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Implicit Caputo-Fabrizio fractional differential equations with delay

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Abstract. This article deals with some existence and uniqueness results for several classes of implicit fractional differential equations with delay. Our results are based on some fixed point theorems. To illustrate our results, we present some examples in the last section.

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1. Introduction

Functional differential equations and inclusions of fractional order have recently been applied in various areas of sciences; see the monographs [1, 2, 3, 20, 24, 28, 25], the papers [5, 8, 26, 27], and the references therein.

The study of implicit differential equations has received great attention in the last years; see [1, 5, 6, 8, 9, 10, 7, 22, 27].

Functional differential equations with delay have received significant attention in recent years. Several authors studied differential equations with delay [1, 4, 8, 13, 14, 15, 16, 17, 18, 19].

In this paper, first we investigate the following class of Caputo-Fabrizio fractional differential equation with finite delay

$$\begin{cases} \wp(t) = \zeta(t); \ t \in [-h, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_t, ({}^{CF}D_0^r \wp)(t)); \ t \in I := [0, T], \end{cases} \tag{1.1}$$

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where $h > 0,\ T > 0,\ \zeta \in \mathcal{C}, f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0, 1]$, and $\mathcal{C} := C([-h, 0], \mathbb{R})$ is the space of continuous functions on $[-h, 0]$. Here, for any $t \in I$, we define $\wp_t$ by

$$\wp_t(s) = \wp(t + s);\ \text{for } s \in [-h, 0].$$

Next, we consider the following infinite delay problem

$$\begin{cases} \wp(t) = \zeta(t);\ t \in \mathbb{R}_- := (-\infty, 0], \\ ({}^{CF}D_0^r\wp)(t) = f(t, \wp_t, ({}^{CF}D_0^r\wp)(t));\ t \in I, \end{cases} \tag{1.2}$$

where $\zeta : \mathbb{R}_- \to \mathbb{R}, f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\mathcal{B}$ is called a phase space that will be specified later. In this case, for any $t \in I$, we let $\wp_t \in \mathcal{B}$ be such that

$$\wp_t(s) = \wp(t + s);\ \text{for } s \in \mathbb{R}_-.$$

In the third section, we investigate the following state-dependent finite delay problem

$$\begin{cases} \wp(t) = \zeta(t);\ t \in [-h, 0], \\ ({}^{CF}D_0^r\wp)(t) = f(t, \wp_{\rho(t,\wp_t)}, (({}^{CF}D_0^r\wp)(t));\ t \in I, \end{cases} \tag{1.3}$$

where $\zeta \in \mathcal{C},\ \rho : I \times \mathcal{C} \to \mathbb{R},\ f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Finally, we study the following class of Caputo-Fabrizio fractional differential equations with state dependent infinite delay

$$\begin{cases} \wp(t) = \zeta(t);\ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^r\wp)(t) = f(t, \wp_{\rho(t,\wp_t)}, (({}^{CF}D_0^r\wp)(t));\ t \in I, \end{cases} \tag{1.4}$$

where $\zeta : \mathbb{R}_- \to \mathbb{R},\ f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In the last section, we present some examples illustrating our presented results.

2. Preliminaries

Let $(C(I), \|\cdot\|_\infty)$ be the Banach space of continuous real functions on I with

$$\|\xi\|_\infty := \sup_{t \in I} |\xi(t)|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous real functions on I, and by $L^1(I)$ we denote the space of measurable real functions on I which are Lebesgue integrable with the norm

$$\|\xi\|_1 = \int_I |\xi(t)| dt.$$

Definition 2.1. [11, 23] The Caputo-Fabrizio fractional integral of order $0 < r < 1$ for a function $w \in L^1(I)$ is defined by

$${}^{CF}I^r w(\tau) = \frac{2(1 - r)}{M(r)(2 - r)} w(\tau) + \frac{2r}{M(r)(2 - r)} \int_0^\tau w(x)dx, \quad \tau \geq 0$$

where $M(r)$ is normalization constant depending on r.

Definition 2.2. [11, 23] The Caputo-Fabrizio fractional derivative for a function $w \in C^1(I)$ of order $0 < r < 1$, is defined by

$$^{CF}D^r w(\tau) = \frac{(2-r)M(r)}{2(1-r)} \int_0^\tau \exp(-\frac{r}{1-r}(\tau - x))w'(x)dx; \ \tau \in I.$$

Note that $(^{CF}D^r)(w) = 0$ if and only if w is a constant function.

Example 2.3. [11]
1- For $h(t) = t$ and $0 < r \leq 1$, we have

$$(^{CF}D^r h)(t) = \frac{M(r)}{r}\left(1 - \exp\left(-\frac{r}{1-r}t\right)\right).$$

2- For $g(t) = e^{\lambda t}, \ \lambda \geq 0$ and $0 < r \leq 1$, we have

$$(^{CF}D^r g)(t) = \frac{\lambda M(r)}{r + \lambda(1-r)} e^{\lambda t}\left(1 - \exp\left(-\lambda - \frac{r}{1-r}t\right)\right).$$

Lemma 2.4. [21] *Let $h \in L^1(I)$. Then the linear problem*

$$\begin{cases} (^{CF}D_0^r \wp)(t) = h(t); \ t \in I := [0, T] \\ \wp(0) = \wp_0, \end{cases} \tag{2.1}$$

has a unique solution given by

$$\wp(t) = \wp_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds, \tag{2.2}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)}.$$

3. Existence of solutions with finite delay

In this section, we establish the existence results for problem (1.1). Consider the Banach space

$$C = \{\wp : (-h, T] \to \mathbb{R}, \ \wp|_{[-h,T]} \equiv \zeta, \ \wp|_I \in C(I)\}.$$

with the norm

$$\|\wp\|_C = \max\{\|\zeta\|_{[-h,0]}, \|\wp\|_\infty\}.$$

Definition 3.1. By a solution of problem (1.1), we mean a function $\wp \in C$ such that

$$\wp(t) = \begin{cases} \zeta(t); \ t \in [-h, 0], \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases}$$

where $g \in C(I)$ with $g(t) = f(t, \wp_t, g(t))$.

The following hypotheses will be used in the sequel.

- (H_1) There exist constants $\omega_1 > 0, \ 0 < \omega_2 < 1$ such that:

  $$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq \omega_1\|\wp_1 - \wp_2\|_{[-h,0]} + \omega_2|\Im_1 - \Im_2|,$$

  for any $\wp_1, \wp_2 \in \mathcal{C}, \ \Im_1, \Im_2 \in \mathbb{R}$, and each $t \in I$.

- (H_2) For any bounded set $B \subset \mathcal{C}$, the set:
$$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r \wp)(t)) : \wp \in B\};$$
is equicontinuous in C.

Theorem 3.2. *If* (H_1) *holds, and*
$$\ell := \frac{\omega_1(2a_r + Tb_r)}{1-\omega_2} < 1, \tag{3.1}$$
then problem (1.1) has a unique solution on $[-h, T]$.

Proof. Consider the operator $N : C \to C$ defined by:
$$(N\wp)(t) = \begin{cases} \zeta(t); \ t \in [-h, 0], \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases} \tag{3.2}$$
where $g \in C(I)$ such that $g(t) = f(t, \wp_t, g(t))$.
Let $u, v \in C(I)$. Then, for each $t \in [-h, 0]$, we have
$$|(N\wp)(t) - (N\Im)(t)| = 0,$$
and for each $t \in I$, we have
$$\begin{aligned} |(N\wp)(t) - (N\Im)(t)| \ &\leq \ a_r|g(0) - h(0)| + a_r|g(t) - h(t)| \\ &\quad + b_r \int_0^t |g(s) - h(s)|ds \end{aligned}$$
where $g, \ h \in C(I)$ such that
$$g(t) = f(t, \wp_t, g(t)) \quad and \quad h(t) = f(t, \Im_t, h(t)).$$
From (H_1), we have
$$\begin{aligned} |g(t) - h(t)| \ &= \ |f(t, \wp_t, g(t)) - f(t, \Im_t, h(t))| \\ &\leq \ \omega_1\|\wp_t - \Im_t\|_{[-h,0]} + \omega_2|g(t) - h(t)|. \end{aligned}$$
This gives,
$$|g(t) - h(t)| \leq \frac{\omega_1}{1-\omega_2}\|\wp_t - \Im_t\|_{[-h,0]}.$$
Thus, for each $t \in I$, we get
$$\begin{aligned} |(N\wp)(t) - (N\Im)(t)| \ &\leq \ 2a_r\tfrac{\omega_1}{1-\omega_2}\|\wp_t - \Im_t\|_{[-h,0]} \\ &\quad + b_r \int_0^t \tfrac{\omega_1}{1-\omega_2}\|\wp_s - \Im_s\|_{[-h,0]}ds \\ &\leq \ 2a_r\tfrac{\omega_1}{1-\omega_2}\|\wp - \Im\|_C + Tb_r\tfrac{\omega_1}{1-\omega_2}\|\wp - \Im\|_C \\ &\leq \ \tfrac{\omega_1(2a_r+Tb_r)}{1-\omega_2}\|\wp - \Im\|_C \\ &\leq \ \ell\|\wp - \Im\|_C. \end{aligned}$$
Hence, we get
$$\|N(\wp) - N(\Im)\|_C \leq \ell\|\wp - \Im\|_C.$$
Since $\ell < 1$, the Banach contraction principle implies that problem (1.1) has a unique solution.

Theorem 3.3. *If* (H_1) *and* (H_2) *hold, and*

$$\frac{\omega_1}{1-\omega_2}(2a_r+Tb_r)<1,$$

then problem (1.1) *has at least one solution on* $[-h,T]$.

Proof. Consider the operator $N:\ C\to C$ defined in (3.2).
Let $R>0$ such that

$$R\geq \max\left\{\|\zeta\|_{C([-h,0])},\ \frac{|\zeta(0)|+\frac{f^*}{1-\omega_2}(2a_r+Tb_r)}{1-\frac{\omega_1}{1-\omega_2}(2a_r+Tb_r)}\right\}, \tag{3.3}$$

where $f^*:=\sup\limits_{t\in I}|f(t,0,0)|$.
Define the ball

$$B_R=\{x\in C(I,\mathbb{R}),\ \|x\|_C\leq R\}.$$

Step 1. N is continuous .
Let $\{\wp_n\}_n$ be a sequence such that $\wp_n\to\wp$ on B_R. For each $t\in[-h,0]$, we have

$$|(N\wp_n)(t)-(N\wp)(t)|=0,$$

and for each $t\in I$, we have

$$\begin{aligned} |(N\wp_n)(t)-(N\wp)(t)| &\leq a_r|g_n(0)-g(0)|+a_r|g_n(t)-g(t)| \\ &\quad +b_r\int_0^t|g_n(s)-g(s)|ds, \end{aligned} \tag{3.4}$$

where $g_n,\ g\in C(I)$ such that

$$g_n(t)=f(t,\wp_{nt},g_n(t))\quad and\quad g(t)=f(t,\wp_t,g(t)).$$

Since $\|\wp_n-\wp\|_C\to 0$ as $n\to\infty$ and f,g and g_n are continuous, then the Lebesgue dominated convergence theorem, implies that

$$\|N(\wp_n)-N(\wp)\|_C\to 0\quad \text{as } n\to\infty.$$

Hence, N is continuous.
Step 2. $N(B_R)\subset B_R$.
Let $\wp\in B_R$, If $t\in[-h,0]$ then $\|(N\wp)(t)\|\leq\|\zeta\|_C\leq R$. From (H_1), for each $t\in I$, we have

$$\begin{aligned} |g(t)| &= |f(t,\wp_t,g(t))| \\ &\leq |f(t,0,0)|+\omega_1\|\wp_t\|_{[-h,0]}+\omega_2|g(t)| \\ &\leq f^*+\omega_1\|\wp\|_C+\omega_2\|g\|_\infty \\ &\leq f^*+\omega_1 R+\omega_2\|g\|_\infty. \end{aligned}$$

Then

$$\|g\|_\infty\leq\frac{f^*+\omega_1R}{1-\omega_2}.$$

Thus,

$$\begin{aligned} |(N\wp)(t)| &\leq |\zeta(0)-a_rg(0)+a_rg(t)+b_r\int_0^t g(s)ds| \\ &\leq |\zeta(0)|+a_r|g(0)|+a_r|g(t)|+b_r\int_0^t|g(s)|ds \\ &\leq |\zeta(0)|+\frac{f^*+\omega_1R}{1-\omega_2}(2a_r+b_r\int_0^t ds) \\ &\leq |\zeta(0)|+\frac{f^*+\omega_1R}{1-\omega_2}(2a_r+Tb_r) \\ &\leq R. \end{aligned}$$

Hence
$$\|N(\wp)\|_C \leq R.$$
Consequently, $N(B_R) \subset B_R$.

Step 3. $N(B_R)$ is equicontinuous.
For $1 \leq t_1 \leq t_2 \leq T$, and $u \in B_R$, we have

$$\begin{array}{rcl} |N(\wp)(t_1) - N(\wp)(t_2)| & \leq & a_r|g(t_2) - g(t_1)| + b_r \int_{t_1}^{t_2} |g(s)|ds| \\ & \leq & a_r|g(t_2) - g(t_1)| + \frac{RKb_r}{1-L}(t_2 - t_1). \end{array}$$

Thus, from (H_2), $a_r|g(t_2) - g(t_1)| + \frac{RKb_r}{1-L}(t_2 - t_1) \to 0$; as $t_2 \to t_1$. This gives the equicontinuity of $N(B_R)$.

From the above steps and the Arzelá-Ascoli theorem, we conclude that N is continuous and compact. Consequently, from Schauder's theorem [12] we deduce that problem (1.1) has at least one solution.

4. Existence of solutions with infinite delay

In this section, we establish some existence results for problem (1.2). Let the space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, T]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato [13] for ordinary differential functional equations:

(A_1). If $\wp : (-\infty, T] \to \mathbb{R}$, and $\wp_0 = \zeta(0) \in \mathcal{B}$, then there exist constants $L, M, H > 0$, such that for each $t \in I$; we have:
- (i). $\wp_t$ *is in* $\mathcal{B}$,
- (ii). $\|\wp_t\|_{\mathcal{B}} \leq K\|\wp_0\|_{\mathcal{B}} + M \sup_{s\in[0,t]} |\wp(s)|$,
- (iii). $\|\wp(t)\| \leq H\|\wp_t\|_{\mathcal{B}}$.

(A_2). For the function $\wp(\cdot)$ *in* $(A1)$, u_t is a $\mathcal{B}-$ valued continuous function on I.
(A_3). The space $\mathcal{B}$ is complete.

Consider the space
$$\Omega = \{\wp : (-\infty, T] \to \mathbb{R},\ \wp|_{\mathbb{R}_-} \in \mathcal{B},\ \wp|_I \in C(I)\}.$$

Definition 4.1. By a solution of problem (1.2), we mean a continuous function $\wp \in \Omega$

$$\wp(t) = \begin{cases} \zeta(t);\ t \in \mathbb{R}_-, \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds;\ t \in I, \end{cases} \tag{4.1}$$

where $g \in C$ such that $g(t) = f(t, \wp_t, g(t))$.

Let us introduce the following hypotheses:

- (H_{01}) The function f satisfies the Lipschitz condition:
$$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq b_1\|\wp_1 - \wp_2\|_{\mathcal{B}} + b_2|\Im_1 - \wp_2|,$$
for any $\wp_1, \Im_1 \in \mathcal{B}$, $\wp_2, \Im_2 \in \mathbb{R}$, and each $t \in I$, where $b_1 > 0$ and $0 < b_2 < 1$.

- (H_{02}) For any bounded set $B_1 \subset \Omega$, the set:
$$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r \wp)(t)) : \wp \in B_1\};$$
is equicontinuous in Ω.

First, we prove an existence and uniqueness result by using the Banach's fixed point theorem.

Theorem 4.2. *Assume that the hypothesis (H_{01}) holds. If*
$$\lambda := (2a_r + Tb_r)\frac{b_1}{1-b_2} < 1, \tag{4.2}$$
then problem (1.2) has a unique solution on $(-\infty, T]$.

Proof. Consider the operator $N_1 : \ \Omega \to \Omega$ defined by:
$$(N_1\wp)(t) = \begin{cases} \zeta(t); \ t \in \mathbb{R}_-, \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases} \tag{4.3}$$
where $g \in C(I)$ such that $g(t) = f(t, \wp_t, g(t))$.
Let $x(\cdot) : (-\infty, T] \to \mathbb{R}$ be a function defined by
$$x(t) = \begin{cases} \zeta(t); & t \in \mathbb{R}_-, \\ \zeta(0)- & t \in I. \end{cases}$$
Then $x_0 = \zeta$, For each $z \in C(I)$, with $z(0) = 0$, we denote by $\overline{z}$ the function defined by
$$\overline{z} = \begin{cases} 0; & t \in t \in \mathbb{R}_-, \\ z(t), & t \in I. \end{cases}$$
If $\wp(\cdot)$ satisfies the integral equation
$$\wp(t) \quad = \quad \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds.$$
We can decompose $\wp(\cdot)$ as $\wp(t) = \overline{z}(t) + x(t)$; for $t \in I$, which implies that $\wp_t = \overline{z}_t + x_t$ for every $t \in I$, and the function $z(\cdot)$ satisfies
$$z(t) = -a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds,$$
where
$$g(t) = f(t, \overline{z}_t + x_t, g(t)); \ t \in I.$$
Set
$$C_0 = \{z \in C(I); \ z_0 = 0\},$$
and let $\|\cdot\|_T$ be the norm in C_0 defined by
$$\|z\|_T = \|z_0\|_{\mathcal{B}} + \sup_{t \in I} |z(t)| = \sup_{t \in I} |z(t)|; \ z \in C_0.$$
C_0 is a Banach space with norm $\|\cdot\|_T$. Define the operator $P : C_0 \to C_0$; by
$$(Pz)(t) = -a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds, \tag{4.4}$$

where

$$g(t) = f(t, \overline{z}_t + x_t, g(t));\ t \in I.$$

We shall show that $P : C_0 \to C_0$ is a contraction map. Let $z, z' \in C_0$, then we have for each $t \in I$

$$|P(z)(t) - P(z')(t)| \quad \leq \quad a_r|g(0) - h(0)| + a_r|g(t) - h(t)| + b_r \int_0^t |g(s) - h(s)|ds, \tag{4.5}$$

where $g,\ h \in C(I)$ such that

$$g(t) = f(t, \overline{z}_t + x_t, g(t)) \quad and \quad h(t) = f(t, \overline{z'}_t + x_t, h(t)).$$

Since, for each $t \in I$, we have

$$|g(t) - h(t)| \leq \frac{b_1}{1 - b_2}\|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}}.$$

Then, for each $t \in I$; we get

$$\begin{aligned} |P(z)(t) - P(z')(t)| &\leq (2a_r + b_r \int_0^t ds)\tfrac{b_1}{1-b_2}\|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}} \\ &\leq (2a_r + Tb_r)\tfrac{b_1}{1-b_2}\|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}} \\ &= \lambda\|\overline{z} - \overline{z'}\|_T. \end{aligned}$$

Thus, we get

$$\|P(z)(t) - P(z')(t)\|_T \leq \lambda\|\overline{z} - \overline{z'}\|_T.$$

Hence, from the Banach contraction principle, the operator P has a unique fixed point. Consequently, N has a unique fixed point which is the unique solution of problem (1.2).

Now, we prove an existence result by using Schaefer's fixed point theorem.

Theorem 4.3. *Assume that the hypotheses* (H_{01}) *and* H_{02} *hold. Then problem (1.2) has at least one solution on* $(-\infty, T]$.

Proof. Let $P : C_0 \to C_0$ defined as in (4.4), For each given $R > 0$, we define the ball

$$B_R = \{x \in C_0,\ \|x\|_T \leq R\}.$$

Step 1. N is continuous.
Let z_n be a sequence such that $z_n \to z$ in C_0. For each $t \in I$, we have

$$\begin{aligned} |(Pz_n)(t) - (Pz)(t)| &\leq a_r|g_n(0) - g(0)| + a_r|g_n(t) - g(t)| \\ &+ b_r \int_0^t |g_n(s) - g(s)|ds, \end{aligned} \tag{4.6}$$

where $g_n,\ g \in C(I)$ such that

$$g_n(t) = f(t, \overline{z}_{nt} + x_t, g_n(t)) \quad and \quad g(t) = f(t, \overline{z}_t + x_t, g(t)).$$

Since $\|z_n - z\|_T \to 0$ as $n \to \infty$ and f, g and g_n are continuous, then

$$\|P(\wp_n) - P(\wp)\|_T \to 0 \quad \text{as } n \to \infty.$$

Hence, P is continuous.

Step 2. $P(B_R)$ is bounded.
Let $z \in B_R$, for each $t \in I$, we have

$$\begin{array}{rcl} |g(t)| & \leq & |f(t, \overline{z}_t + x_t, g(t))| \\ & \leq & |f(t,0,0)| + b_1\|\overline{z}_t + x_t\|_{\mathcal{B}} + b_2|g(t)| \\ & \leq & f^* + b_1\left[\|\overline{z}_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}\right] + b_2\|g\|_\infty \\ & \leq & f^* + b_1MR + b_1K\|\zeta\|_{\mathcal{B}} + b_2\|g\|_\infty. \end{array}$$

Then

$$\|g\|_\infty \leq \frac{f^* + b_1MR + b_1K\|\zeta\|_{\mathcal{B}}}{1-b_2}.$$

Thus,

$$\begin{array}{rcl} |(Pz)(t)| & \leq & a_r|g(0)| + a_r|g(t)| + b_r\int_0^t |g(s)|ds \\ & \leq & (2a_r + b_r\int_0^t ds)\frac{f^*+b_1MR+b_1K\|\zeta\|_{\mathcal{B}}}{1-b_2} \\ & \leq & (2a_r + Tb_r)\frac{f^*+b_1MR+b_1K\|\zeta\|_{\mathcal{B}}}{1-b_2} \\ & := & \ell. \end{array}$$

Hence

$$\|P(z)\|_T \leq \ell.$$

Consequently, P maps bounded sets into bounded sets in C_0.
Step 3. $P(B_R)$ is equicontinuous.
For $1 \leq t_1 \leq t_2 \leq T$, and $z \in B_R$, we have

$$\begin{array}{rcl} |P(z)(t1) - P(z)(t2)| & \leq & a_r|g(t_2) - g(t_1)| + b_r\int_{t_1}^{t_2} |g(s)|ds \\ & \leq & a_r|g(t_2) - g(t_1)| + b_r(t_2 - t_1)\frac{f^*+b_1MR+b_1K\|\zeta\|_{\mathcal{B}}}{1-b_2}. \end{array}$$

By (H_{02}), as $t_2 \to t_1$ the right-hand side of the above inequality tends to zero, we conclude that P maps bounded sets into equicontinuous sets in C_0.
Step 4. The priori bounds.
We prove that the set

$$\mathcal{E} = \{\wp \in C_0 : \Im = \lambda P(\wp);\ for\ some\ \lambda \in (0,1)\}$$

is bounded. Let $z \in C_0$. Let $u \in C_0$, such that $z = \lambda P(z);\ for\ some\ \lambda \in (0,1)$. Then for each $t \in I$, we have

$$z(t) = \lambda(Pz)(t) = \lambda\zeta(0) + \lambda a_r(g(t) - g(0)) + \lambda b_r\int_0^t g(s)ds.$$

From (H_{01}) we have

$$\begin{array}{rcl} |g(t)| & \leq & |f(t, \overline{z}_t + x_t, g(t))| \\ & \leq & f^* + b_1\|\overline{z}_t + x_t\|_{\mathcal{B}} + b_2|g(t)| \\ & \leq & f^* + b_1\left[\|\overline{z}_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}\right] + b_2\|g\|_\infty \\ & \leq & f^* + b_1M\|z\|_T + b_1K\|\zeta\|_{\mathcal{B}} + b_2\|g\|_\infty. \end{array}$$

This gives,

$$\|g\|_\infty \leq \frac{f^* + b_1M\|z\|_T + b_1K\|\zeta\|_{\mathcal{B}}}{1-b_2} := \eta.$$

Thus, for each $t \in I$, we obtain

$$\begin{array}{rcl} |z(t)| & \leq & |\zeta(0)| + a_r|g(0)| + a_r g(t) + b_r \int_0^t |g(s)|ds \\ & \leq & |\zeta(0)| + \eta(2a_r + Tb_r) \\ & := & \eta'. \end{array}$$

Hence

$$\|z\|_T \leq \eta'.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's theorem [12], the operator N has a fixed point which is a solution of problem (1.2).

5. Existence results with state-dependent delay

5.1. The finite delay case

In this section, we establish the existence results for problem (1.3).

Definition 5.1. By a solution of problem (1.3), we mean a continuous function $\wp \in C$ such that

$$\wp(t) = \begin{cases} \zeta(t);\ t \in [-h, 0], \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds;\ t \in I, \end{cases}$$

where $g \in C(I)$ with $g(t) = f(t, \wp_{\rho(t,\wp_t)}, g(t))$.

- (H_4) The function f satisfies the Lipschitz condition:

  $$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq \omega_3\|\wp_1 - \wp_2\|_{[-h,0]} + \omega_4|\Im_1 - \Im_2|,$$

  for any $\wp_1, \Im_1 \in \mathcal{C},\ \wp_2, \Im_2 \in \mathbb{R}$, and each $t \in I$,
  where $\omega_3 > 0,\ 0 < \omega_4 < 1$.
- (H_5) For any bounded set $B_2 \subset \mathcal{C}$, the set:

  $$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r\wp)(t)) : \wp \in B_2\};$$

  is equicontinuous in C.

As in Theorems 3.2 and 3.3, we give without prove, the following results:

Theorem 5.2. *Assume that the hypothesis (H_4) holds. If*

$$(2a_r + Tb_r)\frac{\omega_3}{1 - \omega_4} < 1,$$

then problem (1.2) has a unique solution on $[-h, T]$.

Theorem 5.3. *Assume that the hypotheses (H_4) and (H_5) hold. If*

$$\frac{a_1}{1 - a_2}(2a_r + Tb_r) < 1,$$

then problem (1.3) has at least one solution on $[-h, T]$.

5.2. The infinite delay case

Now, we establish the last problem (1.4).

Definition 5.4. By a solution of problem (1.4), we mean a continuous $\wp \in \Omega$

$$\wp(t) = \begin{cases} \zeta(t); \ t \in \mathbb{R}_-, \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases} \tag{5.1}$$

where $g \in C(I)$ such that $g(t) = f(t, \wp_{\rho(t,\wp_t)}, g(t))$.

Set

$$R' := R'_{\rho^-} = \{\rho(t,\wp) : t \in I, \ \wp \in \mathcal{B} \ \rho(t,\wp) < 0\}$$

We always assume that $\rho : I \times \mathcal{B} \to \mathbb{R}$ is continuous and the function $t \to \wp_t$ is continuous from R' into $\mathcal{B}$. We will need the following hypothesis:
(H_ζ) There exists a continuous bounded function $L : R'_{\rho^-} \to (0, \infty)$ such that

$$\|\zeta_t\|_{\mathcal{B}} \le L(t)\|\zeta\|_{\mathcal{B}}, \ for \ any \ t \in R'.$$

Lemma 5.5. *If $\wp \in \Omega$ then*

$$\|\wp_t\|_{\mathcal{B}} = (M + L')\|\zeta\|_{\mathcal{B}} + K \sup_{\theta \in [0, \max\{0,t\}]} \|\wp(\theta)\|,$$

where

$$L' = \sup_{t \in R'} L(t).$$

- (H_{04}) The function f satisfies the Lipschitz condition:

  $$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \le b_3\|\wp_1 - \wp_2\|_{\mathcal{B}} + b_4|\Im_1 - \Im_2|,$$

  for any $\wp_1, \Im_1 \in \mathcal{B}$, $\wp_2, \Im_2 \in \mathbb{R}$, and each $t \in I$, where $b_3 > 0$ and $0 < b_4 < 1$.
- (H_{05}) For any bounded set $B_2 \subset \Omega$, the set:

  $$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r \wp)(t)) : u \in B_2\};$$

  is equicontinuous in Ω.

As in Theorems 4.2 and 4.3, we give without prove, the following results:

Theorem 5.6. *Assume that the hypothesis (H_{04}) holds. If*

$$(2a_r + Tb_r)\frac{b_3}{1 - b_4} < 1,$$

then problem (1.4) has a unique solution on $(-\infty, T]$.

Theorem 5.7. *Assume that the hypotheses (H_ζ), (H_{04}) and (H_{05}) hold. Then problem (1.4) has at least one solution on $(-\infty, T]$.*

6. Some examples

Example 6.1. Consider the following problem

$$\begin{cases} \wp(t) = 1 + t^2; \ t \in [-1, 0], \\ ({}^{CF}D_0^{1/2}\wp)(t) = \frac{\varsigma}{90(1+\|\wp_t\|)} + \frac{1}{30\left(1+|({}^{CF}D_0^{1/2}\wp(t))|\right)}; \ t \in [0, 2], \end{cases} \tag{6.1}$$

where $\varsigma < \frac{87}{2a_{\frac{1}{2}}+2b_{\frac{1}{2}}}$.

Set

$$f(t, \wp, \Im) = \frac{\varsigma}{90\,(1 + \|\wp\|)} + \frac{1}{30\,(1 + |\Im|)}; \ t \in [1, e], \ \wp \in \mathcal{C}, \ \Im \in \mathbb{R}.$$

Clearly, the function f is continuous. For any $\wp, \widetilde{\wp} \in \mathcal{C}$, $\wp, \widetilde{\wp} \in \mathbb{R}$, and $t \in [0, 2]$, we have

$$|f(t, \wp, \Im) - f(t, \widetilde{\wp}, \widetilde{\Im})| \leq \frac{\varsigma}{90}\|\wp - \widetilde{\wp}\|_{[-1,0]} + \frac{1}{30}|\Im - \widetilde{\Im}|.$$

Hence hypothesis (H_1) is satisfied with

$$\omega_1 = \frac{\varsigma}{90} \quad and \quad \omega_2 = \frac{1}{30}.$$

Next, condition (3.1) is satisfied with $T = 2$ and $r = \frac{1}{2}$. Indeed,

$$\begin{aligned} \tfrac{\omega_1(2a_r+Tb_r)}{1-\omega_2} &= \tfrac{\varsigma\left(2a_{\frac{1}{2}}+2b_{\frac{1}{2}}\right)}{87} \\ &< 1. \end{aligned}$$

Theorem 3.2 implies that problem (6.1) has a unique solution defined on $[-1, 2]$.

Example 6.2. Consider now the following problem

$$\begin{cases} \wp(t) = t; \ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^{2/3}\wp)(t) = \frac{\wp_t e^{-\gamma t+t}}{180(e^t-e^{-t})(1+\|\wp_t\|)} + \frac{\wp(t)e^{-\gamma t+t}}{60(e^t-e^{-t})\left(1+|({}^{CF}D_0^{2/3}\wp(t))|\right)}; \ t \in [0, 1]. \end{cases} \tag{6.2}$$

Let γ be a positive real constant and

$$B_\gamma = \{\wp \in C((-\infty, 1], \mathbb{R},) : \lim_{\theta \to -\infty} e^{\gamma\theta\theta}\wp(\theta) \ exists \ in \ \mathbb{R}\}. \tag{6.3}$$

The norm of B_γ is given by

$$\|\wp\|_\gamma = \sup_{\theta \in (-\infty, 1]} e^{\gamma\theta}|\wp(\theta)|.$$

Let $\wp : \mathbb{R}_- \to \mathbb{R}$ be such that $\wp_0 \in B_\gamma$. Then

$$\begin{aligned} \lim_{\theta \to -\infty} e^{\gamma\theta}\wp_t(\theta) &= \lim_{\theta \to -\infty} e^{\gamma\theta}\wp(t + \theta - 1) = \lim_{\theta \to -\infty} e^{\gamma(\theta-t+1)}\wp(\theta) \\ &= e^{\gamma(-t+1)} \lim_{\theta \to -\infty} e^{\gamma(\theta)}\wp_1(\theta) < \infty. \end{aligned}$$

Hence $\wp_t \in B_\gamma$. Finally we prove that

$$\|\wp_t\|_\gamma \leq K\|\wp_1\|_\gamma + M \sup_{s \in [0,t]} |\wp(s)|,$$

where $K = M = 1$ and $H = 1$. We have

$$\|\wp_t(\theta)\| = |\wp(t+\theta)|.$$

If $t+\theta \leq 1$, we get

$$\|\wp_t(\beta)\| \leq \sup_{s\in\mathbb{R}_-} |\wp(s)|.$$

For $t+\theta \geq 0$, then we have

$$\|\wp_t(\beta)\| \leq \sup_{s\in[0,t]} |\wp(s)|.$$

Thus for all $t+\theta \in I$, we get

$$\|\wp_t(\beta)\| \leq \sup_{s\in\mathbb{R}_-} |\wp(s)| + \sup_{s\in[0,t]} |\wp(s)|.$$

Then

$$\|\wp_t\|_\gamma \leq \|\wp_0\|_\gamma + \sup_{s\in[0,t]} |\wp(s)|.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ a phase space. Set

$$f(t,\wp,\Im) = \frac{e^{-\gamma t+t}}{180\left(e^t - e^{-t}\right)\left(1+\|\wp\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\left(e^t-e^{-t}\right)\left(1+|\Im|\right)};$$
$$t\in[0,1],\ \wp\in B_\gamma, \Im\in\mathbb{R}.$$

We can verify that the hypothesis (H_{01}) is satisfied with

$$B_1 = \frac{1}{180} \quad \text{and} \quad B_2 = \frac{1}{60}.$$

Theorem 4.3 ensures that problem (6.2) has a solution defined on $(-\infty, 1]$.

Example 6.3. We consider the following problem

$$\begin{cases} \wp(t) = 1+t^2;\ t\in[-1,0], \\ ({}^{CF}D_0^{1/2}\wp)(t) = \frac{1}{90(1+|\wp(t-\sigma(\wp(t)))|)} + \frac{1}{30\left(1+|({}^{CF}D_0^{1/2}\wp(t))|\right)};\ t\in[0,1], \end{cases} \tag{6.4}$$

where $\sigma \in C(\mathbb{R},[0,1])$. Set

$$\rho(t,\zeta) = t - \sigma(\zeta(0)),\quad (t,\zeta)\in[0,e]\times C([-1,0],\mathbb{R}),$$

$$f(t,\wp,\Im) = \frac{1}{90(1+|\wp(t-\sigma(\wp(t)))|)} + \frac{1}{30\left(1+|\Im(t)|\right)};\ t\in[1,e],\ \wp\in\mathcal{C},\ \Im\in\mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $\wp,\widetilde{\wp}\in\mathcal{C},\ \Im,\widetilde{\Im}\in\mathbb{R}$ and $t\in[0,1]$, we have

$$|f(t,\wp,\Im) - f(t,\widetilde{\wp},\widetilde{\Im})| \leq \frac{1}{90}\|\wp-\widetilde{\wp}\|_{[-1,0]} + \frac{1}{30}|\Im-\widetilde{\Im}|.$$

Hence hypothesis (H_{04}) is satisfied with

$$\omega_3 = \frac{1}{90} \quad \textit{and} \quad \omega_4 = \frac{1}{30}.$$

From Theorem 5.2, problem (6.4) has a unique solution on $[-1,1]$.

Example 6.4. Consider now the problem

$$\begin{cases} \wp(t) = t^2; \ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^{1/4}\wp)(t) = \frac{\wp(t-\lambda(\wp(t)))e^{-\gamma t+t}}{180(e^t-e^{-t})(1+|\wp(t-\sigma(\wp(t))|)} + \frac{\wp(t)e^{-\gamma t+t}}{60(e^t-e^{-t})\left(1+|({}^{CF}D_0^{1/4}\wp(t))|\right)}; \ t \in [0,3]. \end{cases} \tag{6.5}$$

Let γ be a positive real constant and the phase space B_γ defined in Example 6.2. Define

$$\rho(t,\zeta) = t - \lambda(\zeta(0)), \quad (t,\zeta) \in [0,3] \times B_\gamma,$$

and set

$$f(t,\wp,\Im) = \frac{e^{-\gamma t+t}}{180\,(e^t - e^{-t})\left(1 + \|\wp\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\,(e^t - e^{-t})\,(1 + |\Im|)};$$

$$t \in [0,3], \ \wp \in B_\gamma, \ \Im \in \mathbb{R}.$$

By Theorem 4.3, problem (6.5) has a solution defined on $(-\infty, 3]$.

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References

[1] Abbas, S., Benchohra, M., Graef, J.G., Henderson, J., *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.

[2] Abbas, S., Benchohra, M., N'Guérékata, G.M., *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[3] Abbas, S., Benchohra, M., N'Guérékata, G.M., *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[4] Abbas, S., Benchohra, M., Nieto, J.J., *Functional implicit hyperbolic fractional order differential equations with delay*, Afr. Diaspora J. Math., **15**(2013), no. 1, 74-96.

[5] Abbas, S., Benchohra, M., Vityuk, A.N., *On fractional order derivatives and Darboux problem for implicit differential equations*, Frac. Calc. Appl. Anal., **15**(2012), 168-182.

[6] Albarakati, W., Benchohra, M., Lazreg, J.E., Nieto, J.J., *Anti-periodic boundary value problem for nonlinear implicit fractional differential equations with impulses*, Analele Univ. Oradea Fasc. Mat., **25**(2018), no. 1, 13-24.

[7] Appell, J., *Implicit functions, nonlinear integral equations, and the measure of noncompactness of the superposition operator*, J. Math. Anal. Appl., **83**(1981), 251-263.

[8] Benchohra, M., Bouriah, S., Henderson, J., *Existence and stability results for nonlinear implicit neutral fractional differential equations with finite delay and impulses*, Comm. Appl. Nonlinear Anal., **22**(2015), no. 1, 46-67.

[9] Benchohra, M., Lazreg, J.E., *On stability for nonlinear implicit fractional differential equations*, Matematiche (Catania), **70**(2015), no. 2, 49-61.

[10] Benchohra, M., Lazreg, J.E., *Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivatives*, Stud. Univ. Babeş-Bolyai Math., **62**(2017), no. 1, 27-38.

[11] Caputo, M., Fabrizio, M., *A new definition of fractional derivative without singular kernel*, Prog. Fract. Differ. Appl., **2015**, no. 1, 73-85.

[12] Granas, A., Dugundji, J., *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[13] Hale, J.K., Kato, J., *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac., **21**(1978), 11-41.

[14] Hale, J.K., Verduyn Lunel, S.M., *Introduction to Functional-Differential Equations*, Springer-Verlag, New York, 1991.

[15] Hernández, E., *On abstract differential equations with state dependent non-local conditions*, J. Math. Anal. Appl., **466**(2018), no. 1, 408-425.

[16] Hernández, E., Azevedo, K.A.J., Rolnik, V., *Wellposedness of abstract differential equations with state dependent delay*, Math. Nachrichten, **291**(2018), no. 13, 2045-2056.

[17] Hernández, E., Fernandes, D., Wu, J., *Well-posedness of abstract integro-differential equations with state-dependent delay*, Proc. Amer. Math. Soc., **148**(2020), no. 4, 1595-1609.

[18] Hino, Y., Murakami, S., Naito, T., *Functional Differential Equations with Infinite Delay*, Lecture Notes in Math., **1473**, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[19] Hino, Y., Murakami, S., Naito, T., Minh, N.V., *A variation-of-constants formula for abstract functional differential equations in phase space*, J. Differential Equations, **179**(2002), 336-355.

[20] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[21] Krim, S., Abbas, S., Benchohra, M., Darwish, M.A., *Boundary value problem for implicit Caputo–Fabrizio fractional differential equations*, Int. J. Difference Equ., **15**(2020), no. 2, 493-510.

[22] Kucche, K.D., Nieto, J.J., Venktesh, V., *Theory of nonlinear implicit fractional differential equations*, Differ. Equ. Dynam. Syst., **28**(2020), 117.

[23] Losada, J., Nieto, J.J., *Properties of a new fractional derivative without singular kernel*, Prog. Fract. Differ. Appl., **2015**, no. 1, 87-92.

[24] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam, 1987, Engl. Trans. from the Russian.

[25] Tarasov, V.E., *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[26] Tenreiro Machado, J.A., Kiryakova, V., *The chronicles of fractional calculus*, Fract. Calc. Appl. Anal., **20**(2017), 307-336.

[27] Vityuk, A.N., Mykhailenko, A.V., *The Darboux problem for an implicit fractional-order differential equation*, J. Math. Sci., **175**(2011), no. 4, 391-401.

[28] Zhou, Y., *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

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