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A characterization of relatively compact sets in $L^p(\Omega, B)$

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. We give a characterization of relatively compact sets F in $L^p(\Omega, B)$ for $p \in [1, \infty)$, B a Banach-space, and $\Omega \subset \mathbb{R}^n$. This is a generalization of the results obtained in [12] for the space $L^p((0,T), B)$ with $T > 0$, first to rectangles $\Omega = (a, b) \subset \mathbb{R}^n$ and, under additional conditions, to arbitrary open and bounded subsets of $\mathbb{R}^n$. An application of the main compactness result to a problem arising in homogenization of processes on periodic surfaces is given.

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1. Introduction

In this paper, we prove a Kolmogorov-Riesz-type compactness result for the space $L^p(\Omega, B)$ with $p \in [1, \infty)$, $\Omega \subset \mathbb{R}^n$ open and bounded, and B a Banach space. Such a result was proved in [12] for $\Omega = (0, T)$ with $T > 0$. We generalize this result to rectangles Ω in $\mathbb{R}^n$, see Theorem 2.2, and under additional assumptions to arbitrary open and bounded domains $\Omega \subset \mathbb{R}^n$, see Corollary 2.5.

Similar results in the framework of vector-valued Sobolev and Besov spaces can also be found in [2], see Theorem 5.2 and the proof of Theorem 1.1. There, the compactness result is obtained under the assumption that there exists $\theta > 0$, such that

$$\sup_{h \in \mathbb{R}^n \setminus \{0\}} \frac{\|f(\cdot + h) - f\|_{L^p(\Omega_h, B)}}{|h|^\theta} < \infty.$$

However, our results are proven under the weaker assumption (ii) in Theorem 2.2.

In the homogenization theory, we are often concerned with sequences of functions in the space $L^p((0,T) \times \Omega, B)$, for which we have to show strong convergence. Here, due to lack of regularity, classical results like e. g., the Aubin-Lions Lemma cannot be applied, and the compactness result derived in this paper is an appropriate alternative. In Section 3, we give an application of our main compactness result for a problem arising in homogenization of processes on periodic surfaces.

2. Main result

In this section, we prove our main compactness theorem and related results. The proof is based on the Arzelà-Ascoli theorem, which for the sake of completness is repeated below, and uses similar arguments as in [12].

Lemma 2.1 (Arzelà-Ascoli). *Let T be a compact Hausdorff space and B be a Banach-space. A subset $F \subset C(T, B)$ is relatively compact in $C(T, B)$ iff the following conditions hold:*

(i) For every $x \in T$, the set $F(x) := \{f(x) : f \in F\}$ is relatively compact in B.
(ii) F is uniformly equicontinuous, i. e., for all $\epsilon > 0$ there exists $\eta > 0$ such that

$$\|f(x_2) - f(x_1)\|_B < \epsilon \quad \text{for all } f \in F,\ x_1, x_2 \in T \text{ with } \|x_2 - x_1\| < \eta.$$

Proof. See e. g., [4, Theorem 0.4.11]. □

For an arbitrary set $\Omega \subset \mathbb{R}^n$ and a vector $\xi \in \mathbb{R}^n$, we define

$$\Omega_\xi := \Omega \cap (\Omega - \xi).$$

Further, for $a, b \in \mathbb{R}^n$ we define

$$(a, b) := (a_1, b_1) \times \ldots \times (a_n, b_n),$$

with $(a_i, b_i) := (b_i, a_i)$ if $b_i < a_i$. For $f : \Omega \to B$ and $h \in \mathbb{R}^n$ we define

$$\tau_h f : (\Omega - h) \to B, \quad \tau_h f(x) = f(x + h).$$

We now state our main theorem:

Theorem 2.2. *Let $p \in [1, \infty)$, B be a Banach-space, $\Omega = (a, b)$ with $a, b \in \mathbb{R}^n$ $(a_i < b_i)$, and $F \subset L^p(\Omega, B)$. Then F is relatively compact in $L^p(\Omega, B)$ iff*

(i) for every rectangle $C \subset \Omega$ the set $\left\{\int_C f dx : f \in F\right\}$ is relatively compact in B,
(ii) for $z \in \mathbb{R}^n$ with $0 \le z_i < b_i - a_i$, $i = 1, \ldots, n$ it holds

$$\sup_{f \in F} \|\tau_z f - f\|_{L^p(\Omega_z, B)} \to 0 \quad \text{for } z \to 0.$$

Proposition 2.3. *The condition (ii) in Theorem 2.2 is equivalent to the following one:*

(ii)′ For $i = 1, \ldots, n$ and $s > 0$ it holds

$$\sup_{f \in F} \|\tau_{se_i} f - f\|_{L^p(\Omega_{se_i}, B)} \to 0 \quad \text{for } s \to 0,$$

where e_i is the i-th unit normal vector.

Proof of the Proposition 2.3. It is straightforward that (ii) implies $(ii)'$. For the other implication, we choose $z \in \mathbb{R}^n$ with $z_i \geq 0$ small. Then we have $z = \sum_{i=1}^n z_i e_i$ and we define $z^0 := 0 \in \mathbb{R}^n$ and $z^j := \sum_{i=1}^j z_i e_i$ for $j \in \{1, \ldots, n\}$. Of course $z^n = z$. Now, we use the triangle inequalitys to obtain

$$\begin{aligned}\|\tau_z f - f\|_{L^p(\Omega_z, B)} &\leq \sum_{j=0}^{n-1} \|\tau_{z^{j+1}} f - \tau_{z^j} f\|_{L^p(\Omega_z, B)} \\ &\leq \sum_{i=1}^{n} \|\tau_{z_i e_i} f - f\|_{L^p(\Omega_{z_i e_i}, B)},\end{aligned}$$

where for the last inequality we used for $j = 0, \ldots, n-1$

$$\begin{aligned}\|\tau_{z^{j+1}} f - \tau_{z^j} f\|^p_{L^p(\Omega_z, B)} &= \int_{\Omega_z} \left\| f\left(x + \sum_{i=1}^{j+1} z_i e_i\right) - f\left(x + \sum_{i=1}^{j} z_i e_i\right)\right\|_B^p dx \\ &= \int_{\Omega_z + \sum_{i=1}^j z_i e_i} \left\| f(x + z_{j+1} e_{j+1}) - f(x)\right\|_B^p dx \\ &\leq \int_{\Omega_{z_{j+1} e_{j+1}}} \left\| f(x + z_{j+1} e_{j+1}) - f(x)\right\|_B^p dx.\end{aligned}$$

In the last inequality, we used the inclusion $\Omega_z + \sum_{i=1}^j z_i e_i \subset \Omega_{z_{j+1} e_{j+1}}$. In fact,

$$\Omega_{z_{j+1} e_{j+1}} = \{y \in \mathbb{R}^n : y_{j+1} \in (a_{j+1}, b_{j+1} - z_{j+1}),\ y_i \in (a_i, b_i) \text{ for } i \neq j+1\}$$

and for $y \in \Omega_z + \sum_{i=1}^j z_i e_i = [a, b - z] + \sum_{i=1}^j z_i e_i$, we have

$$\begin{aligned}&y_i \in (a_i + z_i, b_i), \text{ for } i = 1, \ldots, j \\ &y_i \in (a_i, b_i - z_i), \text{ for } i = j+1, \ldots, n.\end{aligned}$$

The claim follows. □

Proof of Theorem 2.2. Assume first that F is relatively compact in $L^p(\Omega, B)$. Then, we can use exactly the same arguments as in the proof of [12, Theorem 1]. In fact, (i) follows from the continuity of the mapping $f \mapsto \int_C f dx$ from $L^p(\Omega, B)$ into B, and (ii) follows, since in metric spaces, relatively compact sets are totally bounded, and the density of $C^0(\overline{\Omega}, B)$ in $L^p(\Omega, B)$.

Conversely, assume that (i) and (ii) hold. Let $f \in F$, and $h \in \mathbb{R}^n$ with $h_i > 0$ for $i = 1, \ldots, n$ (for example choose $h = s\frac{b-a}{2}$ with $s > 0$). Set

$$V_h := |(0, h)| > 0,$$

the measure of $(0, h)$. For $x \in \overline{\Omega_h}$, we have $(x, x + h) \subset \Omega$, and we define the function

$$(M_h f)(x) := \frac{1}{V_h} \int\limits_{(x, x+h)} f(z) dz.$$

We first show that $M_h f \in C\left(\overline{\Omega_h}, B\right)$, and the set

$$M_h F := \{M_h f : f \in F\}$$

is relatively compact in $C(\overline{\Omega_h}, B)$. Let $\epsilon > 0$, and choose $\delta > 0$ so small, that for $i = 1, \dots, n$ it holds

$$\|\tau_{\tilde{\delta} e_i} f - f\|_{L^1(\Omega_{\tilde{\delta} e_i}, B)} < \frac{V_h \epsilon}{n}, \text{ for all } \tilde{\delta} \leq \delta.$$

This is possible due to (ii) and the continuity of the embedding $L^p(\Omega_{\tilde{\delta} e_i}, B)$ into $L^1(\Omega_{\tilde{\delta} e_i}, B)$. Let $x_0 \in \overline{\Omega_h}$ and $x \in B_\delta(x_0) \cap \overline{\Omega_h}$, where the ball is taken with respect to the $\| \cdot \|_\infty$-norm on $\mathbb{R}^n$. Then $x = x_0 + \sum_{i=1}^n \delta_i e_i$ with $\delta_i \in (-\delta, \delta)$. For $j = 1, \dots, n$, we define the vector

$$x^j = x_0 + \sum_{i=1}^{j} \delta_i e_i.$$

Thus, we have $x^n = x$. Now, as in the proof of Proposition 2, we obtain

$$\|M_h f(x) - M_h f(x_0)\|_B \leq \sum_{j=0}^{n-1} \|M_h f(x^{j+1}) - M_h f(x^j)\|_B, \tag{2.1}$$

and we have $x^{j+1} - x^j = \delta_{j+1} e_{j+1}$ for $j = 0, \dots, n-1$. Without loss of generality, we assume that $\delta_i > 0$ for $i = 1, \dots, n$. Otherwise, i. e., for $\delta_i < 0$, we change the role of x^{j+1} and x^j in the following argumentation and for $\delta_i = 0$ it is trivial. It holds that

$$\begin{aligned} \|M_h f(x^{j+1}) - M_h f(x^j)\|_B &= \frac{1}{V_h} \left\| \int_{(x^j, x^j + h)} \left(\tau_{x^{j+1} - x^j} f - f\right)(z) dz \right\|_B \\ &\leq \frac{1}{V_h} \int_{(x^j, x^j + h)} \|\tau_{\delta_{j+1} e_{j+1}} f - f\|_B dz \\ &\overset{(*)}{\leq} \frac{1}{V_h} \|\tau_{\delta_{j+1} e_{j+1}} f - f\|_{L^1(\Omega_{\delta_{j+1} e_{j+1}}, B)} < \frac{\epsilon}{n}, \end{aligned} \tag{2.2}$$

where in $(*)$ we used $(x^j, x^j + h) \subset \Omega_{\delta_{j+1} e_{j+1}}$. In fact, from $x, x_0 \in \overline{\Omega_h}$, it follows by contradiction, that $x^i \in \overline{\Omega_h}$ for $i = 1, \dots, n$. This implies that

$$a_{j+1} \leq x^j_{j+1} \quad \text{and} \quad x^j_{j+1} + h_{j+1} = x^{j+1}_{j+1} - \delta_{j+1} + h_{j+1} \leq b_{j+1} - \delta_{j+1},$$

for $j = 1, \dots, n-1$, and hence, the inclusion $(x^j, x^j + h) \subset \Omega_{\delta_{j+1} e_{j+1}}$. From (2.1) and (2.2), we obtain that $M_h f \in C(\overline{\Omega_h}, B)$, and especially the set $M_h F$ is uniformly equicontinuous in $C(\overline{\Omega_h}, B)$.

For $x \in \Omega_h$ we obtain from the assumption (i) that the set

$$(M_h F)(x) := \left\{ \frac{1}{V_h} \int_{(x, x+h)} f dy : f \in F \right\}$$

is relatively compact in B. From Lemma 2.1 it follows that $M_h F$ is relatively compact in $C(\overline{\Omega_h}, B)$.

The next step in the proof is to show that F is the uniform limit of $M_h F$ in $L^p(\Omega_\xi, B)$ for $h \to 0$, see also [12, (2.2)]. We start from the following relation which holds for $x \in \Omega_h$:

$$(M_h f - f)(x) = \frac{1}{V_h} \int_{(x,x+h)} f(z) - f(x) dz = \frac{1}{V_h} \int_{(0,h)} (\tau_z f - f)(x) dz.$$

With the Jensen-inequality and the Fubini-Theorem we get

$$\begin{aligned} \|M_h f - f\|^p_{L^p(\Omega_h,B)} &= \int_{\Omega_h} \left\| \frac{1}{V_h} \int_{(0,h)} \tau_z f(x) - f(x) dz \right\|^p_B dx \\ &\leq \frac{1}{V_h} \int_{\Omega_h} \int_{(0,h)} \|\tau_z f(x) - f(x)\|^p_B \, dz dx \\ &\leq \sup_{z \in (0,h)} \|\tau_z f - f\|^p_{L^p(\Omega_h,B)}, \end{aligned}$$

and therefore

$$\|M_h f - f\|_{L^p(\Omega_h,B)} \leq \sup_{z \in (0,h)} \|\tau_z f - f\|_{L^p(\Omega_h,B)}.$$

Due to assumption (ii), for every $\epsilon > 0$ we can choose h so small that for every $z \in (0,h)$ and every $f \in F$ we have

$$\|\tau_z f - f\|_{L^p(\Omega_h,B)} \leq \|\tau_z f - f\|_{L^p(\Omega_z,B)} < \epsilon,$$

and we obtain

$$\|M_h f - f\|_{L^p(\Omega_h,B)} < \epsilon.$$

Hence, F is the uniform limit of $M_h F$ in $L^p(\Omega_\xi, B)$ with $\xi = \frac{b-a}{2}$ for $h \to 0$. Since $M_h F$ is relatively compact in $C(\overline{\Omega_\xi}, B)$, it is also relatively compact in $L^p(\Omega_\xi, B)$, because the embedding $C(\overline{\Omega_\xi}, B) \hookrightarrow L^p(\Omega_\xi, B)$ is continuous. From [12, (2.2)] it follows that F is relatively compact in $L^p(\Omega_\xi, B)$.

Until now we have only established that F is relatively compact in $L^p(\Omega_\xi, B)$, but we have to show the result for the whole domain Ω. Let $\Sigma := \{-1, 1\}^n$ and for $z \in \mathbb{R}^n$ we define $z_\sigma := (\sigma_1 z_1, \ldots, \sigma_n z_n)$. Of course, we have $\#\Sigma = 2^n$ and $\overline{\Omega} = \bigcup_{\sigma \in \Sigma} \overline{\Omega_{\xi_\sigma}}$. If additionally $z_i \geq 0$ for $i = 1, \ldots, n$, then we write $z_\sigma^+ := \frac{z_\sigma + z}{2}$ (positive components of z_σ) and $z_\sigma^- := \frac{z_\sigma - z}{2}$ (the negative components of z_σ), such that $z_\sigma = z_\sigma^+ + z_\sigma^-$. For $h \in \mathbb{R}^n$ we write $(x, x + h_\sigma) := (x + h_\sigma^-, x + h_\sigma^+)$.
We define the function $M_{h_\sigma} f$ in the same way as $M_h f$, i. e.,

$$M_{h_\sigma} f(x) := \frac{1}{V_h} \int_{(x,x+h_\sigma)} f(z) dz \quad \text{for} \quad x \in \overline{\Omega_{h_\sigma}},$$

and for all $x \in \overline{\Omega_{h_\sigma}}$ we obtain with the transformation formula

$$(M_{h_\sigma} f - f)(x) = \frac{1}{V_h} \int_{(x,x+h_\sigma)} f(z) - f(x) dz = \frac{1}{V_h} \int_{(0,h_\sigma)} (\tau_z f - f)(x) dz.$$

With Fubini's Theorem, the Jensen-inequality and again by integration by substitution, we get

$$\begin{aligned}\|M_{h_\sigma}f - f\|^p_{L^p(\Omega_{h_\sigma},B)} &\leq \frac{1}{V_h}\int_{\Omega_{h_\sigma}}\int_{(0,h_\sigma)}\|(\tau_z f - f)(x)\|^p_B dz dx\\ &= \frac{1}{V_h}\int_{\Omega_{h_\sigma}}\int_{(0,h)}\|f(x+z_\sigma) - f(x)\|^p_B dz dx\\ &= \frac{1}{V_h}\int_{(0,h)}\int_{\Omega_{h_\sigma}}\|f(x+z^+_\sigma + z^-_\sigma) - f(x)\|^p_B dx dz\\ &= \frac{1}{V_h}\int_{(0,h)}\int_{\Omega_{h_\sigma}+z^-_\sigma}\|f(x+z^+_\sigma) - f(x-z^-_\sigma)\|^p_B dx dz\\ &\leq \frac{1}{V_h}\int_{(0,h)}\int_{\Omega_z}\|f(x+z^+_\sigma) - f(x-z^-_\sigma)\|^p_B dx dz,\end{aligned}$$

where in the last inequality we used $\Omega_{h_\sigma} + z^-_\sigma \subset \Omega_z$ for $z \in (0,h)$. To show this, we consider for $y \in \Omega_{h_\sigma} + z^-_\sigma$, and for $i = 1,\dots,n$ the following two cases:

1) $\sigma_i = 1$: Then $(h_\sigma)_i = h_i$ and $(z^-_\sigma)_i = 0$ and therefore $y_i \in (a_i, b_i - h_i) \subset (a_i, b_i - z_i)$.
2) $\sigma_i = -1$: Then $(h_\sigma)_i = -h_i$ and $(z^-_\sigma)_i = -z_i$ and therefore $y_i \in (a_i + h_i - z_i, b_i - z_i) \subset (a_i, b_i - z_i)$.

Thus, $y_i \in (a_i, b_i - z_i)$ for $i = 1,\dots,n$, i.e., $y \in \Omega_z$. Hence,

$$\begin{aligned}\|M_{h_\sigma}f - f\|_{L^p(\Omega_{h_\sigma},B)} &\leq \sup_{z\in(0,h)}\|\tau_{z^+_\sigma}f - \tau_{-z^-_\sigma}f\|_{L^p(\Omega_z,B)}\\ &\leq \sup_{z\in(0,h)}\|\tau_{z^+_\sigma}f - f\|_{L^p(\Omega_z,B)} + \sup_{z\in(0,h)}\|\tau_{-z^-_\sigma}f - f\|_{L^p(\Omega_z,B)}\\ &\leq \sup_{z\in(0,h)}\|\tau_{z^+_\sigma}f - f\|_{L^p(\Omega_{z^+_\sigma},B)} + \sup_{z\in(0,h)}\|\tau_{-z^-_\sigma}f - f\|_{L^p(\Omega_{-z^-_\sigma},B)}.\end{aligned}$$

With the same arguments as above we obtain that F is relatively compact in $L^p(\Omega_{\xi_\sigma}, B)$ for all $\sigma \in \Sigma$. Hence, $\overline{F}$ is sequentially compact in $L^p(\Omega, B)$ and therefore F is relatively compact in $L^p(\Omega, B)$. □

The next proposition gives us a further characterization of the condition (ii) in Theorem 2.2, where we use a special decomposition of the domain Ω, and consider the shifts on fixed domains. We use the same notation as in the proof of Theorem 2.2, especially we have $\xi = \frac{b-a}{2}$.

Proposition 2.4. *The condition (ii) in Theorem* 2.2 *is equivalent to the following one:*

(ii)″ For $z \in \mathbb{R}^n$ and $z_i \geq 0$ it holds

$$\sup_{f\in F}\|\tau_{z_\sigma}f - f\|_{L^p(\Omega_{\xi_\sigma},B)} \to 0 \textit{ for } z \to 0$$

for all $\sigma \in \Sigma$.

Proof. Let (ii) from Theorem 2.2 be true. We use similar arguments as in the last part of the proof of Theorem 2.2. Let $\epsilon > 0$ and $\delta > 0$ so small that for all $h \in [0,\delta]^n$ the following holds

$$\|\tau_h f - f\|_{L^p(\Omega_h, B)} < \frac{\epsilon}{2}.$$

Now, for $z \in [0,\delta]^n$ it follows by substitution and from

$$\Omega_{\xi_\sigma} + z_\sigma^- \subset \Omega_z \tag{2.3}$$

(which is proved below) that

$$\begin{aligned}
\|\tau_{z_\sigma} f - f\|^p_{L^p(\Omega_{\xi_\sigma}, B)} &= \int_{\Omega_{\xi_\sigma}} \|f(x + z_\sigma) - f(x)\|_B^p dx \\
&= \int_{\Omega_{\xi_\sigma} + z_\sigma^-} \|f(x + z_\sigma^+) - f(x - z_\sigma^-)\|_B^p dx \\
&\le \int_{\Omega_z} \|f(x + z_\sigma^+) - f(x - z_\sigma^-)\|_B^p dx \\
&= \|\tau_{z_\sigma^+} f - \tau_{-z_\sigma^-} f\|^p_{L^p(\Omega_z, B)}.
\end{aligned}$$

Since $z_\sigma^+, -z_\sigma^- \in [0,\delta]^n$, it follows that

$$\begin{aligned}
\|\tau_{z_\sigma} f - f\|_{L^p(\Omega_{\xi_\sigma}, B)} &\le \|\tau_{z_\sigma^+} f - f\|_{L^p(\Omega_z, B)} + \|\tau_{-z_\sigma^-} f - f\|_{L^p(\Omega_z, B)} \\
&\le \|\tau_{z_\sigma^+} f - f\|_{L^p(\Omega_{z_\sigma^+}, B)} + \|\tau_{-z_\sigma^-} f - f\|_{L^p(\Omega_{-z_\sigma^-}, B)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Let us now give the proof of (2.3): For $x \in \Omega_{\xi_\sigma} + z_\sigma^-$ exists $\bar{x} \in \Omega_{\xi_\sigma}$ with $x = \bar{x} + z_\sigma^-$, i. e.,

$$\bar{x}_i \in \begin{cases} \left(a_i, \frac{b_i - a_i}{2}\right) & \text{for } \sigma_i = 1 \\ \left(\frac{b_i - a_i}{2}, b_i\right) & \text{for } \sigma_i = -1 \end{cases}, \quad \text{and} \quad (z_\sigma^-)_i = \begin{cases} 0 & \text{for } \sigma_i = 1 \\ -z_i & \text{for } \sigma_i = -1 \end{cases},$$

for $i = 1, \ldots, n$. Hence, we obtain

$$x_i \in \begin{cases} \left(a_i, \frac{b_i - a_i}{2}\right) & \text{for } \sigma_i = 1 \\ \left(\frac{b_i - a_i}{2} - z_i, b_i - z_i\right) & \text{for } \sigma_i = -1. \end{cases}$$

Since $\Omega_z = \prod_{i=1}^n (a_i, b_i - z_i)$, we obtain $x \in \Omega_z$.

Conversely, let $(ii)''$ hold. For $\epsilon > 0$ choose $\delta > 0$ so small that for all $\sigma \in \Sigma$ and all $h \in [0,\delta]^n$, we have

$$\|\tau_{h_\sigma} f - f\|_{L^p(\Omega_{\xi_\sigma}, B)} < \frac{\epsilon}{2\sqrt[p]{2^n}}.$$

Let $z \in [0,\delta]^n$, then we obtain for $\sigma \in \Sigma$

$$\begin{aligned}\|\tau_z f - f\|^p_{L^p(\Omega_{\xi_\sigma}\cap\Omega_z,B)} &= \int_{\Omega_{\xi_\sigma}\cap\Omega_z} \|f(x+z)-f(x)\|^p_B dx \\ &= \int_{\Omega_{\xi_\sigma}\cap\Omega_z} \|f(x+z^+_\sigma - z^-_\sigma)-f(x)\|^p_B dx \\ &= \int_{(\Omega_{\xi_\sigma}\cap\Omega_z)-z^-_\sigma} \|f(x+z^+_\sigma)-f(x+z^-_\sigma)\|^p_B dx \\ &\leq \|\tau_{z^+_\sigma} f - \tau_{z^-_\sigma} f\|^p_{L^p(\Omega_{\xi_\sigma},B)}.\end{aligned}$$

Further, we have $z^+_\sigma, -z^-_\sigma \in [0,\delta]^n$ and $z^+_\sigma = (z^+_\sigma)_\sigma$ and $z^-_\sigma = (-z^-_\sigma)_\sigma$, what implies

$$\begin{aligned}&\|\tau_z f - f\|_{L^p(\Omega_{\xi_\sigma}\cap\Omega_h,B)} \\ &\qquad \leq \|\tau_{z^+_\sigma} f - f\|_{L^p(\Omega_{\xi_\sigma},B)} + \|\tau_{z^-_\sigma} f - f\|_{L^p(\Omega_{\xi_\sigma},B)} < \frac{\epsilon}{\sqrt[p]{2^n}},\end{aligned}$$

i. e.,

$$\|\tau_z f - f\|^p_{L^p(\Omega_z,B)} = \sum_{\sigma\in\Sigma} \|\tau_z f - f\|^p_{L^p(\Omega_{\xi_\sigma}\cap\Omega_z,B)} < \epsilon^p. \qquad \square$$

Until now we have only considered rectangular domains in $\mathbb{R}^n$. Now we extend our result to more general domains. However, we need an additional assumption to control the functions near the boundary. We use the same notation as above and define for $\delta > 0$ the set $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$ and for $z \in \mathbb{R}^n$ the set

$$\Omega^z_\delta := \{x \in \Omega_\delta : x + z \in \Omega_\delta\} = \{x, x+z \in \Omega_\delta\}.$$

Corollary 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Let $F \subset L^p(\Omega, B)$ for a Banach space B and $p \in [1,\infty)$. Then F is relatively compact in $L^p(\Omega, B)$ iff*

(i) for every measurable set $C \subset \Omega$ the sequence $\left\{\int_C f dx : f \in F\right\}$ is relatively compact in B,

(ii) for all $\delta > 0$ it holds that $\sup_{f\in F} \|\tau_z f - f\|_{L^p(\Omega^z_\delta,B)} \to 0$ for $z \to 0$,

(iii) for $\delta > 0$ it holds that $\sup_{f\in F} \int_{\Omega\setminus\Omega_\delta} |f(x)|^p dx \to 0$ for $\delta \to 0$.

Proof. For F relatively compact in $L^p(\Omega, B)$ the statements (i) - (iii) can be established in a similar way as in Theorem 2.2.

Now assume, that (i) - (iii) hold. Since Ω is bounded, there exists a rectangle $W \subset \mathbb{R}^n$ with $\Omega \subset\subset W$. Extend every function $f \in F$ by zero to a function $\tilde{f} \in L^p(W, B)$ and obtain a set $\tilde{F} \subset L^p(W, B)$. Using the same arguments as in [1, U2.21], we can show that the assumptions of Theorem 2.2 are fulfilled and the claim follows. $\square$

3. Application

We consider an application of the compactness criterion derived din Section 2 to the homogenization of a nonlinear reaction-diffusion-problem on a rapidly oscillating periodic surface. Such problems arise in the mathematical modelling of processes in porous catalysts, see e.g. [7, 9], in biological structures, like e.g. biochemical processes in cells and tissue, see e.g. [6, 8, 11]. The periodically oscillating surface and the so called microscopic or ϵ-problem are given in the following.

Let $Y = (0,1)^n$ with $n \in \mathbb{N}$, $n \geq 3$, and $\Omega = (a,b) \subset \mathbb{R}^n$ with $a,b \in \mathbb{Z}^n$ and $a_i < b_i$ for $i = 1,\dots,n$. We assume that the sequence ϵ fulfills $\epsilon^{-1} \in \mathbb{N}$. Further, let $\Gamma \subset Y$ be a $C^{1,1}$-submanifold, such that

$$\Gamma_\epsilon := \left\{x \in \Omega : x = \epsilon(k+y) \text{ for some } k \in \mathbb{Z}^n,\, y \in \overline{\Gamma}\right\}$$

is connected and of class $C^{1,1}$. Especially, we have $\partial\Gamma_\epsilon \subset \partial\Omega$. On Γ_ϵ we consider the following problem:

$$\begin{aligned} \partial_t u_\epsilon - \Delta_{\Gamma_\epsilon} u_\epsilon &= f(u_\epsilon) && \text{in } (0,T)\times\Gamma_\epsilon, \\ -\nabla_{\Gamma_\epsilon} u_\epsilon \cdot \nu_{\Gamma_\epsilon} &= 0 && \text{on } (0,T)\times\partial\Gamma_\epsilon, \\ u_\epsilon(0) &= u^0 && \text{in } \Gamma. \end{aligned} \tag{3.1}$$

Here, Δ_{Γ_ϵ} denotes the Laplace-Beltrami-operator, $f \in C^{0,1}(\mathbb{R})$, i.e., f is globally Lipschitz-continuous, and $u^0 \in C^1\big(\overline{\Omega}\big)$. For the sake of simplicity the diffusion-coefficient is equal to 1, the nonlinearity f does not depend on a macroscopic or oscillating variable, and on the boundary $\partial\Gamma_\epsilon$, we consider a Neumann-zero condition. However, the following method can easily be generalized to more general problems, e.g., systems of equations and general diffusion-tensors. We are looking for a weak solution of Problem (3.1), i.e., $u_\epsilon \in L^2((0,T),H^1(\Gamma_\epsilon)) \cap H^1((0,T),L^2(\Gamma_\epsilon))$, such that for all $\phi \in H^1(\Gamma_\epsilon)$ we have

$$\int_{\Gamma_\epsilon} \partial_t u_\epsilon \phi \, d\sigma + \int_{\Gamma_\epsilon} \nabla_{\Gamma_\epsilon} u_\epsilon \cdot \nabla_{\Gamma_\epsilon}\phi \, d\sigma = \int_{\Gamma_\epsilon} f(u_\epsilon)\phi \, d\sigma \tag{3.2}$$

almost everywhere in $(0,T)$. With the Galerkin-method we obtain:

Proposition 3.1. *There exists a unique weak solution u_ϵ of Problem (3.1), such that*

$$\|u_\epsilon\|_{L^\infty((0,T),L^2(\Gamma_\epsilon))} + \left\|\nabla_{\Gamma_\epsilon} u_\epsilon\right\|_{L^2((0,T),L^2(\Gamma_\epsilon))} + \|\partial_t u_\epsilon\|_{L^2((0,T)\times\Gamma_\epsilon)} \leq C\epsilon^{-\frac{1}{2}}.$$

This (microscopic) model describes the processes and the medium in a very detailed way. However, due to its high complexity it is not appropriate for practical applications, especially it is not amenable to numerical computations. Therefore, an effective (macroscopic, homogenized) model is needed, which is an approximation of the microscopic one, and consists of equations formulated on a macroscopic scale. The effective model is derived by using methods of periodic homogenization. This consists in showing that for $\epsilon \to 0$, the sequence of solutions (u_ϵ) converges to a limit function u_0, and in the derivation of the limit problem satisfied by u_0.

The appropriate techniques to be used for the derivation of the effective model in our application are the method of two-scale convergence for functions on periodic surfaces introduced in [9], and its equivalent characterisation with the help of the unfolding operator, see e.g. [3, 6]. Based on the estimates (3.1), passing to the limit in the linear terms in the equation (3.2) can be performed like in [5], where a linear problem was considered. Taking the limit in the nonlinear term is however more challenging. To achieve this, we make use of the unfolding operator

$$\mathcal{T}_\epsilon^b : L^2((0,T)\times\Gamma_\epsilon) \to L^2((0,T)\times\Omega\times\Gamma),$$

see [3, 6], defined via

$$\mathcal{T}_\epsilon^b \phi(t,x,y) := \phi\left(t, \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon y\right).$$

Here $[\cdot]$ denotes the Gauß-bracket. Thus, from the theory developed in [5], we obtain the existence of a limit function $u_0 \in L^2((0,T),H^1(\Omega)) \cap H^1((0,T),L^2(\Omega))$ with $u_0(0) = u^0$, such that for all $\phi \in C_0^\infty\big((0,T)\times\overline{\Omega}\big)$ it holds that

$$\begin{aligned} |\Gamma|\int_0^T\int_\Omega \partial_t u_0 \phi dx dt + \int_0^T\int_\Omega D^*\nabla u_0\cdot\nabla\phi dx dt = \\ \lim_{\epsilon\to 0}\int_0^T\int_\Omega\int_\Gamma f\big(\mathcal{T}_\epsilon^b u_\epsilon\big)\mathcal{T}_\epsilon^b\phi d\sigma_y dx dt. \end{aligned} \tag{3.3}$$

The homogenized diffusion-coefficient $D^* \in \mathbb{R}^{n\times n}$ is given by

$$D_{ij}^* = \int_\Gamma \big(\nabla_\Gamma w_i + \nabla_\Gamma y_i\big)\cdot\nabla_\Gamma y_j d\sigma,$$

where w_i for $i \in \{1,\dots,n\}$ are the solutions of the following so called cell problems:

$$\begin{aligned} -\nabla_\Gamma\cdot\big(\nabla_\Gamma w_i + \nabla_\Gamma y_i\big) &= 0 \quad \text{in } \Gamma, \\ -(\nabla_\Gamma w_i + \nabla_\Gamma y_i\big)\cdot\nu &= 0 \quad \text{on } \partial\Gamma, \\ w_i \text{ is } Y\text{-periodic and } \int_\Gamma w_i d\sigma &= 0. \end{aligned}$$

To show the convergence of the nonlinear term we use the fact that

$$\mathcal{T}_\epsilon^b\phi \to \phi \text{ strongly in } L^2((0,T)\times\Omega\times\Gamma),$$

due to the regularity of ϕ. Hence, to go to the limit on the right-hand side in (3.3), it remains to show the weak convergence of $f\big(\mathcal{T}_\epsilon^b u_\epsilon\big)$ to $f(u_0)$ in $L^2((0,T)\times\Omega\times\Gamma)$. Therefore, we show the strong convergence of $\mathcal{T}_\epsilon^b u_\epsilon$ to u_0 in $L^2((0,T)\times\Omega\times\Gamma)$. Then, due to the Lipschitz-regularity of f, we actually obtain the strong convergence of $f\big(\mathcal{T}_\epsilon^b u_\epsilon\big)$ to $f(u_0)$ in $L^2((0,T)\times\Omega\times\Gamma)$. In [11] such a result was proved by showing that $\mathcal{T}_\epsilon^b u_\epsilon$ is a Cauchy-sequence. However, this result strongly relied on the fact, that the diffusion coefficient in the microscopic problem was of order ϵ^2, which led to an equation for $\mathcal{T}_\epsilon^b u_\epsilon$ where all coefficients were of order one. In our paper this is not the case, and the argument with the Cauchy-sequence cannot be applied. Instead, we use the compactness criterion from Section 2. A similar approach was used in [10], where the classical compactness criterion by Kolmogorov, see e. g., [13], for the space $L^2((0,T)\times\Omega\times Z)$, with $Z = (0,1)^{n-1}\times(-1,1)$, was employed. This is not appropriate for the situation in our application since shifts with respect to the surface-variable y make no sense.

In the following, we use the same notations as in Section 2, especially $\xi = \frac{b-a}{2}$.

Lemma 3.2. *Let $l \in \mathbb{N}_0^n$. Then, for all $\epsilon > 0$, such that $|l_i\epsilon| < \left|\frac{b_i-a_i}{2}\right|$ the following estimate holds for all $\sigma \in \Sigma$*

$$\|\tau_{\epsilon l_\sigma}u_\epsilon - u_\epsilon\|_{L^2((0,T)\times(\Gamma_\epsilon)_{\xi_\sigma})} \le C|l|\sqrt{\epsilon}.$$

Proof. We test the variational equation for $\tau_{\epsilon l_\sigma} u_\epsilon - u_\epsilon$ with $\eta^2(\tau_{\epsilon l_\sigma} u_\epsilon - u_\epsilon)$, where $\eta \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function with $0 \leq \eta \leq 1$, $\eta = 1$ in Ω_{ξ_σ}, and zero outside a small neighbourhood of Ω_{ξ_σ}. Then, Gronwall's inequality and the Lipschitz-continuity of u^0 give the desired result. □

Theorem 3.3. *For $\epsilon \to 0$, we have*

$$\mathcal{T}_\epsilon^b u_\epsilon \to u_0 \text{ strongly in } L^2((0,T) \times \Omega \times \Gamma).$$

Especially, we obtain

$$f\big(\mathcal{T}_\epsilon^b u_\epsilon\big) \to f(u_0) \text{ strongly in } L^2((0,T) \times \Omega \times \Gamma).$$

Proof. We consider $\mathcal{T}_\epsilon^b u_\epsilon$ as a function from $L^2(\Omega, L^2((0,T) \times \Gamma))$ and prove the condition (i) in Theorem 2.2 and $(ii)''$ in Proposition 2.4. Let $A \subset \Omega$ measurable, and define $v_A^\epsilon := \int_A \mathcal{T}_\epsilon^b u_\epsilon(\cdot_t, x, \cdot_y)dx$. The a priori estimate in Proposition 3.1 imply that v_A^ϵ is bounded in $L^2((0,T), H^1(\Gamma)) \cap H^1((0,T), L^2(\Gamma))$, and due to the Aubin-Lions Lemma the sequence is relatively compact in $L^2((0,T), L^2(\Gamma))$. It remains to check condition $(ii)''$. For $z \in \mathbb{R}^n$ with $z_i \geq 0$ small, we obtain as in the proof of [10, Theorem 2.3, page 700] for $l(\epsilon, z, m) := m + \left[\frac{z}{\epsilon}\right]$

$$\begin{aligned}
&\left\|\tau_{z_\sigma} \mathcal{T}_\epsilon^b u_\epsilon - \mathcal{T}_\epsilon^b u_\epsilon\right\|^2_{L^2(\Omega_{\xi_\sigma}, L^2((0,T)\times\Gamma))} \\
&\qquad \leq \epsilon \sum_{m \in \{0,1\}^n} \left\|\tau_{\epsilon l(\epsilon,z,m)_\sigma} u_\epsilon - u_\epsilon\right\|^2_{L^2((0,T)\times(\Gamma_\epsilon)_{\xi_\sigma})} \leq C\epsilon^2 |l(\epsilon, z, m)|^2.
\end{aligned}$$

Since $|l(\epsilon, z, m)|\epsilon \to 0$ for $\epsilon \to 0$ and $z \to 0$, condition $(ii)''$ is valid. Hence, Theorem 2.2 and Proposition 2.4 imply the desired result. □

Altogether, we immediately obtain that u_0 fulfills the following variational equation:

$$|\Gamma| \int_\Omega \partial_t u_0 \phi dx + \int_\Omega D^* \nabla u_0 \cdot \nabla \phi dx = |\Gamma| \int_\Omega f(u_0) \phi dx,$$

for all $\phi \in H^1(\Omega)$ and almost everywhere in $(0,T)$. The corresponding inital and boundary value problem is

$$\begin{aligned}
|\Gamma| \partial_t u_0 - \nabla \cdot (D^* \nabla u_0) &= |\Gamma| f(u_0) && \text{in } (0,T) \times \Omega \\
-D^* \nabla u_0 \cdot \nu &= 0 && \text{on } (0,T) \times \partial\Omega \\
u_0(0) &= u^0 && \text{in } \Omega.
\end{aligned}$$

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