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Spectral characterization of new classes of multicone graphs

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Abstract. This paper deals with graphs that are known as multicone graphs. A multicone graph is a graph obtained from the join of a clique and a regular graph. Let w, l, m be natural numbers and k is a natural number. It is proved that any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ is determined by its adjacency spectra as well as its Laplacian spectra, where $ECP_l^k = K_{\underbrace{3^k,\ 3^k,\ \ldots,\ 3^k}_{l\ times}}$. Also, we show that complements of some of these multicone graphs are determined by their adjacency spectra. Moreover, we prove that any connected graph cospectral with these multicone graphs must be perfect. Finally, we pose two problems for further researches.

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1. Introduction

All graphs considered here are simple and undirected. All notions on graphs that are not defined here can be found in [4, 5, 10, 12, 19]. Let Γ be a graph with n vertices, $V(\Gamma)$ and $E(\Gamma)$ be the sets of vertices and edges of Γ, respectively. The complement of a graph Γ, denoted by $\overline{\Gamma}$, is the graph on the vertices set of Γ such that two vertices of $\overline{\Gamma}$, are adjacent if and only if they are not adjacent in Γ. The union of (disjoint) graphs Γ_1 and Γ_2 is denoted by $\Gamma \cup \Gamma_2$, is the graph whose vertices (respectively, edges) set is the union of vertices (respectively, edges) set of Γ_1 and Γ_2. A graph consisting of k disjoint copies of an arbitrary graph Γ will be denoted by $k\Gamma$. The join of two vertex disjoint graphs Γ_1 and Γ_2 is the graph obtained from $\Gamma_1 \cup \Gamma_2$ by joining each vertex in Γ_1 with every vertex in Γ_2. It is denoted by $\Gamma_1 \bigtriangledown \Gamma_2$. Let Γ be a graph with adjacency matrix $A(\Gamma)$. The characteristic polynomial of Γ is $\det(\lambda I - A(\Gamma))$, and denoted by $P_\Gamma(\lambda)$. The roots of $P_\Gamma(\lambda)$ are called the adjaceny eigenvalues of $A(\Gamma)$. The eigenvalues and the spectrum of $A(\Gamma)$ are also called the eigenvalues and the spectrum of Γ, respectively. If we consider a matrix $L = D - A$ instead of A, where D is the diagonal matrix of degree of vertices (in Γ), we get the Laplacian eigenvalues and the Laplacian spectrum, while in the case of matrix $SL(G) = D(\Gamma)+A(\Gamma)$, we get the signless Laplacian eigenvalues and the signless Laplacian spectrum, respectively. Since both matrices $A(\Gamma)$ and $L(\Gamma)$ are real symmetric matrices, their eigenvalues are all real numbers. Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the distinct eigenvalues of Γ with multiplicities $m_1, m_2, \ldots, m_s$, respectively. We denote the adjacency spectrum of Γ by $Spec(\Gamma) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, ..., [\lambda_s]^{m_s}\}$. Two graphs Γ and Λ are called cospectral, if $Spec(\Gamma) = Spec(\Lambda)$. A graph Γ is said to be determined by its spectrum or DS for short, if $Spec(\Gamma) = Spec(\Lambda)$, follows that $\Gamma \cong \Lambda$. About the background of the guestion "which graphs are determined by their spectrums?", we refer to [15]. The friendship graph F_n consists of n edge-disjoint triangles that all of them meeting in one vertex, where n is a natural number (see Figure 1). The friendship (or Dutch windmill or n-fan) graph F_n is the graph that can be constructed by coalesencing n copies of the cycle graph C_3 of length 3 with a common vertex. By construction, the friendship graph F_n is isomorphic to the windmill graph Wd(3,n) [11]. The friendship theorem of Paul Erdös, Alfred Réyni and Vera T. Sós [12], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. In [17, 18], it has been proposed that the friendship graph is DS with respect to its adjacency spectrum. This conjecture studied in [2, 8]. It is claimed in [8] that conjecture is valid. In [7], it is proved that if Γ is any graph cospectral with F_n ($n \neq 16$), then $\Gamma \cong F_n$. Abdollahi and Janbaz [3] precented a proof in special case of this topic. They proved that any connected graph cospectral with F_n is isomorphic to F_n. Abdian and Mirafzal [1] characterized new classes of multicone graphs. In this paper, we present new classes of multicone graphs that friendship graphs are special classes of them and we show these graphs are DS with respect to their spectra. The plan of the present paper is as follows. In Section 2, we review some basic information and preliminaries. In Subsection 3.1, we show that any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ (see Figures 1 and 2, for example) must be regular or bidegreed (Lemma 3.2). In Subsection 3.2, we prove that any connected graphs cospectral with $K_w \bigtriangledown mECP_l^k$ is determined by its adjacency spectra (Theorem 3.4). In Subsection 3.3, we prove that complement of $K_w \bigtriangledown mECP_l^k$ is DS with respect to their adjacency spectra (Theorem 3.7). In Subsection 3.4, we show that graphs $K_w \bigtriangledown mECP_l^k$ are DS with respect to their Laplacian spectra (Theorem 3.8). In Subsection 3.5, we show that any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ must be perfect. We conclude with final remarks and open problems in Section 4.

2. Preliminaries

In this section, we give some facts that will be used in the proof of the main results.

A walk of length m in a graph $\Gamma(V, E)$ is an alternating sequence:

$$v_1 l_1 v_2 l_2 v_3 v_n l_m v_{m+1}$$

of vertices and edges that begins and ends with a vertex and has the added property that l_j is incident with both v_i and v_{i+1}, where $1 \leq i \leq m+1$ and $1 \leq j \leq m$. In graph $\Gamma(V, E)$ a walk of length m is closed, if $v_1 = v_{m+1}$.

Lemma 2.1. ([2, 14]) *Let Γ be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:*

(*i*) *The number of vertices,*
(*ii*) *The number of edges.*

For the adjacency matrix, the following follows from the spectrum:

(*iii*) *The number of closed walks of any length.*
(*iv*) *Being regular or not and the degree of regularity.*
(*v*) *Being bipartite or not.*

For the Laplacian matrix, the following follows from the spectrum:

(*vi*) *The number of spanning trees.*
(*vii*) *The number of components.*
(*viii*) *The sum of squares of degrees of vertices.*

Theorem 2.2. ([5]) *If Γ_1 is r_1-regular with n_1 vertices, and Γ_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $\Gamma_1 \bigtriangledown \Gamma_2$ is given by:*

$$P_{\Gamma_1 \bigtriangledown \Gamma_2(\lambda)} = \frac{P_{\Gamma_1}(\lambda) P_{\Gamma_2}(\lambda)}{(\lambda - r_1)(\lambda - r_2)}((\lambda - r_1)(\lambda - r_2) - n_1 n_2).$$

Proposition 2.3. ([5]) *Let $\Gamma - j$ be the graph obtained from Γ by deleting the vertex j and all edges containing j. Then $P_{\Gamma - j}(\lambda) = P_\Gamma(\lambda) \sum\limits_{i=1}^{m} \frac{\alpha_{ij}^2}{\lambda - \mu_i}$, where m, α_{ij}^2 and $P_\Gamma(\lambda)$ are the number of distinct eigenvalues of graph Γ, the main angle of Γ and the characteristic polynomial of Γ.*

A graph is bidegreed if the set of degrees of its vertices consists of exactly two distinct elements. Also, the spectral radius $\varrho(\Gamma)$ of Γ is the largest eigenvalue of its adjacency matrix $A(\Gamma)$.

Theorem 2.4. ([3]) *Let Γ be a simple graph with n vertices and m edges. Let $\delta = \delta(\Gamma)$ be the minimum degree of vertices of Γ and $\varrho(\Gamma)$ be the spectral radius of the adjacency matrix of Γ. Then*

$$\varrho(\Gamma) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}}.$$

Equality holds if and only if Γ is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.

A t-multipartite graph of order n is $K_{b_1, \ldots, b_t}$, where $b_1 + \ldots + b_t = n$. D. Cvetković, Doob and S. Simić [6] defined a generalized cocktail-party graph, denoted by GCP, as a complete graph with some independent edges removed. A special case of this graph is the well-known cocktail-party graph $CP(t)$ obtained from K_{2t} by removing t disjoint edges.

Theorem 2.5. ([1]) *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

Lemma 2.6. ([1]) *Let* Γ *be a connected non-regular graph with three distinct eigenvalues* $\theta_0 > \theta_1 > \theta_2$. *Then the following hold:*

(i) Γ *has diameter two.*

(ii) *If* θ_0 *is not an integer, then* Γ *is complete bipartite.*

(iii) $\theta_1 \geq 0$ *with equality if and only if* Γ *is complete bipartite.*

(iv) $\theta_2 \leq -\sqrt{2}$ *with equality if and only if* Γ *is the path of length* 2.

Proposition 2.7. ([12]) *For a graph* Γ, *the following statements are equivalent:*

(i) Γ *is d-regular.*

(ii) $\varrho(\Gamma) = d_\Gamma$, *the average vertex degree.*

(iii) G *has* $v = (1, 1, ..., 1)^t$ *as an eigenvector for* $\varrho(\Gamma)$.

Proposition 2.8. ([16]) *Let* Γ *be a disconnected graph that is determined by the Laplacian spectrum. Then the cone over* Γ, *the graph* Λ; *that is, obtained from* Γ *by adding one vertex that is adjacent to all vertices of* Γ, *is also determined by its Laplacian spectrum.*

Lemma 2.9. ([13]) *Let* Γ *be a graph on* n *vertices. Then* n *is Laplacian eigenvalue of* Γ *if and only if* Γ *is the join of two graphs.*

Theorem 2.10. ([13]) *Let* Γ *and* Λ *be two graphs with Laplacian spectrum* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ *and* $\mu_1 \geq \mu_2 \geq .. \geq \mu_m$, *respectively. Then the Laplacian spectra of* $\overline{\Gamma}$ *and* $\Gamma \bigtriangledown \Lambda$ *are* $n - \lambda_1, n - \lambda_2, ..., n - \lambda_{n-1}, 0$ *and* $n + m, m + \lambda_1, ..., m + \lambda_{n-1}, n + \mu_1, ..., n + \mu_{m-1}, 0$, *respectively.*

Lemma 2.11. ([12]) *Let* $G \neq K_1$ *be connected with* $P_\Gamma(\lambda) = \sum\limits_{i=0}^{n} a_i \lambda^{n-i}$ *and* $\lambda = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = \varrho(\Gamma)$, *where* $P_\Gamma(\lambda)$ *is the characteristic polynomial of graph* Γ *and* λ_i *(*$1 \leq i \leq n$*) is eigenvalue of* Γ. *The following are equivalent:*

(i) G *is bipartite.*

(ii) $a_{2i-1} = 0$ *for all* $1 \leq i \leq \lceil \frac{n}{2} \rceil$.

(iii) $\lambda_i = -\lambda_{n+1-i}$ *for* $1 \leq i \leq n$.

(iv) $\varrho(\Gamma) = -\lambda$.

Moreover, $m(\lambda_i) = m(-\lambda_i)$, *where* $m(\lambda_i)$ *denote the multiplicities of* λ_i.

3. Main results

In the following, we show that any connected graph cospectral with multicone graphs $K_w \bigtriangledown mECP_l^k$ are regular or bidegreed.

3.1. Connected bidegreed graph cospectral with multicone graphs $K_w \bigtriangledown mECP_l^k$

Proposition 3.1. *Let* G *be a graph cospectral with multicone graphs* $K_w \bigtriangledown mECP_l^k$. *Then*

$$Spec(G) = \left\{ [0]^{(3^k l - l)m}, [-1]^{w-1}, \left[3^k l - 3^k\right]^{m-1}, \left[-3^k\right]^{lm-m}, \left[\frac{\chi + \sqrt{\chi^2 - 4\Theta}}{2}\right]^1, \left[\frac{\chi - \sqrt{\chi^2 - 4\Theta}}{2}\right]^1 \right\},$$

where $\chi = w - 1 + 3^k l - 3^k$ *and* $\Theta = (w - 1)(3^k l - 3^k) - 3^k lwm$.

Proof. By Theorem 2.2 and $Spec(mECP_l^k) = \left\{ \left[3^k l - 3^k\right]^m, [0]^{3^k lm - lm}, \left[-3^k\right]^{lm-m} \right\}$ the proof is completed. $\square$

In the following, we show that any graph cospectral with a multicone graph $K_w \bigtriangledown mECP_l^k$ must be bidegreed.

Lemma 3.2. *Let* Γ *be a connected graph cospectral with multicone graph* $K_w \bigtriangledown mECP_l^k$. *Then* Γ *is bidegreed in which any vertex of* Γ *is of degree* $w - 1 + 3^k lm$ *or* $3^k l - 3^k + w$.

Proof. It is obvious that Γ cannot be regular; since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the sequence of degrees of vertices of graph Γ consists of at least three number. Hence the equality in Theorem 2.4 cannot happen for any δ. But, if we put $\delta = 3^k l - 3^k + w$, then the equality in Theorem 2.4 holds. So, Γ must be bidegreed. Now, we show that $\Delta = \Delta(\Gamma) = w - 1 + 3^k lm$. By contrary, we suppose that $\Delta < w - 1 + 3^k lm$. Therefore, the equality in Theorem 2.4 cannot hold for any δ. But, if we put $\delta = 3^k l - 3^k + w$, then this equality holds. This is a contradiction and so $\Delta = 3^k l - 3^k + w$. Now, $\delta = 3^k l - 3^k + w$, since Γ is bidegreed and Γ has $w + 3^k lm$, $\Delta = w - 1 + 3^k lm$ and

$$w(w - 1 + 3^k lm) + 3^k lm(3^k l - 3^k + w) = w\Delta + 3^k lm(3^k l - 3^k + w) = \sum_{i=1}^{w+3^k lm} \deg v_i.$$

This completes the proof. $\square$

3.2. Spectral characterization of connected graphs cospectral with multicone graphs $K_1 \bigtriangledown mECP_l^k$.

In this subsection, we show that multicone graphs $K_1 \bigtriangledown mECP_l^k$ are DS.

Lemma 3.3. *Any connected graph cospectral with multicone graph* $K_1 \bigtriangledown mECP_l^k$ *is isomorphic to* $K_1 \bigtriangledown mECP_l^k$.

Proof. Let Γ be a graph cospectral with multicone graph $K_1 \bigtriangledown mECP_l^k$. If $m = 1$ there is nothing to prove. Hence we suppose that $m \neq 1$. It is obvious that in this case Γ cannot be regular. First we show that Γ has one vertex of degree $\Delta = 3^k lm$ and $3^k lm$ vertices of degree $\delta = 3^k l - 3^k + 1$. Let G has t vertex of degree $\Delta = 3^k lm$. Hence

$$t3^k lm + (3^k lm + 1 - t)(3^k l - 3^k + 1) = 3^k lm + 3^k lm(3^k l - 3^k + 1) = \sum_{i=1}^{1+3^k lm} \deg v_i$$

and so $t = 1$. Therefore, Γ has one vertex of degree $\Delta = 3^k lm$, say j. It follows from Proposition 2.3 that

$$\begin{aligned} P_{\Gamma - j}(\lambda) &= (\lambda - \mu_3)^{m-2}(\lambda - \mu_4)^{lm-m-1}(\lambda - \mu_5)^{3^k lm - lm - 1} \\ &\quad \times [\alpha_{1j}^2 F + \alpha_{2j}^2 G + \alpha_{3j}^2 H + \alpha_{4j}^2 I + \alpha_{5j}^2 J], \end{aligned}$$

where

$$\begin{aligned} F &= (\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5), \\ G &= (\lambda - \mu_1)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5), \end{aligned}$$

$$H = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_4)(\lambda - \mu_5),$$

$$I = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_5),$$

$$J = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4),$$

where

$$\mu_1 = \frac{3^k l - 3^k + \sqrt{(3^k l - 3^k)^2 + 4(2^k lm)}}{2},$$

$$\mu_2 = \frac{3^k l - 3^k - \sqrt{(3^k l - 3^k)^2 + 4(2^k lm)}}{2},$$

$$\mu_3 = 3^k l - 3^k, \ \mu_4 = -3^k \text{ and } \mu_5 = 0.$$

It is clear that $P_{\Gamma - j}(\lambda)$ has $3^k lm$ roots. So, we have:

$$\alpha + \beta + \gamma + 3^k l - 3^k = -[(m-2)\mu_3 + (lm - m - 1)\mu_4],$$

$$\alpha^2 + \beta^2 + \gamma^2 + (3^k l - 3^k)^2 = 3^k lm(3^k l - 3^k) - [(m-2)\mu_3^2 + (lm - m - 1)\mu_4^2],$$

$$\alpha^3 + \beta^3 + \gamma^3 + (3^k l - 3^k)^3 = 6m(3^{3k})\tbinom{l}{3} - [(m-2)\mu_3^2 + (lm - m - 1)\mu_4^2],$$

where α, β and γ are the eigenvalues of graph $\Gamma - j$. If we solve the above equations, then we will have: $\alpha = -3^k$, $\beta = 0$ and $\gamma = 3^k l - 3^k$. Therefore,

$$spec(\Gamma - j) = \left\{ \left[3^k l - 3^k\right]^m, [0]^{3^k lm - lm}, \left[-3^k\right]^{lm-m} \right\}.$$

Graph $\Gamma - j$ is regular and degree of its regularity is $3^k l - 3^k$. It follows from Theorem 2.4 that $\Gamma - j = mK_{\underbrace{3^k, \dots, 3^k}_{l\ times}}$ and so $G - j = mECP_l^k$. Hence $\Gamma = K_1 \bigtriangledown mECP_l^k$. This follows the result. □

Up to now, we have shown that the multicone graphs $K_1 \bigtriangledown mECP_l^k$ are DS. The natural question is; what happen for multicone graphs $K_w \bigtriangledown mECP_l^k$? we answer to this question in the following theorem.

Theorem 3.4. *Any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ is isomorphic to $K_w \bigtriangledown mECP_l^k$.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, if $Spec(\Gamma_1) = Spec(K_w \bigtriangledown mECP_l^k)$, then $\Gamma_1 \cong K_w \bigtriangledown mECP_l^k$, where Γ_1 is a graph. We show that, if $Spec(\Gamma) = Spec(K_{w+1} \bigtriangledown mECP_l^k)$, then $\Gamma \cong K_{w+1} \bigtriangledown mECP_l^k$, where Γ is a graph. By Lemma 3.2, Theorem 2.4, Lemma 2.1 (iii) and in a similar manner of Lemma 3.3 for $\Gamma - j$, where j is a vertex of degree $w + 3^k lm$ belonging to Γ, we obtain $Spec(\Gamma - j) = Spec(K_w \bigtriangledown mECP_l^k)$. Therefore, the assertion holds. □

In the following, we give another proof of the above theorem.

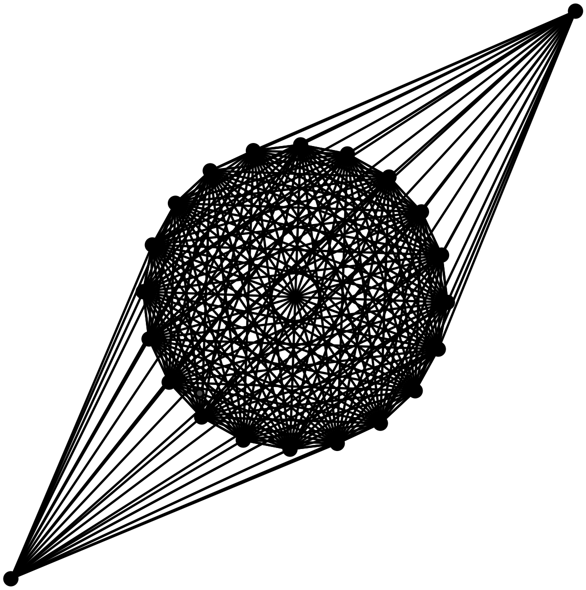

FIGURE 1. Multicone graph $K_{20} \bigtriangledown 2ECP_1^0$

Proof. Let Γ be a connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$. By Lemma 3.2, Γ has subgraph L in which degree of any vertex of L is $w-1+3^k lm$. In other words, $\Gamma \cong K_w \bigtriangledown H$, where H is a subgraph of Γ. Now, we remove the vertices of K_w and we consider $3^k lm$ another vertices. Consider H consisting of these $3^k lm$ vertices. H is regular and degree of its regularity is $3^k l - 3^k$ and multiplicity of $3^k l - 3^k$ is m. By Theorem 2.2, $Spec(H) = \left\{ \left[3^k l - 3^k\right]^m, [0]^{(3^k l - l)m}, \left[-3^k\right]^{(l-1)m} \right\}$. Now, it follows from Theorem 2.5 that $Spec(H) = Spec(mECP_l^k)$. This implies the result. □

Corollary 3.5. *Any connected graph cospectral with multicone graph*

$$K_w \bigtriangledown mECP_1^k = K_w \bigtriangledown mK_{3^k}$$

is DS with respect to their adjacency spectrums.

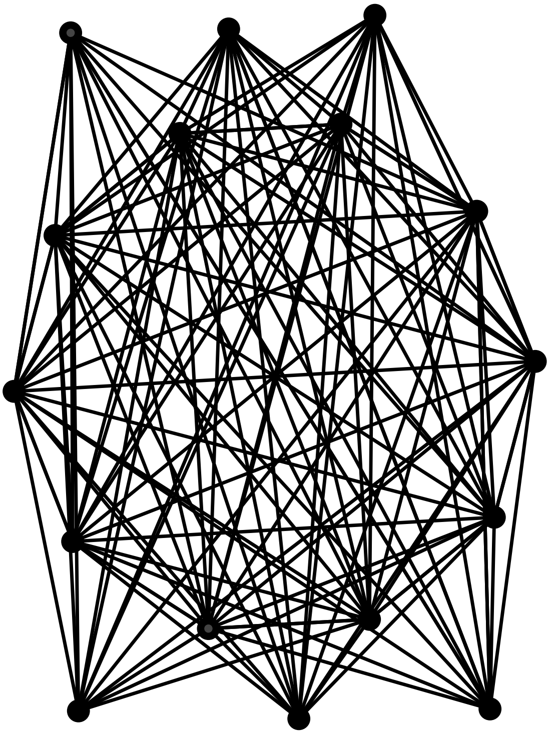

FIGURE 2. Multicone graph $K_{10} \bigtriangledown 2ECP_2^1$

3.3. Some complements of multicone graphs $K_w \bigtriangledown mECP_l^k$ are DS with respect to their spectra.

In this subsection, we show that the complement of multicone graphs $K_w \bigtriangledown mECP_l^k$ are DS with respect to their adjacency spectrum.

Proposition 3.6. *Let* Γ *be cospectral with complement of multicone graphs* $K_w \bigtriangledown mECP_l^k$*. Then*

$$Spec(\Gamma) = \left\{ \left[3^k lm - 3^k l + 3^k - 1\right]^m, \, [-1]^{(3^k-1)lm}, \, \left[3^k - 1\right]^{(l-1)m}, [0]^w \right\}.$$

Proof. Straightforward. ☐

Theorem 3.7. *The complement of multicone graph* $K_w \bigtriangledown ECP_l^k$ *are DS with respect to their adjacency spectrum.*

Proof. The proof of this theorem is the similar of Theorm 5.2 of [1]. Let

$$Spec(\Gamma) = Spec(\overline{K_w \bigtriangledown ECP_l^k}) = \left\{ [-1]^{(3^k-1)l}, \, \left[3^k - 1\right]^l, [0]^w \right\}.$$

If $l = 1$, by Lemma 2.1 ((i), (ii) and (iii)) the proof is clear (Also, by Theorem 2.5 the proof follows). Hence we suppose that $l \neq 1$. It is easy to see that Γ cannot be regular, since regularity of a graph can be determined by its spectrum. By contrary, we suppose that Γ is connected. So, we from Lemma 2.6 and Lemma 2.11 conclude that $k = l = 1$. This is a contradiction. Hence $\Gamma = \Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_h$, where Γ_s is a connected component of Γ and $1 \leq s \leq h$. Now, we show that Γ_s cannot have three distinct eigenvalues. By contrary, we suppose that Γ_i has three distinct eigenvalues. In this case, if we also suppose Γ_s is non-regular, then it follows from Lemma 2.6 that Γ_s is a complete bipartite graph. Hence $l = k = 1$. This is a contradiction. Therefore, if Γ_s has three distinct eigenvalues, then it must be regular. Now, it follows from Theorem 2.5 that $\Gamma_s \cong K_{\underbrace{1,\, 1,\, ...,\, 1}_{3^k\, times}} \cong K_{3^k}$. This is a contradiction. So, Γ_s cannot have three distinct eigenvalues. Therefore, it has one or two eigenvalue(s). Hence, any connected component of Γ is either isolated vertex or a complete graph. Hence $\Gamma \cong wK_1 \cup lK_{3^k}$. This follows the result. □

3.4. The multicone graphs $K_w \bigtriangledown mECP_l^k$ are determined by their Laplacian spectra

In this subsection, we show that any graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ is DS with respect to its Laplacian spectrum.

Theorem 3.8. *Multicone graphs $K_w \bigtriangledown mECP_l^k$ are DS with respect to their Laplacian spectrum.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is,

$$Spec(L(H)) = Spec(L(K_w \bigtriangledown mECP_l^k)$$

$$= \left\{ \left[3^k lm + w\right]^w,\, [w]^{m-1},\, \left[3^k l - 3^k + w\right]^{3^k lm - lm},\, \left[3^k l + w\right]^{lm-m},\, [0]^1 \right\}$$

follows that $H \cong K_w \bigtriangledown mECP_l^k$. We show that the problem is true for $w + 1$; that is, we show that

$$Spec(L(G)) = Spec(L(K_{w+1} \bigtriangledown mECP_l^k))$$

$$= \left\{ \left[3^k lm + w + 1\right]^{w+1}, [w+1]^{m-1}, \left[3^k l - 3^k + w + 1\right]^{3^k lm - lm}, \left[3^k l + w + 1\right]^{lm-m}, [0]^1 \right\}$$

follows that $G \cong K_{w+1} \bigtriangledown mECP_l^k$. It follows from Lemma 2.9 that H and G are the join of two graphs. On the other hand,

$$Spec(L(K_1 \bigtriangledown H)) = Spec(L(G)) = spec(L(K_{w+1} \bigtriangledown mECP_l^k)).$$

Therefore, we must have $G \cong K_1 \bigtriangledown H$. Because, G is the join of two graphs and also according to spectrum of G, must K_1 be joined to H and this is only available state. This completes the proof. □

Corollary 3.9. *Multicone graphs $K_w \bigtriangledown mECP_1^k = K_w \bigtriangledown mK_{3^k}$ are DS with respect to their Laplacian spectrums.*

3.5. Some results about multicone graphs $K_w \bigtriangledown mECP_l^k$

In this subsection, we show that any graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ must be perfect. Also, we prove that any graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ with respect to Laplacian spectrum is perfect. In addition, we show that any graph cospectral with complement of multicone graph $K_w \bigtriangledown mECP_l^k$ is perfect.

Suppose $\chi(\Gamma)$ and $\omega(\Gamma)$ are chromatic number and clique number of graph G, respectively. A graph is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of Γ. It is proved that a graph G is perfect if and only if Γ is Berge; that is, it contains no odd hole or antihole as induced subgraph, where odd hole and antihole are odd cycle, C_m for $m \geq 5$, and its complement, respectively. Also, in 1972 Lovász proved that, a graph is perfect if and only if its complement is perfect (see [22] of [2]). Now, by Theorem 3.4, Theorem 3.7, Theorem 3.8 and by what was said in the previous sections we can conclude the following results.

Theorem 3.10. *Let graph* Γ *be cospectral with multicone graph* $K_w \bigtriangledown mECP_l^k$*. Then* Γ *and* $\overline{\Gamma}$ *are perfect.*

Proof. By what was said in the beginning of this section and Theorem 3.4 the proof is completed. □

Theorem 3.11. *Let* Γ *be a graph and* $Spec(L(\Gamma)) = Spec(L(K_w \bigtriangledown mECP_l^k))$*. Then* Γ *and* $\overline{\Gamma}$ *are perfect.*

Proof. The proof is straightforwad. □

Theorem 3.12. *Let* Γ *be a graph and* $Spec(\Gamma) = Spec(\overline{K_w \bigtriangledown mECP_l^k})$*. Then* Γ *and* $\overline{\Gamma}$ *are perfect.*

Proof. It is obvious. □

In the following, we pose two conjectures.

4. Final remarks and open problems

In this paper, we have shown any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ is DS with respect to its spectra. Also, we have shown in special cases complement of these graphs are DS. In addition, we have proved any connected graph cospectral with these graph is perfect. On the other hand, It is obvious that, F_n are special classes of multicone graphs $K_w \bigtriangledown mECP_l^k$(one can also consider $k = 0$). In addition, F_n are DS with respect to:

(i) Their adjaccency spectrum (if $n \neq 16$).

(ii) Their Laplacian spectrum.

(iii) Their signless Laplacian spectrum. Also, $\overline{F_n}$ are DS with respect to their adjacency spectrum, where $n \neq 2$.

Hence we pose the following conjectures.

Conjecture 4.1. *Multicone graphs* $K_w \bigtriangledown mECP_l^k$ *are DS with respect to their signless Laplacian spectrum.*

Conjecture 4.2. *The complement of multicone graphs* $K_w \bigtriangledown mECP_l^k$ *are DS with respect to their adjacency spectrum.*

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