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Triangular ideal relative convergence on modular spaces and Korovkin theorems

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Abstract. In this paper, we introduce the concept of triangular ideal relative convergence for double sequences of functions defined on a modular space. Based upon this new convergence method, we prove Korovkin theorems. Then, we construct an example such that our new approximation results work. Finally, we discuss the reduced results which are obtained by special choices.

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1. Introduction

Let e_r denote the continuous real functions on $[a,b]$ defined by $e_r(s) = s^r$, $r = 0,1,2$. The Korovkin theorem establishes the uniform convergence in the space $C[a,b]$ for a sequence of positive linear operators $\{L_i\}$ on $C[a,b]$ via the convergence only on the test functions e_r where $C[a,b]$ is the space of all continuous real functions defined on the interval $[a,b]$ ([21]). A more general framework for the Korovkin theorems can be obtained by using different convergence methods. Gadjiev and Orhan [18] developed these theorems by considering statistical convergence ([17], [31]) instead of ordinary convergence in 2002. After these developments Demirci and Dirik [15] have carried this convergence for double sequences of positive linear operators.

The concept of relative uniform convergence given by Moore [25] in 1910, was later investigated in detail by Chittenden [8]. In consideration of these studies, statistical relative convergence for single sequences was defined by Demirci and Orhan [13] and recently this convergence was given for double sequences by Şahin and Dirik [32]

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(see also [14]). Also, Korovkin theorem has been studied on various function spaces via different convergence methods ([5], [7], [11], [16]). Several forms of Korovkin theorems have been examined in modular spaces including as particular case the L_p spaces, Orlicz and Musielak-Orlicz spaces ([12], [13], [14], [20], [30], [34]).

Recently, Bardaro et al. introduced the triangular $A-$statistical convergence which cannot be compared with statistical convergence ([1], [2]) and then, with the help of this definition, triangular $A-$statistical relative uniform convergence has been defined in [9].

Kostyrko et al. [22] presented the definition of ideal convergence which is a more overall method than statistical convergence and it is based on the notion of the ideal I of subsets of the set $\mathbb{N}$, the natural numbers.

In the present paper, we introduce a new form of convergence for double sequence, called triangular ideal relative modular convergence. We will compare this new convergence with triangular statistical modular convergence and obtain more general results.

We now recall some definitions and notations on modular space.

Let $S=[a,b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, we will denote by $X\left(S^2\right)$ the space of all real-valued measurable functions on $S^2=[a,b]\times[a,b]$ provided with equality *a.e.*. A functional

$$\rho : X\left(S^2\right)\rightarrow[0,+\infty]$$

is called a modular on $X\left(S^2\right)$ provided that below conditions hold:

(i) $\rho(h)=0$ if and only if $h=0$ *a.e.* in S^2,

(ii) $\rho(-h)=\rho(h)$ for every $h\in X\left(S^2\right)$,

(iii) $\rho(\alpha h+\beta g)\leq\rho(h)+\rho(g)$ for every $h,g\in X(S^2)$ and for any $\alpha,\beta\geq 0$ with $\alpha+\beta=1$.

A modular ρ is called $N-$quasi convex if there exists a constant $N\geq 1$ such that $\rho(\alpha h+\beta g)\leq N\alpha\rho(Nh)+N\beta\rho(Ng)$ holds for every $h,g\in X\left(S^2\right)$, $\alpha,\beta\geq 0$ with $\alpha+\beta=1$. In particular, if $N=1$, then ρ is said to be convex. A modular ρ is called $N-$quasi semiconvex if there exists a constant $N\geq 1$ such that $\rho(ah)\leq Na\rho(Nh)$ holds for every $h\in X\left(S^2\right)$ and $a\in(0,1]$. Note that if $\beta=0$, then every $N-$quasi convex modular is $N-$quasi semiconvex (see for details, [5, 6]).

Now, we recall vector subspaces of $X(S^2)$ defined via a modular functional by:

The modular spaces $L^\rho\left(S^2\right)$ generated by ρ is defined by

$$L^\rho\left(S^2\right):=\left\{h\in X(S^2):\lim_{\lambda\rightarrow 0^+}\rho(\lambda h)=0\right\},$$

and the space of the finite elements of $L^\rho\left(S^2\right)$ is given by

$$E^\rho\left(S^2\right):=\left\{h\in L^\rho\left(S^2\right):\rho(\lambda h)<+\infty\ \text{ for all } \lambda>0\right\}.$$

Recognize that if ρ is $N-$quasi semiconvex, then the space

$$\left\{h\in X\left(S^2\right):\rho(\lambda h)<+\infty\ \text{ for some } \lambda>0\right\}$$

coincides with $L^\rho\left(S^2\right)$. The notions about modulars are introduced in [27] and developed in [6] (see also [23, 28]).

Bardaro and Mantellini [4] introduced some Korovkin theorems through the notions of modular convergence and strong convergence. Afterwards Karakuş et al. [20] investigated the modular Korovkin theorem via statistical convergence and then, Orhan and Demirci [30] extended these type of approximation for double sequences of positive linear operators on modular space. In [14], Demirci and Orhan presented the notion of statistical relative modular (or strong) convergence for double sequences.

Let's first express the concept of statistical convergence given for double sequences by Moricz in [27].

Let $A \subseteq \mathbb{N}^2$ be a two-dimensional subset of positive integers, then A_{ij} denotes the set $\{(m,n) \in A : m \leq i,\ n \leq j\}$ and $|A_{ij}|$ denotes the cardinality of A_{ij}. The double natural density of A is given by

$$\delta_2(A) := P - \lim_{i,j} \frac{1}{ij} |A_{ij}|,$$

if it exists. The number sequence $x = \{x_{i,j}\}$ is said to be statistically convergent to l provided that for every $\varepsilon > 0$, the set

$$A_{m,n}(\varepsilon) := \{m \leq i,\ n \leq j : |x_{i,j} - l| \geq \varepsilon \}$$

has natural density zero; in that case, we write $st_2 - \lim_{i,j} x_{i,j} = l$ (see [27]).

Now we recall the above mentioned convergence methods on modular spaces:

Definition 1.1. [14] Let $\{h_{i,j}\}$ be a double function sequence whose terms belong to $L^\rho(S^2)$. Then, $\{h_{i,j}\}$ is statistically relatively modularly convergent to a function $h \in L^\rho(S^2)$ if there exists a function σ, called a scale function $\sigma \in X(S^2)$, $|\sigma(s,t)| \neq 0$ such that

$$st_2 - \lim_{i,j} \rho\left(\lambda_0 \left(\frac{h_{i,j} - h}{\sigma}\right)\right) = 0, \quad \text{for some } \lambda_0 > 0. \tag{1.1}$$

Also, $\{h_{i,j}\}$ is statistically relatively $F-$norm convergent (or, statistically relatively strongly convergent) to h if

$$st_2 - \lim_{i,j} \rho\left(\lambda \left(\frac{h_{i,j} - h}{\sigma}\right)\right) = 0, \quad \text{for every } \lambda > 0. \tag{1.2}$$

It is known from [14] that (1.1) and (1.2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e., there exists a constant $M > 0$ such that $\rho(2h) \leq M\rho(h)$ for every $h \in X(S^2)$.

2. Triangular ideal relative modular convergence

In this section, we introduce the notion of the triangular ideal relative modular (or strong) convergence for double sequences. Let us first recall the notion of ideal convergence and some of its main features that are required for this article.

If K is a non-empty set, a class I of subsets of K is called an ideal in K if

i) $\varnothing \in I$,

ii) $A, B \in I$ implies $A \cup B \in I$,

iii) for each $A \in I$ and $B \subset A$ we have $B \in I$ ([22]).

The ideal I is called non-trivial if $I \neq \{\varnothing\}$ and $K \notin I$. A non-trivial ideal I is called admissible if $\{x\} \in I$ for each $x \in K$.

A sequence $\{x_i\}$ is said to $I-$convergent to l if for any $\varepsilon > 0$,

$$A(\varepsilon) = \{i \in \mathbb{N} : \ |x_i - l| \geq \varepsilon\} \in I.$$

We write $I - \lim_i x_i = l$ ([22]).

Now, we introduce the following ideal type convergence.

Definition 2.1. The double sequence $x = \{x_{i,j}\}$ is triangular ideal convergent to l provided that for every $\varepsilon > 0$ the set

$$B_i(\varepsilon) := \{j \in \mathbb{N} : \ j \leq i, \ |x_{i,j} - l| \geq \varepsilon\} \in I.$$

We set $I^T - \lim_i x_{i,j} = l$.

It is worthwhile to point out that, the triangular density defined $\mathbb{N}$ [1] as follows.

Let $B \subset \mathbb{N}^2$ be a nonempty set, and for every $i \in \mathbb{N}$, let $B_i = \{j \in \mathbb{N} : j \leq i\}$. Let $|B_i|$ be the cardinality of B_i. The triangular density of B is defined by

$$\delta^T(B) = \lim_i \frac{1}{i} |B_i|$$

provided that the limit on the right-hand side exists in $\mathbb{R}$.

Let $I_\delta^T = \{B : \ \delta^T(B) = 0\}$. I_δ^T is a non-trivial admissible ideal in $\mathbb{N}$ then I_δ^T-convergence coincides with the triangular statistical convergence in [1], [2]. Also, it is clear that $I_\delta^T \subset I$.

Similar to [10], the triangular ideal limit superior and inferior can be define. Given a double sequence $x = \{x_{i,j}\}$, put

$$A_x := \{a \in \mathbb{R} : \{j \in \mathbb{N} : j \leq i, \ x_{i,j} < a\} \notin I\},$$
$$C_x := \{c \in \mathbb{R} : \{j \in \mathbb{N} : j \leq i, \ x_{i,j} > c\} \notin I\},$$

and define

$$I^T - \limsup_i \ x_{i,j} = \begin{cases} \sup C_x, & \text{if } C_x \neq \varnothing, \\ -\infty, & \text{if } C_x = \varnothing, \end{cases}$$

$$I^T - \liminf_i \ x_{i,j} = \begin{cases} \inf A_x, & \text{if } A_x \neq \varnothing, \\ +\infty, & \text{if } A_x = \varnothing. \end{cases}$$

We also have the following theorem from [10]:

Theorem 2.2. *i)If $\beta = I^T - \limsup_i x_{i,j}$ is finite, then for every positive number ε*

$$\{j : \ j \leq i, \ x_{i,j} > \beta - \varepsilon\} \notin I \ \text{and} \ \{j : \ j \leq i, \ x_{i,j} > \beta + \varepsilon\} \in I. \tag{2.1}$$

Conversely, if (2.1) holds for every positive ε, then $\beta = I^T - \limsup_i x_{i,j}$.

ii) If $\alpha = I^T - \liminf_i x_{i,j}$ is finite, then for every positive number ε

$$\{j : \ j \leq i, \ x_{i,j} < \alpha + \varepsilon\} \notin I \ \text{and} \ \{j : \ j \leq i, \ x_{i,j} < \alpha - \varepsilon\} \in I. \tag{2.2}$$

Conversely, if (2.2) holds for every positive ε, then $\alpha = I^T - \liminf_i x_{i,j}$.

We can now introduce our new convergence methods:

Definition 2.3. Let $\{h_{i,j}\}$ be a double function sequence whose terms belong to $L^{\rho}\left(S^2\right)$. Then, $\{h_{i,j}\}$ is triangular ideal relatively modularly convergent to a function $h \in L^{\rho}\left(S^2\right)$ if there exists a scale function σ such that

$$I^T - \lim_i \rho\left(\lambda_0\left(\frac{h_{i,j}-h}{\sigma}\right)\right) = 0, \text{ for some } \lambda_0 > 0. \tag{2.3}$$

And also, $\{h_{i,j}\}$ is triangular ideal relatively modularly strongly convergent (or, triangular ideal relatively $F-$ norm convergent) to a function $h \in L^{\rho}\left(S^2\right)$ if

$$I^T - \lim_i \rho\left(\lambda\left(\frac{h_{i,j}-h}{\sigma}\right)\right) = 0, \text{ for every } \lambda > 0. \tag{2.4}$$

It is worthwhile to point out that (2.3) and (2.4) are equivalent if and only if the modular ρ satisfies the Δ_2-condition.

Below we present an interesting example of a double sequence which is triangular ideal relatively modularly convergent but not triangular statistically modularly convergent.

Example 2.4. Take $S = [0,1]$ and let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous function for which the following conditions hold:

- *φ is convex,*
- *$\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$.*

Let be functional ρ^{φ} on $X(S^2)$ defined by

$$\rho^{\varphi}(h) := \int_0^1 \int_0^1 \varphi(|h(s,t)|)\, dsdt \text{ for } h \in X\left(S^2\right). \tag{2.5}$$

Then, ρ^{φ} is a convex modular on $X\left(S^2\right)$, which satisfies all the assumptions stated previous section. Let us consider the Orlicz space generated by φ as follows:

$$L^{\rho}_{\varphi}(S^2) := \left\{h \in X\left(S^2\right) : \rho^{\varphi}(\lambda h) < +\infty \quad \text{for some } \lambda > 0\right\}.$$

Let $I = I^T_{\delta}$ and $B := \{(i,j) : \ j \le i\}$ be a infinite set. For each $(i,j) \in \mathbb{N}^2$ define $g_{i,j} : [0,1] \times [0,1] \to \mathbb{R}$ by

$$g_{i,j}(s,t) = \begin{cases} 1, & i \text{ and } j \text{ are square,} \\ i^3 j^3 st, & \begin{array}{c}(i,j) \in B,\ i \text{ and } j \text{ are not square,} \\ (s,t) \in \left(0, \frac{1}{i}\right) \times \left(0, \frac{1}{j}\right)\end{array} \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

If $\varphi(x) = x^p$ for $1 \le p < \infty$, $x \ge 0$, then $L^{\rho}_{\varphi}\left(S^2\right) = L_p\left(S^2\right)$. Moreover, we have for any function $h \in L^{\rho}_{\varphi}\left(S^2\right)$

$$\rho^{\varphi}(h) = \|h\|^p_{L_p}.$$

We can verify that $\{g_{i,j}\}$ does not converge triangular statistically modularly however converges to $g = 0$ triangular statistically modularly relatively to the scale function

$$\sigma(s,t) = \begin{cases} \frac{1}{s^2t^2}, & \text{if } (s,t) \in (0,1] \times (0,1], \\ 1, & \text{otherwise,} \end{cases}$$

on $L_1(S)$. Indeed, for some $\lambda_0 > 0$, when we take $p = 1$, we have $\rho^{\varphi}(.) = \|.\|_{L_1}$,

$$\begin{aligned} \rho(\lambda_0(g_{i,j} - g)) &= \|\lambda_0(g_{i,j} - g)\|_{L_1} \\ &= \lambda_0 \begin{cases} 1, & i \text{ and } j \text{ are square}, \\ \frac{ij}{4}, & (i,j) \in B \ i \text{ and } j \text{ are not square}, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{2.7}$$

For every $\varepsilon \in \left(0, \frac{1}{9}\right]$, we have

$$\lim_i \frac{1}{i} \left|\{j \in \mathbb{N} : \ j \leq i, \ \rho(\lambda_0(g_{i,j} - g)) \geq \varepsilon\}\right| = 1$$

Clearly, $\{j \in \mathbb{N} : \ j \leq i, \ \rho(\lambda_0(g_{i,j} - g)) \geq \varepsilon\} \notin I_\delta^T$. So, $\{g_{i,j}\}$ does not converge triangular statistically modularly to $g = 0$ (see details, [2]). Using the scale function σ,

$$\rho\left(\lambda_0\left(\frac{g_{i,j} - g}{\sigma}\right)\right) = \lambda_0 \begin{cases} \frac{1}{9}, & i \text{ and } j \text{ are square}, \\ \frac{1}{16ij}, & (i,j) \in B \ i \text{ and } j \text{ are not square}, \\ 0, & \text{otherwise}. \end{cases}$$

for every $\varepsilon \in \left(0, \frac{1}{9}\right]$, and since

$$\lim_i \frac{1}{i} \left|\left\{j \in \mathbb{N} : \ j \leq i, \ \rho\left(\lambda_0\left(\frac{g_{i,j} - g}{\sigma}\right)\right) \geq \varepsilon\right\}\right| = 0,$$

then we get,

$$I_\delta^T - \lim_i \rho\left(\lambda_0\left(\frac{g_{i,j} - g}{\sigma}\right)\right) = 0.$$

Prior to expressing the next theorem, we will need below assumptions on a modular ρ :

(a) ρ is monotone, i.e., $\rho(h) \leq \rho(g)$ whenever $|h(s,t)| \leq |g(s,t)|$ for any $(s,t) \in S^2$ and $h, g \in X\left(S^2\right)$. Further, ρ is finite if the characteristic function $\chi_B \in L^\rho\left(S^2\right)$ whenever B is measurable subset of S^2.

(b) ρ is absolutely finite i.e., ρ is finite and for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda \chi_B) < \varepsilon$ for any measurable subset $B \subset S^2$ with $\mu(B) < \delta$. Also, we say that ρ is strongly finite, i.e., $\chi_{S^2} \in E^\rho\left(S^2\right)$.

(c) ρ is absolutely continuous, i.e. there exists $\alpha > 0$ such that for every h in $X\left(S^2\right)$, with $\rho(h) < +\infty$, the following condition holds: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(\alpha h \chi_B) < \varepsilon$ for any measurable subset $B \subset S^2$ with $\mu(B) < \delta$.

As usual, let $C\left(S^2\right)$ be the space of all continuous real-valued functions, and $C^\infty\left(S^2\right)$ be the space of all infinitely differentiable functions on S^2. Based upon the above concepts (see [4, 5]) if a modular ρ is monotone and finite, then we have $C(S^2) \subset L^\rho\left(S^2\right)$. Similarly, if ρ is monotone and strongly finite, then $C(S^2) \subset E^\rho\left(S^2\right)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^\infty(S^2)} = L^\rho\left(S^2\right)$. (For more details see [3, 6, 24, 28]).

Here and in the sequel, we use I as a non-trivial admissible ideal on $\mathbb{N}$.

3. Korovkin theorems

In this section, we apply our definition of triangular ideal relative modular convergence for double sequences of positive linear operators to prove the Korovkin type approximation theorems.

Let ρ be a monotone and finite modular on $X\left(S^2\right)$. Assume that D is a set satisfying $C^{\infty}\left(S^2\right) \subset D \subset L^{\rho}\left(S^2\right)$. Assume further that $\mathbb{L} := \{L_{i,j}\}$ is a sequence of positive linear operators from D into $X\left(S^2\right)$ for which there exists a subset $X_{\mathbb{L}} \subset D$ containing $C^{\infty}\left(S^2\right)$ and $\sigma \in X\left(S^2\right)$ is an unbounded function satisfying $|\sigma(s,t)| \neq 0$ such that

$$I^T - \limsup_i \rho\left(\lambda\left(\frac{L_{i,j}(h)}{\sigma}\right)\right) \leq R\rho(\lambda h) \tag{3.1}$$

holds for every $h \in X_{\mathbb{L}}$, $\lambda > 0$ and for an absolutely positive constant R.
Let $\mathbb{L}$ be a linear operator from $C\left(S^2\right)$ into itself . It is called positive, if $L_{i,j}(h) \geq 0$, for all $h \geq 0$. Also, we denote the value of $L_{i,j}(h)$ at a point $(s,t) \in S^2$ by $L_{i,j}(h; s,t)$.

Now we have the following Korovkin theorem for triangular ideal relative modular convergence that is our main theorem.

Theorem 3.1. *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular on $X\left(S^2\right)$. Let $\mathbb{L} := \{L_{i,j}\}$ be a double sequence of positive linear operators from D into $X\left(S^2\right)$ satisfying (3.1) and suppose that σ_r is an unbounded function satisfying $|\sigma_r(s,t)| \geq \alpha_r > 0$ $(r = 0,1,2,3)$. Assume that*

$$I^T - \lim_i \rho\left(\lambda\left(\frac{L_{i,j}(e_r) - e_r}{\sigma_r}\right)\right) = 0, \text{ for every } \lambda > 0 \text{ and } r = 0,1,2,3, \tag{3.2}$$

where $e_0(s,t) = 1$, $e_1(s,t) = s$, $e_2(s,t) = t$, $e_3(s,t) = s^2 + t^2$. Now let h be any function belonging to $L^{\rho}\left(S^2\right)$ such that $h - g \in X_{\mathbb{L}}$ for every $g \in C^{\infty}\left(S^2\right)$. Then, we have

$$I^T - \lim_i \rho\left(\lambda_0\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) = 0, \text{ for some } \lambda_0 > 0 \tag{3.3}$$

where $\sigma(s,t) = \max\{|\sigma_r(s,t)| : r = 0,1,2,3\}$.

Proof. We first claim that

$$I^T - \lim_i \rho\left(\eta\left(\frac{L_{i,j}(g) - g}{\sigma}\right)\right) = 0 \text{ for every } g \in C\left(S^2\right) \cap D \text{ and every } \eta > 0. \tag{3.4}$$

To see this, assume that g belongs to $g \in C\left(S^2\right) \cap D$. By the continuity of g on S^2, given $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all $(u,v), (s,t) \in S^2$ satisfying $|u - s| < \delta$ and $|v - t| < \delta$ we have

$$|g(u,v) - g(s,t)| < \varepsilon. \tag{3.5}$$

Also we obtain for all $(u,v), (s,t) \in S^2$ satisfying $|u - s| > \delta$ and $|v - t| > \delta$ that

$$|g(u,v) - g(s,t)| \leq \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\} \tag{3.6}$$

where $M := \sup_{(s,t)\in S^2} |g(s,t)|$. Combining (3.5) and (3.6) we have for $(u,v), (s,t) \in S^2$ that

$$|g(u,v) - g(s,t)| < \varepsilon + \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\}.$$

Namely,

$$\begin{aligned} & -\varepsilon - \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\} \\ < \quad & g(u,v) - g(s,t) < \varepsilon + \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\}. \end{aligned} \tag{3.7}$$

Since $L_{i,j}$ is linear and positive, by applying $L_{i,j}$ to (3.7) for every $i,j \in \mathbb{N}$ we get

$$\begin{aligned} & -\varepsilon L_{i,j}(e_0;s,t) - \frac{2M}{\delta^2} L_{i,j}\left((u-s)^2 + (v-t)^2; s,t\right) \\ < \quad & L_{i,j}(g;s,t) - g(s,t) L_{i,j}(e_0;s,t) \\ < \quad & \varepsilon L_{i,j}(e_0;s,t) + \frac{2M}{\delta^2} L_{i,j}\left((u-s)^2 + (v-t)^2; s,t\right) \end{aligned}$$

and hence,

$$\begin{aligned} |L_{i,j}(g;s,t) - g(s,t)| \quad \leq \quad & |L_{i,j}(g;s,t) - g(s,t) L_{i,j}(e_0;s,t)| \\ & + |g(s,t) L_{i,j}(e_0;s,t) - g(s,t)| \\ \leq \quad & \varepsilon L_{i,j}(e_0;s,t) + M |L_{i,j}(e_0;s,t) - (e_0;s,t)| \\ & + \frac{2M}{\delta^2} L_{i,j}\left((u-s)^2 + (v-t)^2; s,t\right) \end{aligned}$$

holds for every $s,t \in S$ and $i,j \in \mathbb{N}$. The above inequality implies that

$$\begin{aligned} |L_{i,j}(g;s,t) - g(s,t)| \quad \leq \quad & \varepsilon + \left\{\varepsilon + M + \frac{4M}{\delta^2} E^2\right\} |L_{i,j}(e_0;s,t) - (e_0;s,t)| \\ & + \frac{4M}{\delta^2} E |L_{i,j}(e_1;s,t) - (e_1;s,t)| \\ & + \frac{4M}{\delta^2} E |L_{i,j}(e_2;s,t) - (e_2;s,t)| \\ & + \frac{2M}{\delta^2} E |L_{i,j}(e_3;s,t) - (e_3;s,t)| \end{aligned}$$

where $E := \max\{|t| : \; t \in S\}$. Now, we multiply the both-sides of the above inequality by $\frac{1}{|\sigma(s,t)|}$ and for every $\eta > 0$, the last inequality gives that:

$$\begin{aligned} \eta \left| \frac{L_{i,j}(g;s,t) - g(s,t)}{\sigma(s,t)} \right| \quad \leq \quad & \frac{\eta\varepsilon}{|\sigma(s,t)|} + K\eta \left\{ \left| \frac{L_{i,j}(e_0;s,t) - (e_0;s,t)}{\sigma(s,t)} \right| \right. \\ & + \left| \frac{L_{i,j}(e_1;s,t) - (e_1;s,t)}{\sigma(s,t)} \right| \\ & + \left| \frac{L_{i,j}(e_2;s,t) - (e_2;s,t)}{\sigma(s,t)} \right| \\ & \left. + \left| \frac{L_{i,j}(e_3;s,t) - (e_3;s,t)}{\sigma(s,t)} \right| \right\}, \end{aligned}$$

where

$$K := \max\left\{\varepsilon + M + \frac{4M}{\delta^2}E^2, \frac{4M}{\delta^2}E, \frac{2M}{\delta^2}\right\}.$$

Now, applying the modular ρ to both-sides of the above inequality, since ρ is monotone and

$$\sigma(s,t) = \max\{|\sigma_r(s,t)| \ ; r = 0,1,2,3\},$$

we have

$$\begin{aligned}\rho\left(\eta\left(\frac{L_{i,j}(g)-g}{\sigma}\right)\right) &\leq \rho\left(\eta\frac{\varepsilon}{|\sigma|} + \eta K\left|\frac{L_{i,j}(e_0)-e_0}{\sigma_0}\right| + \eta K\left|\frac{L_{i,j}(e_1)-e_1}{\sigma_1}\right|\right. \\ &\quad \left. + \eta K\left|\frac{L_{i,j}(e_2)-e_2}{\sigma_2}\right| + \eta K\left|\frac{L_{i,j}(e_3)-e_3}{\sigma_3}\right|\right).\end{aligned}$$

Since ρ is a $N-$quasi semiconvex and strongly finite, also assuming $0 < \varepsilon \leq 1$, we can write

$$\begin{aligned}\rho\left(\eta\left(\frac{L_{i,j}(g)-g}{\sigma}\right)\right) &\leq N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right) + \rho\left(5\eta K\left(\frac{L_{i,j}(e_0)-e_0}{\sigma_0}\right)\right) \\ &\quad + \rho\left(5\eta K\left(\frac{L_{i,j}(e_1)-e_1}{\sigma_1}\right)\right) \\ &\quad + \rho\left(5\eta K\left(\frac{L_{i,j}(e_2)-e_2}{\sigma_2}\right)\right) \\ &\quad + \rho\left(5\eta K\left(\frac{L_{i,j}(e_3)-e_3}{\sigma_3}\right)\right).\end{aligned}$$

For a given $t > 0$, choose an $\varepsilon \in (0,1]$ such that $N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right) < t$. Let's define the following sets:

$$\begin{aligned}D_\eta &: = \left\{j \in \mathbb{N} \ : \ j \leq i, \ \rho\left(\eta\left(\frac{L_{i,j}(g)-g}{\sigma}\right)\right) > t\right\}, \\ D_{\eta,r} &: = \left\{j \in \mathbb{N} \ : \ j \leq i, \ \rho\left(\eta\left(\frac{L_{i,j}(e_r)-e_r}{\sigma_r}\right)\right) > \frac{t - N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)}{4}\right\},\end{aligned}$$

where $r = 0,1,2,3$. It is a simple matter to see that $D_\eta \subset \bigcup_{r=0}^{3} D_{\eta,r}$. So, by (3.2) we have $D_{\eta,r} \in I$ for $r = 0,1,2,3$. Hence, by definition of an ideal $\bigcup_{r=0}^{3} D_{\eta,r} \in I$, $D_\eta \in I$. So we get $I^T - \lim_i \rho\left(\eta\left(\frac{L_{i,j}(g)-g}{\sigma}\right)\right) = 0$ which proves our claim (3.4). Obviously (3.4) also holds for every $g \in C^\infty(S^2)$. Let $h \in L^\rho(S^2)$ satisfying $h - g \in X_T$ for every $g \in C^\infty(S^2)$. Since $\mu(S^2) < \infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X(S^2)$. Using these properties of the modular ρ, it is known from [6, 24] that the space $C^\infty(S^2)$ is modular dense in $L^\rho(S^2)$, i.e., there exists a sequence $\{g_{i,j}\} \subset C^\infty(S^2)$ such that

$$P - \lim_{i,j} \rho(3\lambda_0^*(g_{i,j} - h)) = 0 \text{ for some } \lambda^* > 0.$$

This means that, for every $\epsilon > 0$, there is positive number $k_0 = k_0(\varepsilon)$ such that

$$\rho\left(3\lambda_0^*\left(g_{i,j} - h\right)\right) < \varepsilon \text{ for every } i, j \geq k_0. \tag{3.8}$$

Otherwise, by the linearity and positivity of the operators $L_{i,j}$ we can write that

$$\begin{aligned}\lambda_0^*\left|L_{i,j}(h; s,t) - h(s,t)\right| \leq\ & \lambda_0^*\left|L_{i,j}\left(h - g_{k_0,k_0}; s,t\right)\right| \\ & +\lambda_0^*\left|L_{i,j}\left(g_{k_0,k_0}; s,t\right) - g_{k_0,k_0}(s,t)\right| \\ & +\lambda_0^*\left|g_{k_0,k_0}(s,t) - h(s,t)\right|\end{aligned}$$

holds for every $s, t \in S$ and $i, j \in \mathbb{N}$. Applying the modular ρ in the last enequality and using the monotonicity of ρ and moreover multiplying the both-sides of above inequality by $\frac{1}{|\sigma(s,t)|}$, the last inequality leads to

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) \leq\ & \rho\left(3\lambda_0^*\frac{L_{i,j}\left(h - g_{k_0,k_0}\right)}{\sigma}\right) \\ & +\rho\left(3\lambda_0^*\left(\frac{L_{i,j}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right) \\ & +\rho\left(3\lambda_0^*\left(\frac{g_{k_0,k_0} - h}{\sigma}\right)\right).\end{aligned}$$

Hence, observing that $|\sigma| \geq \alpha > 0$ ($\alpha = \max\{\alpha_r : r = 0,1,2,3\}$) we can write

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) \leq\ & \rho\left(3\lambda_0^*\frac{L_{i,j}\left(h - g_{k_0,k_0}\right)}{\sigma}\right) \\ & +\rho\left(3\lambda_0^*\left(\frac{L_{i,j}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right) \\ & +\rho\left(\frac{3\lambda_0^*}{\alpha}\left(g_{k_0,k_0} - h\right)\right).\end{aligned} \tag{3.9}$$

Then, it follows from (3.8) and (3.9) that

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) \leq\ & \varepsilon + \rho\left(3\lambda_0^*\frac{L_{i,j}\left(h - g_{k_0,k_0}\right)}{\sigma}\right) \\ & +\rho\left(3\lambda_0^*\left(\frac{L_{i,j}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right).\end{aligned} \tag{3.10}$$

So, taking triangular ideal limit superior as $i \to \infty$ in the both-sides of (3.10) and also using the facts that $g_{k_0,k_0} \in C^\infty\left(S^2\right)$ and $h - g_{k_0,k_0} \in X_T$, we get from (3.1) that

$$\begin{aligned}I^T - \limsup_i \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) \leq\ & \varepsilon + R\rho\left(3\lambda_0^*\left(h - g_{k_0,k_0}\right)\right) \\ & + I^T - \limsup_i \rho\left(3\lambda_0^*\left(\frac{L_{i,j}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right)\end{aligned}$$

which gives

$$\begin{aligned} & I^T - \limsup_i \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h)-h}{\sigma}\right)\right) \\ \leq\ & \varepsilon(R+1) + I^T - \limsup_i \rho\left(3\lambda_0^*\left(\frac{L_{i,j}(g_{k_0,k_0})-g_{k_0,k_0}}{\sigma}\right)\right). \end{aligned} \tag{3.11}$$

By (3.4), we get

$$I^T - \limsup_i \rho\left(3\lambda_0^*\left(\frac{L_{i,j}(g_{k_0,k_0})-g_{k_0,k_0}}{\sigma}\right)\right) = 0. \tag{3.12}$$

Combining (3.11) with (3.12), from Theorem 2.2 we conclude that

$$I^T - \limsup_i \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h)-h}{\sigma}\right)\right) \leq \varepsilon(R+1).$$

Since $\varepsilon > 0$ is arbitrary, we find

$$I^T - \lim_i \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h)-h}{\sigma}\right)\right) = 0.$$

Thus, the assertion follows. □

Now, we give an example that shows that our triangular ideal relative modular Korovkin theorem is stronger than the Korovkin theorem in [2].

Example 3.2. Take $S = [0,1]$ and $I = I_\delta^T$. Also, φ, σ, ρ^φ, $L_\varphi^\rho\left(S^2\right)$ and B be as in Example 2.4. Then consider the following bivariate Bernstein-Kantorovich operator $\mathbb{U} := \{U_{i,j}\}$ on the space $L_\varphi^\rho\left(S^2\right)$ which is defined by:

$$\begin{aligned} U_{i,j}(h;s,t) &= \sum_{m=0}^{i}\sum_{n=0}^{j} p_{m,n}^{(i,j)}(s,t)(i+1)(j+1) \\ &\quad \times \int_{m/(i+1)}^{(m+1)/(i+1)} \int_{n/(j+1)}^{(n+1)/(j+1)} h(t,s)\,ds dt \end{aligned} \tag{3.13}$$

for $s,t \in S$, where $p_{m,n}^{(i,j)}(s,t)$ defined by

$$p_{m,n}^{(i,j)}(s,t) = \binom{i}{m}\binom{j}{n} s^m t^n (1-s)^{i-m}(1-t)^{j-n}.$$

Also it is clear that,

$$\sum_{m=0}^{i}\sum_{n=0}^{j} p_{m,n}^{(i,j)}(s,t) = 1. \tag{3.14}$$

Observe that the operators $U_{i,j}$ maps $L_{\varphi}^{\rho}\left(S^2\right)$ into itself. In view of (3.14), as in the proof of Lemma 5.1 [4] and also similar to Example 1 [30], we can use the Jensen inequality in order to obtain that for every $h \in L_{\varphi}^{\rho}\left(S^2\right)$ and $i,j \in \mathbb{N}$ there is an absolute constant $M > 0$ such that

$$\rho^{\varphi}\left(U_{i,j}(h)\right) \leq M\rho^{\varphi}\left(h\right).$$

It is worthwhile to point out that, for any function $h \in L_{\varphi}^{\rho}\left(S^2\right)$ such that $h - g \in X_{\mathbb{L}}$ for every $g \in C^{\infty}\left(S^2\right)$, $\{U_{i,j}\}$ is modularly convergent to h. If $\varphi\left(x\right) = x^p$ for $1 \leq p < \infty$, $x \geq 0$, then $L_{\varphi}^{\rho}\left(S^2\right) = L_p\left(S^2\right)$. Moreover we have $\rho^{\varphi}\left(.\right) = \left\|.\right\|_{L_p}^{p}$. For $p = 1$, we have $\rho^{\varphi}\left(.\right) = \left\|.\right\|_{L_1}$. In what follows, using the operators $U_{i,j}$, we can obtain the sequence of positive operators $\mathbb{V} := \{V_{i,j}\}$ on $L_1\left(S^2\right)$ as follows:

$$\begin{aligned} V_{i,j}\left(h;s,t\right) &= \left(1+g_{i,j}\left(s,t\right)\right)U_{i,j}\left(h;s,t\right) \\ \text{for } h &\in L_1\left(S^2\right), \ \left(s,t\right) \in S^2 \text{ and } i,j \in \mathbb{N} \end{aligned} \tag{3.15}$$

where $\{g_{i,j}\}$ is the same as in (2.6) and we choose $\sigma_r = \sigma$ $\left(r = 0,1,2,3\right)$, where

$$\sigma\left(s,t\right) = \begin{cases} \frac{1}{s^2t^2}, & \text{if } \left(s,t\right) \in (0,1] \times (0,1], \\ 1, & \text{otherwise.} \end{cases}$$

As in the proof of Lemma 5.1 [4] and similar to Example 1 [30], we get, for every $h \in L_1\left(S^2\right)$, $\lambda > 0$ and for positive constant C, that

$$I_{\delta}^{T} - \limsup_{i} \left\|\lambda\left(\frac{V_{i,j}\left(h\right)}{\sigma}\right)\right\|_{L_1} \leq C\left\|\lambda h\right\|_{L_1}. \tag{3.16}$$

We now claim that

$$I_{\delta}^{T} - \lim_{i} \left\|\lambda\left(\frac{V_{i,j}\left(e_r\right) - e_r}{\sigma}\right)\right\|_{L_1} = 0, \ r = 0,1,2,3. \tag{3.17}$$

Indeed, first observe that,

$$\begin{aligned} V_{i,j}\left(e_0;s,t\right) &= 1 + g_{i,j}\left(s,t\right), \\ V_{i,j}\left(e_1;s,t\right) &= \left(1+g_{i,j}\left(s,t\right)\right)\left(\frac{is}{i+1} + \frac{1}{2\left(i+1\right)}\right), \\ V_{i,j}\left(e_2;s,t\right) &= \left(1+g_{i,j}\left(s,t\right)\right)\left(\frac{jt}{j+1} + \frac{1}{2\left(j+1\right)}\right), \\ V_{i,j}\left(e_3;s,t\right) &= \left(1+g_{i,j}\left(s,t\right)\right)\left(\frac{i\left(i-1\right)s^2}{\left(i+1\right)^2} + \frac{2is}{\left(i+1\right)^2} + \frac{1}{3\left(i+1\right)^2}\right. \\ &\quad \left. \frac{j\left(j-1\right)t^2}{\left(j+1\right)^2} + \frac{2jt}{\left(j+1\right)^2} + \frac{1}{3\left(j+1\right)^2}\right). \end{aligned}$$

We can easily calculate, for any $\lambda > 0$, that

$$\left\| \lambda \left(\frac{V_{i,j}(e_0) - e_0}{\sigma} \right) \right\|_{L_1} = \lambda \begin{cases} \frac{1}{9}, & \text{if } i \text{ and } j \text{ are square,} \\ \frac{1}{16ij}, & \text{if } (i,j) \in B \ i \text{ and } j \text{ are not square,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

Now, since

$$\lim_i \frac{1}{i} \left| \left\{ j \in \mathbb{N} : \ j \leq i, \ \left\| \lambda \left(\frac{V_{i,j}(e_0) - e_0}{\sigma} \right) \right\|_{L_1} \geq \varepsilon \right\} \right| = 0,$$

we get

$$I_\delta^T - \lim_i \left\| \lambda \left(\frac{V_{i,j}(e_0) - e_0}{\sigma} \right) \right\|_{L_1} = 0,$$

which guarantees that (3.17) holds true for $r = 0$.
Also, we have

$$\begin{aligned} \left\| \lambda \left(\frac{V_{i,j}(e_1) - e_1}{\sigma} \right) \right\|_{L_1} &= \lambda \int_0^1 \int_0^1 \left| \frac{V_{i,j}(e_1; s,t) - e_1(s,t)}{\sigma(s,t)} \right| dsdt \\ &\leq \lambda \int_0^1 \int_0^1 \left| \frac{g_{i,j}(s,t)}{\sigma(s,t)} \left(\frac{is}{i+1} + \frac{1}{2(i+1)} \right) \right| dsdt \\ &+ \lambda \int_0^1 \int_0^1 \left| \frac{s^2t^2 - 2s^3t^2}{2(i+1)} \right| dsdt \\ &< \left\| \lambda \frac{g_{i,j}}{\sigma} \right\|_{L_1} + \frac{\lambda}{36(i+1)}, \end{aligned}$$

because of

$$\left\{ j \in \mathbb{N} : \ j \leq i, \ \left\| \lambda \frac{g_{i,j}}{\sigma} \right\|_{L_1} \geq \varepsilon \right\} \in I_\delta^T$$

and

$$\lim_i \frac{1}{i} \left| \left\{ j \in \mathbb{N} : \ j \leq i, \ \frac{\lambda}{36(i+1)} \geq \varepsilon \right\} \right| = 0,$$

we get

$$I_\delta^T - \lim_i \left\| \lambda \left(\frac{V_{i,j}(e_1) - e_1}{\sigma} \right) \right\|_{L_1} = 0.$$

Hence (3.17) is valid for $r = 1$. Similarly, we have

$$I_\delta^T - \lim_i \left\| \lambda \left(\frac{V_{i,j}(e_2) - e_2}{\sigma} \right) \right\|_{L_1} = 0.$$

Finally, since

$$\left\|\lambda\left(\frac{V_{i,j}(e_3)-e_3}{\sigma}\right)\right\|_{L_1} = \lambda\int_0^1\int_0^1\left|\frac{V_{i,j}(e_3;s,t)-e_3(s,t)}{\sigma(s,t)}\right|dsdt$$

$$\leq \lambda\int_0^1\int_0^1\left|\frac{g_{i,j}(s,t)}{\sigma(s,t)}\left(\frac{i(i-1)s^2}{(i+1)^2}+\frac{2is}{(i+1)^2}\right.\right.$$

$$\left.\left.+\frac{1}{3(i+1)^2}+\frac{j(j-1)t^2}{(j+1)^2}+\frac{2jt}{(j+1)^2}+\frac{1}{3(j+1)^2}\right)\right|dsdt$$

$$+\lambda\int_0^1\int_0^1\left|\frac{(3i+1)s^4t^2}{(i+1)^2}+\frac{(3j+1)s^2t^4}{(j+1)^2}+\frac{2is^3t^2}{(i+1)^2}+\frac{2js^3t^2}{(j+1)^2}\right.$$

$$\left.+s^2t^2\left(\frac{1}{3(i+1)^2}+\frac{1}{3(j+1)^2}\right)\right|dsdt$$

$$< 6\left\|\lambda\frac{g_{i,j}}{\sigma}\right\|_{L_1}+\frac{\lambda(3i+1)}{15(i+1)^2}+\frac{\lambda(3j+1)}{15(j+1)^2}+\frac{\lambda i}{6(i+1)^2}+\frac{\lambda j}{6(j+1)^2}$$

$$+\frac{\lambda}{9}\left(\frac{1}{3(i+1)^2}+\frac{1}{3(j+1)^2}\right),$$

then we have

$$I_\delta^T-\lim_i\left\|\lambda\left(\frac{V_{i,j}(e_3)-e_3}{\sigma}\right)\right\|_{L_1}=0.$$

So, our claim (3.17) is valid for each $i=0,1,2,3$ and for any $\lambda>0$. Also, from (3.16) and (3.17), we observe that our sequence $\mathbb{V}=\{V_{i,j}\}$ defined by (3.15) satisfies all assumptions of Theorem 3.1 and

$$I_\delta^T-\lim_i\left\|\lambda\left(\frac{V_{i,j}(h)-h}{\sigma}\right)\right\|_{L_1}=0,$$

holds for any $h\in L_1\left(S^2\right)$ such that $h-g\in X_T=L_1\left(S^2\right)$ for every $g\in C^\infty\left(S^2\right)$. However, in view of (2.7), since

$$\lim_i\frac{1}{i}\left\{j\in\mathbb{N}:\ j\leq i,\ \left\|\lambda\left(V_{i,j}(e_0)-e_0\right)\right\|_{L_1}\ \geq\varepsilon\right\}=1,$$

$(V_{i,j}(e_0)-e_0)$ does not triangular statistically modularly convergent. The Korovkin theorem in [2], does not work for the sequence $\mathbb{V}=\{V_{i,j}\}$.

As indicated earlier, if the modular ρ satisfies the Δ_2-condition then the space $C^\infty\left(S^2\right)$ is dense in $L^\rho\left(S^2\right)$ ([4]). Hence, we get the following result from Theorem 3.1.

Theorem 3.3. *Let $\mathbb{L}:=\{L_{i,j}\}$, ρ and σ be the same as in Theorem 3.1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $I^T-\lim_i\rho\left(\lambda\left(\frac{L_{i,j}(e_r)-e_r}{\sigma_r}\right)\right)=0$, *for every* $\lambda>0$, $r=0,1,2,3$,

(b) $I^T - \lim_i \rho\left(\lambda\left(\frac{L_{i,j}(h)-h}{\sigma}\right)\right) = 0$, *for every* $\lambda > 0$, *provided that* h *is any fuction belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

If one replaces the scale function by nonzero constant, then the condition (3.1) reduces to

$$I^T - \limsup_i \rho\left(\lambda\left(L_{i,j}(h)\right)\right) \leq R\rho(\lambda h) \tag{3.19}$$

for every $h \in X_{\mathbb{L}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, the following results immediately follows from our Theorem 3.1 and Theorem 3.3.

Corollary 3.4. *Let* ρ *be a monotone, strongly finite, absolutely continuous and* $N-$*quasi semiconvex modular on* $X\left(S^2\right)$. *Let* $\mathbb{L} := \{L_{i,j}\}$ *be a double sequence of positive linear operators from* D *into* $X\left(S^2\right)$ *satisfying (3.19). If* $\{L_{i,j}(e_r)\}$ *is triangular ideal strongly convergent to* e_r *for each* $r = 0, 1, 2, 3$, *then* $\{L_{i,j}h\}$ *triangular ideal modularly convergent to* h *provided that* h *is any function belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

Corollary 3.5. $\mathbb{L} := \{L_{i,j}\}$ *and* ρ *be the same as in Corollary 3.4. If* ρ *satisfies the* Δ_2-*condition, then the following statements are equivalent:*

(a) $\{L_{i,j}(e_r)\}$ *is triangular ideal strongly convergent to* e_r *for each* $r = 0, 1, 2, 3$,

(b) $\{L_{i,j}(h)\}$ *is triangular ideal strongly convergent to* h *provided that* h *is any fuction belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

If we take $I = I_\delta^T$, then the condition (3.1) reduces to

$$st^T - \limsup_i \rho\left(\lambda\left(\frac{L_{i,j}(h)}{\sigma}\right)\right) \leq R\rho(\lambda h) \tag{3.20}$$

for every $h \in X_{\mathbb{L}}$, $\lambda > 0$ and for an absolute positive constant R. In this case the following results immediately follows from our Theorem 3.1 and Theorem 3.3.

Corollary 3.6. *Let* ρ *be a monotone, strongly finite, absolutely continuous and* $N-$*quasi semiconvex modular on* $X\left(S^2\right)$. *Let* $\mathbb{L} := \{L_{i,j}\}$ *be a double sequence of positive linear operators from* D *into* $X\left(S^2\right)$ *satisfying (3.20). Moreover suppose that* σ_r *is an unbounded function satisfying* $|\sigma_r(s,t)| \geq \alpha_r > 0$ $(r = 0, 1, 2, 3)$. *If* $\{L_{i,j}(e_r)\}$ *is triangular statistically relatively strongly convergent to* e_r *for each* $r = 0, 1, 2, 3$, *then* $\{L_{i,j}(h)\}$ *triangular statistically relatively modularly convergent to* h *provided that* h *is any function belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

Corollary 3.7. $\mathbb{L} := \{L_{i,j}\}$, ρ *and* σ_r $(r = 0, 1, 2, 3)$ *be the same as in Corollary 3.6. If* ρ *satisfies the* Δ_2-*condition, then the following statements are equivalent:*

(a) $\{L_{i,j}(e_r)\}$ *is triangular statistically relatively strongly convergent to* e_r *for each* $r = 0, 1, 2, 3$,

(b) $\{L_{i,j}(h)\}$ *is triangular statistically relatively strongly convergent to* h *provided that* h *is any fuction belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

4. Concluding remarks

Now, we give some reduced results showing the importance of Theorem 3.1 and Theorem 3.3 in approximation theory with special choices:

1. If we take $I = I_\delta^T$ and the scale function is a non-zero constant, triangular ideal relative modular convergence given in the Definition 2.1 reduces to the triangular statistical modular convergence form in [2]. So, from Theorem 3.1 and Theorem 3.3 we immediately get the triangular statistical modular Korovkin theorems for double sequences in [2].

2. As it is well known, if $(X, \|.\|)$ is a normed space, then $\rho(.) = \|.\|$ is a convex modular in X. So, by choosing $\rho(.) = \|.\|$, then from Theorem 3.1 and Theorem 3.3, the followings are obtained on normed spaces:

i) We get the triangular ideal relative convergence for double sequences on normed spaces by choosing $\rho(.) = \|.\|$.

ii) If we take $I = I_\delta^T$, then we immediately get the triangular statistical relative convergence for double sequences on normed spaces and in addition, we immediately get the triangular statistical relative Korovkin theorems for double sequences on normed spaces in [9].

iii) If we take $I = I_\delta^T$ and the scale function is a non-zero constant, then we get triangular statistical convergence for double sequences on normed spaces and in addition, we immediately get the triangular statistical Korovkin theorems for double sequences on normed spaces in [1].

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