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On some classes of holomorphic functions whose derivatives have positive real part

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Abstract. In this paper we discuss about normalized holomorphic functions whose derivatives have positive real part. For this class of functions, denoted R, we present a general distortion result (some upper bounds for the modulus of the k-th derivative of a function). We present also some remarks on the functions whose derivatives have positive real part of order α, $\alpha \in (0,1)$. More details about these classes of functions can be found in [6], [8], [7, Chapter 4] and [4]. In the last part of this paper we present two new subclasses of normalized holomorphic functions whose derivatives have positive real part which generalize the classes R and $R(\alpha)$. For these classes we present some general results and examples.

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1. Introduction

In this paper we denote $U = U(0,1)$ the open unit disc in the complex plane, $\mathcal{H}(U)$ the family of all holomorphic functions on the unit disc and S the family of all univalent normalized ($f(0) = 0$ and $f'(0) = 1$) functions on the unit disc. Also, let us denote

$$\mathcal{P} = \left\{p \in \mathcal{H}(U) : p(0) = 1 \text{ and } \mathrm{Re}[p(z)] > 0, \quad z \in U\right\}$$

the *Carathéodory class* and

$$R = \{f \in \mathcal{H}(U) : f(0) = 0, f'(0) = 1 \text{ and } \mathrm{Re}[f'(z)] > 0, \quad z \in U\}$$

the *class of normalized functions whose derivative has positive real part.* For more details about these classes, one may consult [1], [2, Chapter 7], [3, Chapter 2] or [7, Chapter 3].

Remark 1.1. Notice that, according to a result due to Noshiro and Warschawski (see [1, Theorem 2.16], [6] or [7, Theorem 4.5.1]), we have that each function from R is also univalent on the unit disc U. Hence, $R \subseteq S$.

Remark 1.2. Another important result (see [7, p. 87]) says that $f \in R$ if and only if $f' \in \mathcal{P}$.

Remark 1.3. During this paper, we use the following notations for the series expansions of $p \in \mathcal{P}$ and $f \in S$:

$$p(z) = 1 + p_1 z + p_2 z^2 + ... + p_n z^n + ... \tag{1.1}$$

and

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ..., \tag{1.2}$$

for all $z \in U$.

2. Preliminaries

First, we present some classical results regarding to the coefficient estimations and distortion results for the Carathéodory class $\mathcal{P}$. For details and proofs, one may consult [2, Chapter 7], [3, Chapter 2], [6, Lemma 1] or [7, Chapter 3].

Proposition 2.1. *Let $p \in \mathcal{P}$. Then*

$$|p_n| \leq 2, \quad n \geq 1, \tag{2.1}$$

$$\frac{1-|z|}{1+|z|} \leq \mathrm{Re}[p(z)] \leq |p(z)| \leq \frac{1+|z|}{1-|z|} \tag{2.2}$$

and

$$|p'(z)| \leq \frac{2}{(1-|z|)^2}, \tag{2.3}$$

for all $z \in U$. These estimates are sharp. The extremal function is $p : U \to \mathbb{C}$ given by

$$p(z) = \frac{1+z}{1-z}, \quad z \in U. \tag{2.4}$$

The next result is another important result regarding to the coefficient estimations and distortion results for the class R. For more details and proofs, one may consult [6, Theorem 1], [7, Chapter 4] or [8, Theorem A].

Proposition 2.2. *Let $f \in R$. Then*

$$|a_n| \leq \frac{2}{n}, \quad n \geq 2, \tag{2.5}$$

$$\frac{1-|z|}{1+|z|} \leq \mathrm{Re}\big[f'(z)\big] \leq |f'(z)| \leq \frac{1+|z|}{1-|z|}. \tag{2.6}$$

and

$$-|z| + 2\log(1+|z|) \leq |f(z)| \leq -|z| - 2\log(1-|z|). \tag{2.7}$$

for all $z \in U$. These estimates are sharp. The extremal function is $f : U \to \mathbb{C}$ given by

$$f(z) = -z - \frac{2}{\lambda}\log(1-\lambda z), \quad |\lambda| = 1, \quad z \in U. \tag{2.8}$$

Remark 2.3. Let $r = |z| < 1$. Then, for every $k \in \mathbb{N}^*$, the following relation hold

$$T_k = \frac{1}{(1-r)^k} = \sum_{p=0}^{\infty} \frac{(k+p-1)! \cdot r^p}{p! \cdot (k-1)!}. \tag{2.9}$$

This remark will be used in the next section as part of the proofs of the main results.

Proof. Let us consider the following Taylor series expansion

$$\frac{1}{1-r} = 1 + r + r^2 + ... + r^n + ..., \quad -1 < r < 1.$$

Then

$$\frac{1}{(1-r)^2} = \frac{\partial}{\partial r}\left[\frac{1}{1-r}\right] = 1 + 2r + 3r^2 + ... + nr^{n-1} + ..., \ -1 < r < 1.$$

It is easy to prove relation (2.9) using mathematical induction. For this, let us consider

$$P(k) : \frac{1}{(1-r)^k} = \sum_{p=0}^{\infty} \frac{(k+p-1)! \cdot r^p}{p! \cdot (k-1)!}, \quad k \geq 1.$$

Assume that $P(k)$ is true and let us prove that $P(k+1)$ is also true, where

$$P(k+1) : \frac{1}{(1-r)^{k+1}} = \sum_{p=0}^{\infty} \frac{(k+p)! \cdot r^p}{p! \cdot k!}.$$

Indeed,

$$\frac{k}{(1-r)^{k+1}} = \frac{\partial}{\partial r}\left[\frac{1}{(1-r)^k}\right] = \frac{\partial}{\partial r}\left[\sum_{p=0}^{\infty} \frac{(k+p-1)! \cdot r^p}{p! \cdot (k-1)!}\right]$$
$$= \sum_{p=1}^{\infty} \frac{(k+p-1)! \cdot p \cdot r^{p-1}}{p! \cdot (k-1)!} = \sum_{p=0}^{\infty} \frac{(k+p)! \cdot r^p}{p! \cdot (k-1)!}$$

and then

$$\frac{1}{(1-r)^{k+1}} = \sum_{p=0}^{\infty} \frac{(k+p)! \cdot r^p}{p! \cdot k!}, \ r > 1.$$

Hence, $P(k)$ is true for all $k \geq 1$ and the relation (2.9) holds. □

3. General distortion result for the class R

Starting from the previous proposition, we give a general distortion result (some upper bounds for the modulus of the k-th derivative) for the frunction from the class R.

Theorem 3.1. *If $f \in R$, then the following estimate hold:*

$$|f^{(k)}(z)| \leq \frac{2(k-1)!}{(1-|z|)^k}, \quad z \in U, \quad k \geq 1.$$

Proof. It is clear that R is a subclass of class S. Then the k-th derivative of a function $f \in R$ has the form

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n, \quad z \in U. \tag{3.1}$$

Let $|z| \le r < 1$. In view of relations (2.5) and (3.1) we obtain that

$$\begin{aligned} |f^{(k)}(z)| &= \left| \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n \right| \le \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} |a_{k+n}| \cdot |z^n| \\ &\le \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} \cdot \frac{2}{k+n} r^n = 2 \cdot \sum_{n=0}^{\infty} \frac{(k+n-1)! r^n}{n!} \\ &= 2(k-1)! \cdot \sum_{n=0}^{\infty} \frac{(k+n-1)! r^n}{n!(k-1)!} \\ &= 2(k-1)! \cdot \frac{1}{(1-r)^k} = \frac{2(k-1)!}{(1-r)^k}. \end{aligned}$$

Hence, we obtain that

$$|f^{(k)}(z)| \le \frac{2(k-1)!}{(1-r)^k}, \quad k \in \mathbb{N}^*, \quad |z| \le r < 1. \qquad \square$$

Remark 3.2. Notice that the above result is not sharp for $k = 1$ (in view of relation (2.6)), but it is sharp for $k \ge 2$ and the extremal function is given by (2.8).

4. Some remarks on the class $R(\alpha)$

Let $\alpha \in [0, 1)$. Then

$$R(\alpha) = \{f \in \mathcal{H}(U) : f(0) = 0, f'(0) = 1, \mathrm{Re}\big[f'(z)\big] > \alpha, z \in U\}$$

denotes the *class of functions whose derivative has positive real part of order* α. For more details about this class, one may consult [4] and [5].

Remark 4.1. It is easy to prove that $f \in R(\alpha)$ if and only if $g \in \mathcal{P}$, where $g : U \to \mathbb{C}$ is given by

$$g(z) = \frac{1}{1-\alpha}\Big(f'(z) - \alpha\Big), \quad z \in U. \tag{4.1}$$

Proposition 4.2. *Let* $\alpha \in [0, 1)$ *and* $f \in R(\alpha)$. *Then*

$$|a_n| \le \frac{2(1-\alpha)}{n}, \quad n \ge 2, \tag{4.2}$$

and these estimates are sharp. The equality holds for the function $f : U \to \mathbb{C}$ *given by*

$$f(z) = \frac{(2\alpha - 1)\lambda z - 2(1-\alpha)\log(1 - \lambda z)}{\lambda} \tag{4.3}$$

with $|\lambda| = 1$.

Proof. Let $f \in R(\alpha)$ be of the form (1.2). Then

$$f'(z) = 1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} z^n, \quad z \in U.$$

Let us consider the function $g : U \to \mathbb{C}$ given by

$$g(z) = \frac{1}{1-\alpha}\Big(f'(z) - \alpha\Big), \quad z \in U.$$

Then $g \in \mathcal{P}$ and

$$g(z) = \frac{f'(z)-\alpha}{1-\alpha} = \frac{1-\alpha+\sum_{n=1}^{\infty}(n+1)a_{n+1}z^n}{1-\alpha} = 1 + \sum_{n=1}^{\infty} \frac{(n+1)}{1-\alpha} a_{n+1} z^n$$

or, equivalent

$$g(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad \text{where} \quad p_n = \frac{n+1}{1-\alpha} a_{n+1}. \tag{4.4}$$

Taking into account the relations (2.1) and (4.4) we obtain that

$$\left|\frac{n+1}{1-\alpha} a_{n+1}\right| \leq 2 \Leftrightarrow |a_{n+1}| \leq \frac{2(1-\alpha)}{n+1}, \quad \forall\, n \geq 1.$$

So we obtain that

$$|a_n| \leq \frac{2(1-\alpha)}{n}, \quad \forall\, n \geq 2.$$

The function given by relation (4.3) is obtained from the extremal function of the Carathédory class. We have the following Taylor expansion

$$f(z) = z + (1-\alpha)\lambda z^2 + \frac{2}{3}(1-\alpha)\lambda^2 z^3 + \ldots$$

leading to the estimates

$$|a_2| = \Big|(1-\alpha)\lambda\Big| = 1-\alpha$$

$$|a_3| = \left|\frac{2}{3}(1-\alpha)\lambda\right| = \frac{2(1-\alpha)}{3}$$

and the equalities hold for every $n \geq 2$. □

Remark 4.3. The previous result can be found also in [5, Theorem 3.5] with another version of the proof.

Next, we present a growth and distortion result for the class $R(\alpha)$. Starting from this theorem we give also a general distortion result (some upper bounds for the modulus of the k-th derivative) for the class $R(\alpha)$.

Theorem 4.4. *Let $\alpha \in [0,1)$ and $f \in R(\alpha)$. Then*

$$|f(z)| \leq (2\alpha-1)|z| + 2(\alpha-1)\log(1-|z|), \tag{4.5}$$

$$|f(z)| \geq -|z| - 2(\alpha-1)\log(1+|z|) \tag{4.6}$$

and

$$\frac{1-2\alpha-|z|}{1+|z|} \leq |f'(z)| \leq \frac{1+(1-2\alpha)|z|}{1-|z|}, \tag{4.7}$$

for all $z \in U$. These estimates are sharp. The extremal function is $f : U \to \mathbb{C}$ given by

$$f(z) = (2\alpha-1)z - \frac{2(1-\alpha)\log(1-\lambda z)}{\lambda}, \quad |\lambda| = 1, \quad z \in U. \tag{4.8}$$

Proof. Let $\alpha \in [0,1)$ and $f \in R(\alpha)$. In view of Remark 4.1 and Proposition 2.1, we obtain that

$$\left|\frac{1}{1-\alpha}\left[f'(z)-\alpha\right]\right| \leq \frac{1+|z|}{1-|z|}$$

$$\left|f'(z)-\alpha\right| \leq \frac{(1-\alpha)(1+|z|)}{1-|z|}$$

Then

$$|f'(z)| \leq \frac{(1-\alpha)(1+|z|)}{1-|z|} + \alpha = \frac{1+(1-2\alpha)|z|}{1-|z|}$$

On the other hand,

$$\left|\frac{1}{1-\alpha}\left[f'(z)-\alpha\right]\right| \geq \frac{1-|z|}{1+|z|}$$

$$\left|f'(z)-\alpha\right| \geq \frac{(1-\alpha)(1-|z|)}{1+|z|}$$

Then

$$|f'(z)| \geq \frac{(1-\alpha)(1-|z|)}{1+|z|} - \alpha = \frac{1-2\alpha-|z|}{1+|z|}$$

Hence, we obtain relations (4.7). Finally, to obtain the relations (4.5) and (4.6), it is enough to integrate the relation (4.7). □

Theorem 4.5. *Let $\alpha \in [0,1)$ and $f \in R(\alpha)$. Then the following estimate hold:*

$$|f^{(k)}(z)| \leq \frac{2(1-\alpha)(k-1)!}{(1-|z|)^k}, \quad z \in U, \quad k \geq 1.$$

Proof. Let $\alpha \in [0,1)$. It is clear that $R(\alpha)$ is a subclass of class S. Then the k-th derivative of a function $f \in R(\alpha)$ has the form

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n, \quad z \in U. \tag{4.9}$$

Let $|z| \leq r < 1$. According to the relations (4.2) and (4.9) we obtain that

$$\begin{aligned} |f^{(k)}(z)| &= \left|\sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n\right| \leq \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} |a_{k+n}| \cdot |z^n| \\ &\leq \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} \cdot \frac{2(1-\alpha)}{k+n} r^n = 2(1-\alpha) \cdot \sum_{n=0}^{\infty} \frac{(k+n-1)! r^n}{n!} \\ &= 2(1-\alpha)(k-1)! \cdot \sum_{n=0}^{\infty} \frac{(k+n-1)! r^n}{n!(k-1)!} = \frac{2(1-\alpha)(k-1)!}{(1-r)^k}, \end{aligned}$$

Hence, we obtain that

$$|f^{(k)}(z)| \leq \frac{2(1-\alpha)(k-1)!}{(1-r)^k}, \quad k \in \mathbb{N}^*, \quad |z| \leq r < 1. \qquad \square$$

Remark 4.6. Notice that, for $k = 1$, the previous result is not sharp. The sharpness is obtained if $k \geq 2$ for the function f defined by (4.8).

Remark 4.7. It is clear that if $\alpha = 0$, then $R(0) = R$ and we obtain the classical results from the previous section.

5. The class R_p

Let $p \in \mathbb{N}^*$. Starting from the well-known class R, we define

$$R_p = \{f \in \mathcal{H}(U) : f(0) = 0, f'(0) = 1, f^{(p)}(0) = 1, \mathrm{Re}\big[f^{(p)}(z)\big] > 0, z \in U\}$$

the *class of normalized functions whose p-th derivative has positive real part*. This is the natural extension of the class R (extension which preserves the connection with the Carathéodory class). We present for this class some important results, a few examples and structure formulas (in the particular cases $p = 2$ and $p = 3$). It is clear that if $p = 1$, then $R_1 = R$.

Remark 5.1. In previous definition we have the following equivalent conditions

$$f^{(p)}(0) = 1 \Leftrightarrow a_p = \frac{1}{p!}, \tag{5.1}$$

for $p \in \mathbb{N}^*$ arbitrary fixed. Indeed, if $f \in R_p$, then

$$f^{(p)}(z) = \sum_{n=0}^{\infty} \frac{(n+p)!}{n!} a_{n+p} z^n = p! \cdot a_p + \frac{(p+1)!}{1!} a_{p+1} z + \frac{(p+2)!}{2!} a_{p+2} z^2 + \ldots$$

For $z = 0$ we obtain

$$f^{(p)}(0) = p! \cdot a_p.$$

Hence

$$f^{(p)}(0) = 1 \Leftrightarrow p! \cdot a_p = 1 \Leftrightarrow a_p = \frac{1}{p!}, \quad p \geq 1.$$

Remark 5.2. Let $p \in \mathbb{N}^*$ be arbitrary fixed. In view of above definition we deduce that

$$f \in R_p \Leftrightarrow f^{(p)} \in \mathcal{P},$$

so we can use the properties of Carathéodory class $\mathcal{P}$ to describe the function $f^{(p)}$ and then we can obtain some properties for $f \in R_p$.

Proposition 5.3. *Let $p \in \mathbb{N}^*$ and $f \in R_p$. Then the following relation hold:*

$$|a_n| \leq \frac{2(n-p)!}{n!}, \quad n \geq p, \tag{5.2}$$

Proof. Let $f \in R_p$. Then

$$f^{(p)}(z) = \sum_{n=0}^{\infty} \frac{(n+p)!}{n!} a_{n+p} z^n, \quad z \in U.$$

Taking into account Remark 5.2 and Proposition 2.1 we have that

$$f^{(p)} \in \mathcal{P},$$

and

$$\left| \frac{(n+p)!}{n!} a_{n+p} \right| \leq 2, \quad \forall\, n \geq 2.$$

In view of above relations we obtain

$$|a_{n+p}| \leq \frac{2 \cdot n!}{(n+p)!}$$

or, an equivalent form

$$|a_n| \leq \frac{2(n-p)!}{n!}, \quad \forall\, n \geq p.$$

□

Theorem 5.4. *Let $p \in \mathbb{N}^*$ and $f \in R_p$. Then the following estimate hold:*

$$|f^{(k)}(z)| \leq \frac{2(k-p)!}{(1-|z|)^{k-p+1}}, \quad z \in U, \quad k \geq p. \tag{5.3}$$

Proof. Let $f \in R_p$. Then

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{n+k} z^n, \quad z \in U. \tag{5.4}$$

Let $|z| \leq r < 1$. Using relations (5.2) and (5.4) we obtain

$$\begin{aligned} |f^{(k)}(z)| &= \left| \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n \right| \leq \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} |a_{k+n}| \cdot |z^n| \\ &\leq \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} \cdot \frac{2(n+k-p)!}{(k+n)!} r^n = 2 \cdot \sum_{n=0}^{\infty} \frac{(n+k-p)! r^n}{n!} \\ &= 2(k-p)! \cdot \sum_{n=0}^{\infty} \frac{(k+n-p)! r^n}{n!(k-p)!} = \frac{2(k-p)!}{(1-r)^{k-p+1}}. \end{aligned}$$

Hence,

$$|f^{(k)}(z)| \leq \frac{2(k-p)!}{(1-|z|)^{k-p+1}}, \quad z \in U, \quad k \geq p.$$

□

Remark 5.5. In estimates (5.3) we have the following existence condition:

$$\forall\, k, p \in \mathbb{N}^* : \quad k \geq p.$$

In other words, for $p \in \mathbb{N}^*$ arbitrary fixed we can estimate the derivatives of order k with $k \geq p$ (the derivatives of order at least p). In particular, for $p = 1$ (i.e. for the class R) we can estimate all derivatives of order at least 1.

Remark 5.6. For the bounds of the modulus of the first $(p-1)$ derivatives of a function $f \in R_p$ we can apply the following argument

$$\forall\, j \in \{0, ..., p-1\} : \quad |f^{(j)}(z)| \leq \underbrace{\int_0^r ... \int_0^r}_{(p-j) \text{ times}} \left[\frac{1+\rho}{1-\rho}\right] d\rho \tag{5.5}$$

In particular,

$$|f^{(p-1)}(z)| \leq -|z| - 2\log(1-|z|)$$

and

$$|f^{(p-2)}(z)| \leq \frac{-|z|(|z|-4)}{2} - 2(|z|-1)\log(1-|z|).$$

Hence, for $f \in R_p$ we obtain general upper bounds, as follows:

- if $0 \leq k < p$, we use relation (5.3);
- if $k \geq p$, we use relation (5.5).

Remark 5.7. If $p = 1$, then $R_1 = R$ and we obtain the result (*general result of distortion*) from Theorem 3.1.

In following results we discuss about the relation between two consecutive classes of order p, respectively $p+1$, for $p \in \mathbb{N}^*$ arbitrary choosen.

Proposition 5.8. *Let $p \in \mathbb{N}^*$. Then $R_p \cap R_{p+1} \neq \emptyset$.*

For $p \in \mathbb{N}^*$ we can find a function f which belongs to both class R_p and R_{p+1}. We present two examples to illustrate this proposition (first for the case $p = 1$ and second for the general case $p \geq 2$).

Example 5.9. Let $f : U \to \mathbb{C}$ be given by $f(z) = \frac{1}{2}z^2 + z$, $z \in U$. Then $f \in R_1 \cap R_2$.

Proof. Indeed, we have

$$f(0) = 0$$
$$f'(z) = z + 1$$
$$f''(z) = 1, \quad z \in U.$$

For $z = 0$ we obtain

$$f'(0) = f''(0) = 1 \quad \text{and} \quad \text{Re} f''(z) = 1 > 0, \quad \forall\, z \in U.$$

Then, in view of definition, $f \in R_2$. On the other hand,

$$f'(0) = 1 \quad \text{and} \quad \text{Re} f'(z) = \text{Re}(z+1) = 1 + \text{Re} z > 0, \quad \forall\, z \in U,$$

and this means that $f \in R_1$. □

Example 5.10. Let $p \geq 2$ and let $f : U \to \mathbb{C}$ be given by

$$f(z) = z + \frac{1}{p!}z^p + \frac{1}{(p+1)!}z^{p+1}, \quad z \in U.$$

Then $f \in R_p \cap R_{p+1}$.

Proposition 5.11. *Let $p \in \mathbb{N}^*$. In general, $R_p \not\subseteq R_{p+1}$.*

For $p \in \mathbb{N}^*$ we can find a function f which belongs to the class R_p, but does not belong to the class R_{p+1}. We present two examples to illustrate this statement.

Example 5.12. Let $f : U \to \mathbb{C}$ be given by $f(z) = z$, $z \in U$. Then $f \in R = R_1$, but $f \notin R_2$.

Example 5.13. Let $p \geq 2$ and let $f : U \to \mathbb{C}$ be given by $f(z) = z + \frac{1}{p!}z^p$, $z \in U$. Then $f \in R_p$, but $f \notin R_{p+1}$.

Remark 5.14. The above example can be generalized by adding the terms between z and $\frac{1}{p!}z^p$. We can consider the function $f : U \to \mathbb{C}$ given by

$$f(z) = z + \sum_{n=2}^{p-1} a_n z^n + \frac{1}{p!}z^p, \quad z \in U.$$

For $n \in \{2, 3, ..., p-1\}$ the coefficients a_n can be real or complex numbers, but $a_1 = 1$ and $a_p = \dfrac{1}{p!} \in \mathbb{R}$.

Proposition 5.15. *Let $p \in \mathbb{N}^*$. In general, $R_{p+1} \not\subseteq R_p$.*

For $p \in \mathbb{N}^*$ we can find a function f which belongs to the class R_{p+1}, but does not belong to the class R_p. We present also two examples to illustrate this statement.

Example 5.16. Let $f : U \to \mathbb{C}$ be given by $f(z) = z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3$, $z \in U$. Then $f \in R_2$, but $f \notin R_1$.

Proof. Indeed, we have

$$f(0) = 0, \quad f'(z) = 1 + z + \frac{z^2}{2} \quad \text{and} \quad f''(z) = 1 + z, \quad z \in U.$$

Then

$$f'(0) = f''(0) = 1 \quad \text{and} \quad \text{Re} f''(z) = 1 + \text{Re} z > 0, \quad z \in U.$$

Hence, in view of definition, $f \in R_2$. But,

$$\text{Re} f'(z) = 1 + \text{Re} z + \frac{1}{2}\text{Re} z^2 > -\frac{1}{2}, \quad z \in U.$$

Then $\text{Re} f'(z) \not> 0$, $z \in U$ and hence $f \notin R_1$. □

Example 5.17. Let $p \geq 2$ and let $f : U \to \mathbb{C}$ be given by $f(z) = z + \frac{1}{(p+1)!}z^{p+1}$, $z \in U$. Then $f \in R_{p+1}$, but $f \notin R_p$.

Remark 5.18. Let $p \in \mathbb{N}^*$. Then

1. $R_p \not\subseteq R_{p+1}$;
2. $R_p \not\supseteq R_{p+1}$;
3. $R_p \cap R_{p+1} \neq \emptyset$.

Remark 5.19. Let $p \geq 2$ and consider the polynomial

$$q(z) = z + a_2 z^2 + a_3 z^3 + ... + a_{p-1} z^{p-1} + a_p z^p, \quad z \in U.$$

Then $q \in R_p$ if and only if $a_p = \dfrac{1}{p!}$.

5.1. Structure formula for $p = 2$ and $p = 3$

Proposition 5.20. *Let $f : U \to \mathbb{C}$. Then $f \in R_2$ if and only if there exists a function μ measurable on $[0, 2\pi]$ such that*

$$f(z) = -\frac{z^2}{2} - 2 \cdot \int_0^{2\pi} e^{it}\big[(z - e^{it}) \log(1 - ze^{-it}) - z\big] d\mu(t),$$

where $\log 1 = 0$.

Proof. According to Remark 5.2 we have that $f'' \in \mathcal{P}$. Hence, in view of Herglotz formula we obtain that

$$f''(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t), \quad \mu \in [0, 2\pi].$$

Then,

$$f(z) = \int_0^z \left(\int_0^z \int_0^{2\pi} \frac{e^{it} + s}{e^{it} - s} d\mu(t) ds\right) ds = \int_0^z \left[\int_0^{2\pi} \left(\int_0^z \frac{e^{it} + s}{e^{it} - s} ds\right) d\mu(t)\right] ds.$$

Using [7, Theorem 3.2.2] we know that

$$f(z) = \int_0^z \big[-\zeta - 2\int_0^{2\pi} e^{it} \log(1 - \zeta e^{-it}) d\mu(t)\big] d\zeta,$$

so we obtain

$$f(z) = -\frac{z^2}{2} - 2 \cdot \int_0^{2\pi} e^{it}\big[(z - e^{it}) \log(1 - ze^{-it}) - z\big] d\mu(t). \qquad \square$$

Remark 5.21. It is possible to obtain a structure formula for the case $p = 3$:

$$f(z) = -\frac{z^3}{6} - 2 \cdot \int_0^{2\pi} e^{it}\left[\left(\frac{z^2}{2} + e^{-it} - e^{it}(z - e^{it})\right) \log(1 - ze^{-it}) - 2z - \frac{z^2}{2}\right] d\mu(t),$$

where $\log 1 = 0$.

6. The class $R_p(\alpha)$

Let $\alpha \in [0, 1)$ and $p \in \mathbb{N}^*$. Then we define

$$R_p(\alpha) = \{f \in \mathcal{H}(U) : f(0) = 0, f'(0) = 1, f^{(p)}(0) = 1, \mathrm{Re}\big[f^{(p)}(z)\big] > \alpha, z \in U\}.$$

the *class of normalized functions whose p-th derivative has positive real part of order* α.

Remark 6.1. Let $\alpha \in [0, 1)$ and $p \in \mathbb{N}^*$. Then $f \in R_p(\alpha)$ if and only if $g \in \mathcal{P}$, where $g : U \to \mathbb{C}$ is given by

$$g(z) = \frac{f^{(p)}(z) - \alpha}{1 - \alpha}, \quad z \in U.$$

Proposition 6.2. *Let $\alpha \in [0, 1)$ and $p \in \mathbb{N}^*$. If $f \in R_p(\alpha)$, then the following relation hold:*

$$|a_n| \le \frac{2(1 - \alpha)(n - p)!}{n!}, \quad n \ge p, \tag{6.1}$$

Proof. Similar to the proof of Proposition 4.2. $\square$

Theorem 6.3. *Let $\alpha \in [0,1)$ and $p \in \mathbb{N}^*$. If $f \in R_p(\alpha)$, then the following estimate hold for all $k \in \mathbb{N}^*$ with $k \geq p$:*

$$|f^{(k)}(z)| \leq \frac{2(1-\alpha)(k-p)!}{(1-|z|)^{k-p+1}}, \quad z \in U. \tag{6.2}$$

Proof. Similar to the proof of Theorem 4.5. □

Remark 6.4. If $\alpha = 0$, then $R_p(0) = R_p$ and we obtain Proposition 5.3 and Theorem 5.4 from previous section. If, in addition, $p = 1$, then $R_1(0) = R$ and we obtain the coefficient estimates, respectively the growth and distortion result regarded to the class R.

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