let $\xi \geq 1$ and $\xi + \zeta \geq \dfrac{3(|a| - 1) + \sqrt{9|a|^2 + 2|a| + 1}}{2}$, then $\mathcal{J}^{\xi}_{\zeta,a}$ is convex in $\mathscr{D}$.*

Proof. Let $q(z)$ be a function defined by the equality $q(z) = z(\mathbb{J}^{\xi}_{\zeta,a}(z))'/\mathbb{J}^{\xi}_{\zeta,a}(z)\ \ z \in \mathscr{D}$. Since $\mathbb{J}^{\xi}_{\zeta,a}(z)/z \neq 0, \ \ z \in \mathscr{D}$, the function q is analytic in $\mathscr{D}$ and $q(0) = 1$. To prove the result, we need to show that $\Re(q(z)) > 0$ which follows if we show $|q(z) - 1| < 1$.

For $\xi \geq 1$, it is easy to see that $\Gamma(\zeta + n + 1) \leq \Gamma(\xi n + \zeta + 1)$, $n \in \mathbb{N}$, holds and is equivalent to

$$\frac{1}{(\zeta + 1)(\zeta + 2)...(\zeta + n)} \geq \frac{\Gamma(\zeta + 1)}{\Gamma(\xi n + \zeta + 1)}, \quad n \in \mathbb{N}. \tag{3.1}$$

If $z \in \mathscr{D}$, then using (1.6) and (3.1), we obtain

$$\begin{aligned}\left|(\mathbb{J}^{\xi}_{\zeta,a}(z))' - \frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}\right| &= \left|\sum_{n=1}^{\infty} \frac{(-a)^n n z^n \Gamma(\zeta+1)}{n!\Gamma(\xi n+\zeta+1)}\right| \\ &\le \sum_{n=1}^{\infty} \frac{|a|^n n}{n!(\zeta+1)(\zeta+2)(\zeta+n)} \\ &< \frac{|a|}{(\zeta+1)} \sum_{n=0}^{\infty} \left(\frac{|a|}{\zeta+2}\right)^n \\ &= \frac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)} \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}\right| &\ge 1 - \left|\sum_{n=1}^{\infty} \frac{(-a)^n z^n \Gamma(\zeta+1)}{n!\Gamma(\xi n+\zeta+1)}\right| \\ &\ge 1 - \sum_{n=1}^{\infty} \frac{|a|^n}{n!(\zeta+1)(\zeta+2)...(\zeta+n)} \\ &> 1 - \frac{|a|}{(\zeta+1)} \sum_{n=0}^{\infty} \left(\frac{|a|}{\zeta+2}\right)^n \\ &= 1 - \frac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)} \\ &= \frac{(\zeta+1)(\zeta+2-|a|) - |a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)}. \qquad (3.3)\end{aligned}$$

From (3.2) and (3.3), we have for $z \in \mathscr{D}$

$$\begin{aligned}|q(z)-1| &= \left|\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}^{\xi}_{\zeta,a}(z)} - 1\right| = \left|\frac{(\mathbb{J}^{\xi}_{\zeta,a}(z))' - \frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}}{\frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}}\right| \\ &< \frac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|) - |a|(\zeta+2)}. \qquad (3.4)\end{aligned}$$

This implies that if $(\zeta+1)(\zeta+2-|a|) - |a|(\zeta+2) \ge |a|(\zeta+2)$, then $\Re(q(z)) > 0$, hence $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in $\mathscr{D}$, but the inequality $(\zeta+1)(\zeta+2-|a|) - 2|a|(\zeta+2) \ge 0$ is a consequence of the hypothesis $\zeta \ge \dfrac{3(|a|-1) + \sqrt{9|a|^2 + 2|a| + 1}}{2}$. This shows that $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in $\mathscr{D}$. □

(ii) In view of the hypothesis the inequality $\Gamma(n+\xi+\zeta+1) \le \Gamma(\xi n+\xi+\zeta+1)$ holds. If $z \in \mathscr{D}$, then a calculation similar to (3.2) and (3.3) gives

$$\left|z\left(\mathbb{V}^{\xi}_{\xi+\zeta,a}(z)\right)'\right| < \frac{|a|(\xi+\zeta+2)}{(\xi+\zeta+1)(\xi+\zeta+2-|a|)}$$

and
$$\left|\mathbb{V}^{\xi}_{\xi+\zeta,a}(z)\right| > \frac{(\xi+\zeta+1)(\xi+\zeta+2-|a|)-|a|(\xi+\zeta+2)}{(\xi+\zeta+1)(\xi+\zeta+2-|a|)}.$$
From these inequalities and (1.11), we obtain
$$\left|\frac{z\left(\mathbb{V}^{\xi}_{\zeta,a}(z)\right)''}{\left(\mathbb{V}^{\xi}_{\zeta,a}(z)\right)'}\right| = \left|\frac{z\left(\mathbb{V}^{\xi}_{\xi+\zeta,a}(z)\right)'}{\left(\mathbb{V}^{\xi}_{\xi+\zeta,a}(z)\right)}\right|$$
$$< \frac{|a|(\xi+\zeta+2)}{(\xi+\zeta+1)(\xi+\zeta+2-|a|)-|a|(\xi+\zeta+2)} \; (z\in\mathscr{D}).$$
This means that, if $(\xi+\zeta+1)^2+(\xi+\zeta+1)(1-3|a|)-2|a|\geq 0$, then by definition $\mathbb{V}^{\xi}_{\zeta,a}$ is convex in $\mathscr{D}$. But this inequality is true under the condition
$$\xi+\zeta > \frac{3(|a|-1)+\sqrt{9|a|^2+2|a|+1}}{2}.$$
Hence, $\mathbb{V}^{\xi}_{\zeta,a}$ is convex in $\mathscr{D}$.
(iii) The function $\mathscr{J}^{\xi}_{\zeta,a}(z)$ is convex iff $z(\mathscr{J}^{\xi}_{\zeta,a}(z))'$ is starlike, but from (1.10)
$$z(\mathscr{J}^{\xi}_{\zeta,a}(z))' = \mathbb{J}^{\xi}_{\xi+\zeta,a}(z).$$
This in view of part (i) of the theorem completes the proof.

Remark 3.2. If we put $a=1$ and $\xi=1$, then we obtain part(a) and part(b) of Corollary 2.8 of [15]. Similarly if we put $a=-1$ and $\xi=1$ and using Lemma 2.4, we obtain part(c) of Corollary 2.8 of [15].

Theorem 3.3. (i) *Let $\xi\geq 1$ and $\zeta\geq \frac{-3+2|a|+\sqrt{1+4|a|^2}}{2}$, then $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in the disk $\mathscr{D}_{1/2}$.*
(ii) *Let $\xi\geq 1$ and $\zeta\geq \dfrac{3(|a|-1)+\sqrt{9|a|^2+2|a|+1}}{2}$, then $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is convex in the disk $\mathscr{D}_{1/2}$.*
(iii) *Let $\xi\geq 1$ and $\zeta>\zeta^*$, where ζ^* is positive root of the equation*
$$\zeta^2+\zeta(3-(1+\sqrt{5})|a|)+(2-(1+2\sqrt{5})|a|)=0,$$
then $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in the disk $\mathscr{D}$.

Proof. On performing calculations, we have
$$\left|\frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}-1\right| = \left|\frac{1}{z}\left\{z+\sum_{n=1}^{\infty}\frac{(-a)^n\Gamma(\zeta+1)z^{n+1}}{n!\Gamma(\xi n+\zeta+1)}\right\}-1\right|$$
$$\leq \sum_{n=1}^{\infty}\frac{|a|^n}{n!}\frac{1}{(\zeta+1)(\zeta+2)...(\zeta+n)} < \frac{|a|}{(\zeta+1)}\sum_{n=0}^{\infty}\left(\frac{|a|}{\zeta+2}\right)^n$$
$$= \frac{|a|}{(\zeta+1)}\frac{(\zeta+2)}{(\zeta+2-|a|)}$$

In view of Lemma 2.1, $\mathbb{J}^{\xi}_{\zeta,a}$ is starlike in $\mathscr{D}_{1/2}$, if $(\zeta+1)^2+(\zeta+1)(1-2|a|)-|a|\geq 0$, but this is true in view of the hypothesis. Hence, the result is proved.

(ii) Using Lemma 2.2, we obtain

$$\left|(\mathbb{J}^{\xi}_{\zeta,a}(z))'-1\right| = \left|\sum_{n=1}^{\infty}\frac{(-a)^n(n+1)\Gamma(\zeta+1)z^n}{n!\Gamma(\xi n+\zeta+1)}\right| \tag{3.5}$$

$$\leq \sum_{n=1}^{\infty}\frac{|a|^n n\Gamma(\zeta+1)}{n!\Gamma(\xi n+\zeta+1)}+\sum_{n=1}^{\infty}\frac{|a|^n\Gamma(\zeta+1)}{n!\Gamma(\xi n+\zeta+1)}$$

$$\leq \sum_{n=1}^{\infty}\frac{2|a|^n}{(\zeta+1)(\zeta+2)...(\zeta+n)} < \frac{2|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)} \leq 1. \tag{3.6}$$

This shows that $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is convex in $\mathscr{D}_{1/2}$.
(iii) Using the Lemma 2.3 and equation (3.6), we see that $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in $\mathscr{D}_{1/2}$, if

$$\zeta^2+\zeta(3-(1+\sqrt{5})|a|)+(2-(1+2\sqrt{5})|a|)>0.$$

This proves the result. □

Remark 3.4. Setting $\xi=1$, and $a=1$ in Theorem 3.3, we obtain Part (a), (b) and (c) of Corollary 2.10 of [15].

Theorem 3.5. *Let $\xi\geq 1$ and $0\leq\eta<1$. Suppose also that*

$$\psi(\eta)=\frac{(3|a|-1)-\eta(2|a|-1)+\sqrt{\eta^2(4|a|^2+1)-2\eta(6|a|^2+|a|+1)+(9|a|^2+2|a|+1)}}{2(1-\eta)}.$$

(i)*Let $\zeta\geq\psi(\eta)$, then $\mathbb{J}^{\xi}_{\zeta,a}(z)\in\mathscr{S}^*(\eta)$.*
(ii) *Let $\zeta+\xi\geq\psi(\eta)$, then $\mathbb{V}^{\xi}_{\zeta,a}(z)\in\mathscr{K}^*(\eta)$.*
(iii) Let $\zeta+\xi\geq\psi(\eta)$, then $\mathcal{J}^{\xi}_{\zeta,a}(z)\in\mathscr{K}^*(\eta)$.

Proof. Following the proof of Theorem 3.1, we find that $\mathbb{J}^{\xi}_{\zeta,a}(z)\in\mathscr{S}^*(\eta)$ii, if

$$\frac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)-|a|(\zeta+2)}\leq(1-\eta),$$

but this inequality is a direct consequence of hypothesis. Hence the result. Remaining part can be shown similarly. □

Theorem 3.6. *Let $\xi\geq 1$ and ζ^* be the positive root of the cubic equation*

$$\zeta^3+\zeta^2(2-3|a|)-3\zeta(1+2|a|)-(6+|a|)\geq 0,$$

then $\mathbb{J}^{\xi}_{\zeta,a}$ is close-to-convex with respect to $\mathbb{J}_{\zeta}$ in $\mathscr{D}$, provided $\zeta>\max\{\zeta^*,|a|-2,\sqrt{3}\}$.

Proof. Using definition, we need to show $\exists h\in\mathcal{S}^*$, such that

$$\Re\left(\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{h(z)}\right)>0\quad(z\in\mathscr{D}),$$

this can be easily shown by proving

$$\left|\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{h(z)}-1\right|<1, \quad z\in\mathscr{D}.$$

If $z\in\mathscr{D}$, then a computation gives

$$\begin{aligned}\left|(\mathbb{J}^{\xi}_{\zeta,a}(z))'-\frac{\mathbb{J}_{\xi}(z)}{z}\right| &\leq \sum_{n=1}^{\infty}\frac{\Gamma(\zeta+1)}{n!}\left|\frac{(-a)^n(n+1)}{\Gamma(\xi n+\zeta+1)}-\frac{(-1)^n}{\Gamma(n+\zeta+1)}\right|\\
&\leq \sum_{n=1}^{\infty}\frac{\Gamma(\zeta+1)}{n!}\left|\frac{(a)^n(n+1)}{\Gamma(\xi n+\zeta+1)}+\frac{1}{\Gamma(n+\zeta+1)}\right|\\
&\leq \sum_{n=1}^{\infty}\frac{\Gamma(\zeta+1)}{n!}\left[\frac{|a|^n(n+1)+1}{\Gamma(n+\zeta+1)}\right]\\
&\leq \sum_{n=1}^{\infty}\frac{2|a|^n}{(\zeta+1)(\zeta+2)...(\zeta+n)}+\sum_{n=1}^{\infty}\frac{1}{(\zeta+1)(\zeta+2)...(\zeta+n)}\\
&< \frac{1}{(\zeta+1)}\sum_{n=0}^{\infty}\frac{(2|a|^{n+1}+1)}{(\zeta+2)^n}=\frac{(\zeta+2)[|a|(2\zeta+1)+\zeta+2]}{(\zeta+2-|a|)(\zeta+1)^2} \qquad (3.7)\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\mathbb{J}_{\zeta}(z)}{z}\right| &\geq 1-\sum_{n=1}^{\infty}\frac{1}{n!(\zeta+1)(\zeta+2)...(\zeta+n)}\\
&> 1-\frac{1}{\zeta+1}\sum_{n=0}^{\infty}\left(\frac{1}{\zeta+2}\right)^n=\frac{\zeta^2+\zeta-1}{(\zeta+1)^2}. \qquad (3.8)\end{aligned}$$

From (3.7) and (3.8)

$$\begin{aligned}\left|\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}_{\zeta}(z)}-1\right| &= \left|\frac{(\mathbb{J}^{\xi}_{\zeta,a}(z))'-\frac{\mathbb{J}_{\xi}(z)}{z}}{\frac{\mathbb{J}_{\zeta}(z)}{z}}\right|\\
&< \frac{(\zeta+2)}{(\zeta+2-|a|)(\zeta^2+\zeta-1)}[|a|(2\zeta+1)+\zeta+2]\leq 1, \quad z\in\mathscr{D}.\end{aligned}$$

This shows that $\Re\left(z(\mathbb{J}^{\xi}_{\zeta,a}(z))'/\mathbb{J}_{\zeta}(z)\right)>0$ and hence $\mathbb{J}^{\xi}_{\zeta,a}$ is close-to-convex in $\mathscr{D}$. Starlikeness of $\mathbb{J}_{\zeta}$ can be deduced from Theorem 3.1 for $a=1$ and it comes out $\xi\geq 1$ and $\zeta\geq\sqrt{3}$. □

For a non-zero complex number δ, we define an integral operator $\mathcal{F}_{\delta}:\mathscr{D}\to\mathbb{C}$, by

$$\mathcal{F}_{\delta}(z)=\left\{\delta\int_0^z t^{\delta-2}\mathbb{J}^{\xi}_{\zeta,a}(t)dt\right\}^{\frac{1}{\delta}}, \quad z\in\mathscr{D}. \qquad (3.9)$$

Note that $\mathcal{F}_{\delta}(z)\in\mathscr{A}$. In the next theorem, we find conditions so that $\mathcal{F}_{\delta}$ is univalent in $\mathscr{D}$.

Theorem 3.7. *Let* $\xi > -1$, $\zeta > -1$, $\kappa = \dfrac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)-|a|(\zeta+2)}$ *and* $L \in \mathbb{R}^+$ *such that* $\left|\mathbb{J}^{\xi}_{\zeta,a}(z)\right| \le L$ *in* $\mathscr{D}$, *then following results holds*
(i) *If* $\kappa + |\delta - 1| + L/\delta \le 1$, *then* $\mathcal{F}_\delta$ *is univalent in* $\mathscr{D}$.
(ii) *If* $d \in \mathbb{C}$ *with* $|d| \le 1$, $d \ne -1$ *and* $|d| + \kappa/|\delta| \le 1$, *then* $\mathcal{F}_\delta$ *is univalent in* $\mathscr{D}$.

Proof. (i) A simple calculation gives us

$$\frac{z\mathcal{F}''_\delta(z)}{\mathcal{F}'_\delta(z)} = \frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}_\zeta(z)} + \frac{z^{\delta-1}}{\delta}\mathbb{J}^{\xi}_{\zeta,a}(z) + \delta - 2, \quad z \in \mathscr{D}. \tag{3.10}$$

Since $\mathbb{J}^{\xi}_{\zeta,a} \in \mathscr{A}$, so using Schwarz Lemma, we obtain $\left|\mathbb{J}^{\xi}_{\zeta,a}(z)\right| \le L|z|$ in $\mathscr{D}$.
Now using (3.4) and the triangle inequality ($|z_1 + z_2| \le |z_1| + |z_2|$), we obtain

$$\begin{aligned}(1-|z|^2)\left|\frac{z\mathcal{F}''_\delta(z)}{\mathcal{F}'_\delta(z)}\right| &\le (1-|z|^2)\left\{|\delta-1| + \left|\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}^{\xi}_{\zeta,a}(z)} - 1\right| + \frac{|z|^{\Re(\delta)}}{|\delta|}\left|\frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}\right|\right\}\\ &< (1-|z|^2)\left\{\zeta + |\delta-1| + \frac{L}{|\delta|}\right\} \le 1.\end{aligned}$$

This implies that $\mathcal{F}_\delta$ satisfy Becker's criterion for univalence, hence $\mathcal{F}_\delta$ is univalent in $\mathscr{D}$.
(ii) Let us consider the function

$$\mathcal{G}(z) = \int_0^z \frac{\mathbb{J}^{\xi}_{\zeta,a}(t)}{t}dt, \quad z \in \mathscr{D}.$$

Observe that, $\mathcal{G} \in \mathscr{A}$. Using (3.4) and the triangle inequality, we get

$$\begin{aligned}\left|d|z|^{2\delta} + (1-|z|^{2\delta})\frac{z\mathcal{G}''(z)}{\delta\mathcal{G}'(z)}\right| &\le \left|d|z|^{2\delta} + (1-|z|^{2\delta})\frac{1}{\delta}\left(\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}_\zeta(z)} - 1\right)\right|\\ &\le |d| + \frac{\zeta}{|\delta|} \le 1 \quad \text{(using the hypothesis of Theorem 3.7).}\end{aligned}$$

This in view of Lemma 2.5, implies that $\mathcal{F}_\delta(z)$ defined by

$$\mathcal{F}_\delta(z) = \left\{\delta\int_0^z t^{\delta-1}\mathcal{G}'(t)dt\right\}^{1/\delta} = \left\{\delta\int_0^z t^{\delta-2}\mathbb{J}^{\xi}_{\zeta,a}(t)dt\right\}^{1/\delta} \quad (z \in \mathscr{D}). \tag{3.11}$$

is univalent in $\mathscr{D}$. □

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