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Notes on various operators of fractional calculus and some of their implications for certain analytic functions

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Abstract. The main purpose of this note is firstly to present certain information in relation with some elementary operators created by the well-known fractional calculus, also to determine a number of applications of them for certain complex function analytic in the open unit disc, and then to reveal (*or* point out) some implications of the fundamental results of this research.

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1. Introduction and preliminaries

Fractional Calculus has important roles in both applied studies and theoretical researches. We also know that various operators have been defined by the help of Fractional Calculus. This scientific note is an example for one of such theoretical investigations. In this present investigation, only three elementary operators of fractional calculus, *which* are frequently encountered in the literature, will be considered for determining a number of results relating to certain complex functions. They are well-known operators which are also called as Fractional Integral Operator, Fractional Derivative Operator and Tremblay Operator in the literature. Specially, as we just have indicated just above, these mentioned operators will be taken into consideration for certain analytic functions. For their details and also some extra examples, one

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may refer to the earlier works in [5] and [26]-[28], and also see [1]-[3], [11]-[10], [19], [18], [27] and [13] in the references of this investigation.

Let us now recall certain notations, notions and also some extra information in relation with the mentioned operators in certain domains of the complex plane, *which* there will need for our investigation.

Firstly, *here and also in parallel with this research,* let

$$\mathbb{C} \quad , \quad \mathbb{R} \quad , \quad \mathbb{N} \quad \text{and} \quad \mathbb{U}$$

be the set of *complex numbers,* be the set of *real numbers,* be the set of *positive integers,* and the *open unit* disc, *namely,* the well-known *open* set given by

$$\Big\{z \,:\, z \in \mathbb{C} \text{ and } |z| < 1\Big\},$$

respectively.

Also let

$$\mathbb{R}^* := \mathbb{R} - \{0\} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Moreover, by the notation $\mathcal{A}(n)$ denote the family of the functions $f(z)$ normalized by the following Taylor-Maclaurin series:

$$f(z) = z + q_{n+1}z^{n+1} + q_{n+2}z^{n+2} + \cdots \quad \big(q_{n+1} \in \mathbb{C};\ n \in \mathbb{N}\big). \tag{1.1}$$

Secondly, for an analytic function $f(z)$, *the fractional integral of order* λ is then defined by

$$\mathcal{D}_z^{-\lambda}\{f\}(z) = \frac{1}{\Gamma(\lambda)}\int_0^z f(T)\big(z-T\big)^{\lambda-1}dT \quad \big(\lambda > 0\big), \tag{1.2}$$

where the multiplicity of $\big(z-T\big)^{\lambda-1}$ is removed by requiring $log\big(z-T\big)$ to be real *when* $z-T>0$.

For an analytic function $f(z)$, *the fractional derivative of order* λ is also defined by

$$\mathcal{D}_z^{\lambda}\{f\}(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z f(T)\big(z-T\big)^{-\lambda}dT \quad \big(0 \le \lambda < 1\big), \tag{1.3}$$

where is constrained, and the multiplicity of $\big(z-T\big)^{-\lambda}$ is removed, as in the definition of the fractional integral operator accentuated as (1.2).

Under the hypotheses of the definition of the fractional derivative of order λ, emphasized as (1.3), for an analytic function $f(z)$, *the fractional derivative of order* $m+\lambda$ is defined by

$$\mathcal{D}_z^{m+\lambda}\{f\}(z) = \frac{d^m}{dz^m}\Big(\mathcal{D}_z^{\lambda}\{f\}(z)\Big), \tag{1.4}$$

where $0 \le \lambda < 1$ and $m \in \mathbb{N}_0$.

In the light of the fractional derivative operator, given by (1.2), for an analytic function $f(z)$, the Tremblay operator is also defined by

$$\mathcal{T}_{\tau,\lambda}\{f\}(z) = \frac{\Gamma(\lambda)}{\Gamma(\tau)}\, z^{1-\lambda}\,\mathcal{D}_z^{\tau-\lambda}\big\{z^{\tau-1}f\big\}(z), \tag{1.5}$$

where $0 < \tau \le 1$, $0 < \lambda \le 1$, $0 \le \tau - \lambda < 1$ and $z \in \mathbb{U}$.

In consideration of the fractional integral operator (1.2), fractional derivative operator (1.3) and Tremblay operator (1.4), for an elementary-analytic function given by

$$\varphi := \varphi(z) = z^{\kappa},$$

we remark in passing that the following-special results can be easily assigned as the forms:

$$\mathcal{D}_z^{-\lambda}\{\varphi\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\lambda+1)} z^{\kappa+\lambda} \quad \big(\lambda > 0\big), \tag{1.6}$$

$$\mathcal{D}_z^{\lambda}\{\varphi\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\lambda+1)} z^{\kappa-\lambda} \quad \big(0 \leq \lambda < 1\big), \tag{1.7}$$

$$\mathcal{D}_z^{m+\lambda}\{\varphi\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-m-\lambda+1)} z^{\kappa-m-\lambda} \quad \big(0 \leq \lambda < 1; m \in \mathbb{N}_0\big), \tag{1.8}$$

and

$$\mathcal{T}_{\tau,\lambda}\{\varphi\} = \frac{\Gamma(\kappa+\tau)\Gamma(\lambda)}{\Gamma(\kappa+\lambda)\Gamma(\tau)} z^{\kappa} \quad \big(0 < \tau \leq 1; 0 < \lambda \leq 1; 0 \leq \tau-\lambda < 1\big). \tag{1.9}$$

In terms of our purposes, in special, by means of the assertions presented in (1.6)-(1.9), for a function $f(z)$ belonging to the family $\mathcal{A}(n)$, there is a need to state certain results which are given by the following relations:

$$\mathcal{D}_z^{-\lambda}\{f\}(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\lambda+1)} q_k z^{k+\lambda} \tag{1.10}$$

$$\big(\lambda > 0; z \in \mathbb{U}\big)$$

$$\mathcal{D}_z^{\lambda}\{f\}(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} q_k z^{k-\lambda} \tag{1.11}$$

$$\big(0 \leq \lambda < 1; z \in \mathbb{U}\big)$$

and

$$\mathcal{T}_{\tau,\lambda}\{f\}(z) = \frac{\tau}{\lambda} z + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+\tau)\Gamma(\lambda)}{\Gamma(k+\lambda)\Gamma(\tau)} q_k z^{k} \tag{1.12}$$

$$\big(0 < \tau \leq 1; 0 < \lambda \leq 1; 0 \leq \tau-\lambda < 1; z \in \mathbb{U}\big),$$

and, from (11) and (12), the following results are easily determined:

$$\mathcal{D}_z^{1+\lambda}\{f\}(z) = \frac{1}{\Gamma(1-\lambda)} z^{-\lambda} + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda)} q_k z^{k-\lambda-1} \tag{1.13}$$

$$\big(0 \leq \lambda < 1; z \in \mathbb{U}\big)$$

and

$$\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big) = \frac{\tau}{\lambda} + \sum_{k=n+1}^{\infty} \frac{k\Gamma(k+\tau)\Gamma(\lambda)}{\Gamma(k+\lambda)\Gamma(\tau)} q_k z^{k-1} \tag{1.14}$$

$$\big(0 < \tau \leq 1; 0 < \lambda \leq 1; 0 \leq \tau-\lambda < 1; z \in \mathbb{U}\big).$$

The following assertion, namely, Lemma 1.1 (below) will be required for stating and proving of our main results. By considering the well-known result (see [22] and

[24]), it was earlier proven by Nunokawa [23]. In addition, it has been also used by a great number of researchers for their studies. For some of them, for instance, it can be checked some of results given by [13]-[16] in the references. Specially, in the recent time, by making use of the same assertions considered in [17]-[20], they have used those for their earlier results and they have also obtained various results identified by the assertions relating to the mentioned operators and their applications given by (1.12)-(1.16). Moreover, we point that some (special) results can be compared with certain earlier results obtained in [20], [16] and [18].

Lemma 1.1. *Let $p(z)$ be an analytic function in the open set $\mathbb{U}$ with $p(0)=1$ and also suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\Re e\Big(p(z)\Big) > 0 \quad when \quad |z| < |z_0| < 1\,, \tag{2.1}$$

$$\Re e\Big(p(z)\big|_{z:=z_0}\Big) = 0 \tag{2.2}$$

and

$$p(z)\big|_{z:=z_0} \neq 0\,. \tag{2.3}$$

Then,

$$p(z)\big|_{z:=z_0} = is \quad \big(s \in \mathbb{R}^*\big) \tag{2.4}$$

and

$$zp'(z)\big|_{z:=z_0} = ic\left(s+s^{-1}\right)p(z)\big|_{z:=z_0} \quad \big(s \in \mathbb{R}^*\big) \tag{2.5}$$

for all c in $\big[1/2,\infty\big)$.

2. The main results and their implications

In this section, by considering certain necessity and sufficiency, which are terms used to describe a conditional (*or* implicational) relationship between two statements in mathematics, various comprehensive theories consisting of some complex-valued-exponential forms constituted by the operator given by (1.3)-(1.5) will be presented and they will be then proven.

Theorem 2.1. *For admissible values of the parameters given by*

$$0 < \tau \leq 1 \quad , \quad 0 < \lambda \leq 1 \quad , \quad 0 \leq \tau - \lambda < 1 \quad and \quad 0 \leq \alpha < 1\,, \tag{2.6}$$

if the following statement:

$$Arg\left(\frac{z\frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}{\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}\right) \notin \begin{cases} \big(-\pi,-\frac{\pi}{2}\big] \cup \big[\frac{\pi}{2},\pi\big) & if \quad \omega>0 \\ \big[-\frac{\pi}{2},0\big) \cup \big(0,\frac{\pi}{2}\big] & if \quad \omega<0 \end{cases} \tag{2.7}$$

is true, then

$$\Re e\left\{\Big[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\Big]^{\omega}\right\} > \alpha\Big(\frac{\tau}{\lambda}\Big)^{\omega} \quad \big(\omega \in \mathbb{R}^*\big) \tag{2.8}$$

is also true, where $\omega \in \mathbb{R}^*$, $z \in \mathbb{U}$ *and* $f(z) \in \mathcal{A}(n)$, *and, here and through the proof of this theorem and its implications, each one of the values of the complex expressions like*

$$\left[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\mu}\{f\}(z)\Big)\right]^{\omega} \qquad \big(\omega \in \mathbb{R}^*\big)$$

is taken to be as its principal value.

Proof. By the help of the application (of Tremblay operator in (1.5)) given by (1.14) and under the conditions given in (2.6), for a function $f(z) \in \mathcal{A}(n)$, let us then consider a function $p(z)$ in the implicit form, given in

$$\left[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\right]^{\omega} = \Big(\frac{\tau}{\lambda}\Big)^{\omega}\big[\alpha + (1-\alpha)p(z)\big]\,, \tag{2.9}$$

where $0 \leq \alpha < 1$, $\omega \in \mathbb{R}^*$ and $z \in \mathbb{U}$. By a simple focusing, clearly, the function $p(z)$ satisfies the condition $p(0) = 1$ in the hypothesis of Lemma 1.1.

By differentiating the both sides of the definition in (2.9) with respect to the complex variable z, it can be easily obtained that

$$\omega\, z\, \frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\left[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\right]^{\omega-1} = (1-\alpha)\Big(\frac{\tau}{\lambda}\Big)^{\omega} zp'(z)\,, \tag{2.10}$$

and, by combining (2.9) and (2.10), the following statement:

$$\omega \cdot \frac{z\frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}{\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)} = \frac{(1-\alpha)zp'(z)}{\alpha + (1-\alpha)p(z)} \tag{2.11}$$

is also received, *where,* of course,

$$0 \leq \alpha < 1\ ,\ \omega \in \mathbb{R}^*\ ,\ f(z) \in \mathcal{A}(n)\ \text{ and }\ \frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big) \neq 0\ \ \big(\forall z \in \mathbb{U}\big)\,.$$

For the proof, suppose now that there exists a point z belonging to $\mathbb{U}$, which satisfies the condition:

$$\Re e\big(p(z)\big) = 0 \quad \Big(z_0 \in \mathbb{U}; p\big(z_0\big) \neq 0\Big)\,,$$

indicated by (1.6) of Lemma 1.1. Then, by applying of the assertions of Lemma 1.1, given in (2.4) and (2.5) to the result given by (2.11), the following-special result:

$$\begin{aligned}
&Arg\left(-\omega \cdot \left.\frac{z\frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\mu}\{f\}(z)\Big)}{\frac{d}{dz}\Big(\mathcal{T}_{\tau,\mu}\{f\}(z)\Big)}\right|_{z:=z_0}\right) = Arg\left(\left.\frac{\big(\alpha-1\big)zp'(z)}{\alpha + \big(1-\alpha\big)p(z)}\right|_{z:=z_0}\right) \\
&= Arg\Big(\ \big(\alpha-1\big)zp'(z)\big|_{z:=z_0}\Big) - Arg\Big(\ \alpha + (1-\alpha)p(z)\big|_{z:=z_0}\Big) \\
&= Arg\Big(c(1-\alpha)\big(1+a^2\big)\Big) - Arg\Big(\alpha + ia(1-\alpha)\Big) \\
&= -Arg\Big(\alpha + ia(1-\alpha)\Big) \in \begin{cases} \big[-\frac{\pi}{2}, 0\big) & \text{if} \quad a < 0 \\ \big(0, \frac{\pi}{2}\big] & \text{if} \quad a > 0 \end{cases}
\end{aligned}$$

is easily obtained, *which* contradicts the result given in (2.7), of course, after some calculations.

This shows that there is no any point $z_0 \in \mathbb{U}$ satisfying the condition given in (2.2). This means that

$$\Re e\Big(p(z)\Big) > 0 \quad \big(\forall z \in \mathbb{U}\big).$$

Therefore, the special definition in (2.9) immediately yields that the inequality in (2.8). The desired proof is also completed. □

In this section, as we know, an extensive-main result has been constituted by applying one of elementary operators of fractional calculus, which is introduced by (1.5) (*or,* (1.12) and (1.14)), to a function $f(z)$ belonging to the family $\mathcal{A}(n)$. In consideration of the main result, as certain conclusions and recommendations, by considering those extensive information about all operators (together with combining some of them given in (1.2)-(1.4)), it can be easily redetermined several new results like Theorem 2.1 again. With the help of such information, the main theorem can help us to recompose many new-comprehensive results and also to reveal a great number of some important-specific results will be obtained by the possible results. In these determinations *or* constructions, we want to give some suggestions to the relevant researchers for stating and proving of new possible theorems *or* their special results.

As first suggestion, in view of the results determined by (1.6)-(1.14), several new theorems, *which* are similar to Theorem 2.1, can be also reconstituted. For it and its proof, it is enough to redefine a similar type function like $p(z)$, *which* is defined as (2.9) and plays an important role in the creation and the proof of Theorem 2.1. As example, if one takes into account the related function $p(z)$, which also consists of fractional fractional derivative(s), given as the following-implicit form:

$$\left[\frac{d}{dz}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)\right]^{\omega} = \Big(\frac{1}{\Gamma(2-\lambda)}\Big)^{\omega}\big[\alpha+(1-\alpha)p(z)\big]$$

$$\big(0 \leq \alpha < 1; 0 \leq \lambda < 1; \omega \in \mathbb{R}^*; f(z) \in \mathcal{A}(n)\big)$$

and also follows the similar manner in the proof of Theorem 2.1, the following theorem can be then demonstrated. Its detail is excluded here.

Theorem 2.2. *Let* $0 \leq \alpha < 1$, $\omega \in \mathbb{R}^*$ *and* $z \in \mathbb{U}$. *For a function* $f(z) \in \mathcal{A}(n)$, *if the statement:*

$$Arg\left(\frac{z\frac{d^2}{dz^2}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)}{\frac{d}{dz}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)}\right) \notin \begin{cases} \big(-\pi,-\frac{\pi}{2}\big]\cup\big[\frac{\pi}{2},\pi\big) & if \quad \omega>0 \\ \big[-\frac{\pi}{2},0\big)\cup\big(0,\frac{\pi}{2}\big] & if \quad \omega<0 \end{cases}$$

is true, then

$$\Re e\left\{\left[\frac{d}{dz}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)\right]^{\omega}\right\} > \alpha\big[\Gamma(2-\lambda)\big]^{-\omega} \quad \big(\omega \in \mathbb{R}^*\big)$$

is also true, where the value of the complex power given by

$$\left[\frac{d}{dz}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)\right]^{\omega} \quad \big(\omega \in \mathbb{R}^*\big)$$

is considered to be as its principal value.

As second suggestion, it will be the suggested determinations of both the new theorems to be determined by the researchers and the specific results of the main results determined by us. In order to reveal them, it will be sufficient to select the suitable values of the related parameters. We also leave to reveal the others to the attentions of the interested researchers. For this, as examples, we want also to present only two of them as propositions.

The first special result is one of the main result, *which* is Proposition 2.3 (below). It can be also constituted by choosing the value of ω as $\omega := 1$ in Theorem 2.1.

Proposition 2.3. *For a function* $f(z) \in \mathcal{A}(n)$, *if the statement*

$$Arg\left(\frac{z\frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}{\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}\right) \in \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big) \cup \{\pm\pi\}$$

holds, then

$$\Re e\left[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\right] > \alpha\frac{\tau}{\lambda}$$

holds for all $z \in \mathbb{U}$ *and also for some of the admissible values of the parameters given by* $0 < \tau \leq 1$, $0 < \lambda \leq 1$, $0 \leq \tau - \lambda < 1$ *and* $0 \leq \alpha < 1$.

By taking the values of the parameters ω, τ and λ as $\omega := 1$, $\tau := 1$ and $\mu := 1$ in Theorem 2.1 (*or,* by selecting the values of the parameters of τ and λ as $\tau := 1$ and $\lambda := 1$ in Proposition 2.3), for a function $f(z) \in \mathcal{A}(n)$, the second special result is then received. In this case, as a special consequence of the main result, *which* relates to (Analytic and) Geometric Function Theory (see, for details, [6]), it can be easily identified by the following-well-known result (Proposition 2.4 below).

Proposition 2.4. *Let the function* $f(z)$ *be in* $\mathcal{A}(n)$. *Then, the following statement is true:*

$$Arg\left(\frac{zf''(z)}{f'(z)}\right) \in \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big) \cup \{\pm\pi\} \quad \Rightarrow \quad \Re e\Big(f'(z)\Big) > \alpha.$$

It also shows that the function $f(z)$ *is a close-to-convex of order* α $(0 \leq \alpha < 1)$ *in the open disc* $\mathbb{U}$.

As concluding remark, all other results (and, of course, their possible consequences), which will be new (*or* known) for the literature and are also omitted in this scientific note, are presented to reveal to the attention of the researchers who have been working on the topics of this investigation. In particular, for the related researchers, we believe that it would be useful to focus on the results determined in the papers given by the references in [17-20], in terms of highlighted results and even their specific implications. As an example in relation with geometric properties of our works, Proposition 2.4 (above) has been presented. In the same time, extra simple examples can be also determined for those results (and also their special forms). These are also left to interested researchers. In addition, as was indicated before, some examples of certain applications of fractional calculations in different disciplines are especially emphasized in the references.

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