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On a Fredholm-Volterra integral equation

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Abstract. In this paper we give conditions in which the integral equation

$$x(t) = \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds + g(t),\ t \in [a,b],$$

where $a < c < b$, $K \in C([a,b] \times [a,c] \times \mathbb{B}, \mathbb{B})$, $H \in C([a,b] \times [a,b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a,b], \mathbb{B})$, with $\mathbb{B}$ a (real or complex) Banach space, has a unique solution in $C([a,b], \mathbb{B})$. An iterative algorithm for this equation is also given.

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1. Introduction

The following type of integral equation was studied by several authors (see [11], [2], [3], [6], [1], [5], [10], [7], ...),

$$x(t) = \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds + g(t),\ t \in [a,b], \tag{1.1}$$

where $a < c < b$, $K \in C([a,b] \times [a,c] \times \mathbb{B}, \mathbb{B})$, $H \in C([a,b] \times [a,b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a,b], \mathbb{B})$, with $(\mathbb{B}, |\cdot|)$ a (real or complex) Banach space.

The aim of this paper is to give some conditions on K and H in which the equation (1.1) has a unique solution in $C([a,b], \mathbb{B})$. To do this, we shall use the contraction principle, the fiber contraction principle ([9], [13], [10], [11]) and a variant of Maia fixed point theorem given in [8] (see also [4]).

2. Preliminaries

Let us recall some notions, notations and fixed point results which will be used in this paper.

2.1. Picard operators and weakly Picard operators

Let $(X,\to)$ be an L-space $((X,d),\overset{d}{\to};(X,\tau),\overset{\tau}{\to};(X,\|\cdot\|),\overset{\|\cdot\|}{\to},\rightharpoonup;\ldots)$. An operator $A:(X,\to)\to(X,\to)$ is called weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x\in X$ and the limit (which generally depends on x) is a fixed point of A.

If an operator A is WPO and the fixed point set of A is a singleton, i.e.,

$$F_A=\{x^*\},$$

then, by definition, A is called Picard operator (PO).

For a WPO, $A:(X,\to)\to(X,\to)$, we define the limit operator $A^\infty:(X,\to)\to(X,\to)$, by $A^\infty(x):=\lim\limits_{n\to\infty}A^n(x)$. We remark that, $A^\infty(X)=F_A$, i.e., A^∞ is a set retraction of X on F_A.

2.2. Fiber contraction principle

Regarding this principle, some important results were given in [12] and [13].

Fiber Contraction Theorem. *Let $(X,\to)$ be an L-space, (Y,d) be a metric space, $B:X\to X$, $C:X\times Y\to Y$ and $A:X\times Y\to X\times Y$, $A(x,y):=(B(x),C(x,y))$. We suppose that:*

(i) (Y,d) is a complete metric space;
(ii) B is a WPO;
(iii) $C(x,\cdot):Y\to Y$ is an l-contraction, for all $x\in X$;
(iv) $C:X\times Y\to Y$ is continuous.

Then A is a WPO. Moreover, if B is a PO, then A is a PO.

Generalized Fiber Contraction Theorem. *Let $(X,\to)$ be an L-space and (X_i,d_i), $i=\overline{1,m}$, $m\geq 1$ be metric spaces. Let $A_i:X_0\times\ldots\times X_i\to X_i$, $i=\overline{0,m}$, be some operators. We suppose that:*

(i) (X_i,d_i), $i=\overline{1,m}$, are complete metric spaces;
(ii) A_0 is a WPO;
(iii) $A_i(x_0,\ldots,x_{i-1},\cdot):X_i\to X_i$, $i=\overline{1,m}$, are l_i-contractions;
(iv) A_i, $i=\overline{1,m}$, are continuous.

Then the operator $A:X_0\times\ldots\times X_m\to X_0\times\ldots\times X_m$, defined by

$$A(x_0,\ldots,x_m):=(A_0(x_0),A_1(x_0,x_1),\ldots,A_m(x_0,\ldots,x_m))$$

is a WPO. Moreover, if A_0 is a PO, then A is a PO.

2.3. A variant of Maia fixed point theorem

We recall here the following variant of Maia fixed point theorem, given by I.A. Rus in [8]:

Theorem 2.1. *Let X be a nonempty set, d and ρ be two metrics on X and $A:X\to X$ be an operator. We suppose that:*

(1) there exists $c>0$ such that $d(A(x),A(y))\leq c\rho(x,y)$, for all $x,y\in X$;
(2) (X,d) is a complete metric space;
(3) $A:(X,d)\to(X,d)$ is continuous;

(4) $A : (X, \rho) \to (X, \rho)$ *is an* l*-contraction.*

Then:

(i) $F_A = \{x^*\}$;

(ii) $A : (X, d) \to (X, d)$ *is PO.*

3. Operatorial point of view on equation (1.1)

Let $X := C([a, b], \mathbb{B})$ and $T : X \to X$ be defined by

$$T(x)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds + g(t),\ t \in [a, b].$$

For $x \in X$, we denote by $u := x\big|_{[a,c]}$ and $v := x\big|_{[c,b]}$. If x is a solution of the equation (1.1) (i.e. a fixed point of T), then

$$u(t) = \int_a^c K(t, s, u(s))ds + \int_a^t H(t, s, u(s))ds + g(t),\ t \in [a, c] \tag{3.1}$$

and

$$v(t) = \int_a^c K(t, s, u(s))ds + \int_a^c H(t, s, u(s))ds + \int_c^t H(t, s, v(s))ds + g(t),\ t \in [c, b]. \tag{3.2}$$

Let $X_1 := C([a, c], \mathbb{B})$, $X_2 := C([c, b], \mathbb{B})$ and
$T_1 : X_1 \to X_1$, $T_1(u)(t) :=$ *the second part of* (3.1),
$T_2 : X_1 \times X_2 \to X_2$, $T_2(u, v)(t) :=$ *the second part of* (3.2).
The mappings T_1 and T_2 allow us to construct the triangular operator

$$\tilde{T} : X_1 \times X_2 \to X_1 \times X_2,\ \tilde{T}(u, v) := (T_1(u), T_2(u, v)),\ \text{for all } (u, v) \in X_1 \times X_2.$$

Remark 3.1. If $(u^*, v^*) \in F_{\tilde{T}}$, then $u^*(c) = v^*(c)$. So the function $x^* \in X$, defined by

$$x^*(t) := \begin{cases} u^*(t),\ t \in [a, c] \\ v^*(t),\ t \in [c, b] \end{cases}$$

is a fixed point of T, i.e., a solution of (1.1).

Remark 3.2. For $(u_0, v_0) \in X_1 \times X_2$ we consider the successive approximations corresponding to the operator $\tilde{T}$, $(u_{n+1}, v_{n+1}) = \tilde{T}(u_n, v_n)$, $n \in \mathbb{N}$. We observe that, for $n \in \mathbb{N}^*$, $u_n(c) = v_n(c)$. So, the function x_n, defined by

$$x_n(t) := \begin{cases} u_n(t),\ t \in [a, c] \\ v_n(t),\ t \in [c, b] \end{cases}$$

is in X.

Remark 3.3. Let $Y \subset X_1 \times X_2$ be defined by

$$Y := \{(u,v) \in X_1 \times X_2 \mid u(c) = v(c)\}.$$

The operator $R : X \to Y$, defined by $R(x) := (x\big|_{[a,c]}, x\big|_{[c,b]})$ is a bijection. From the above definitions, it is clear that $T(x) = (R^{-1}\tilde{T}R)(x)$ and the n^{th} iterate of T is $T^n = R^{-1}\tilde{T}^n R$.

In conclusion, to study the equation (1.1) (which is equivalent with $x = T(x)$) it is sufficient to study the fixed point of the operator $\tilde{T}$. If $(u^*, v^*) \in F_{\tilde{T}}$ then $R^{-1}(u^*, v^*) \in F_T$.

4. Existence and uniqueness of solution of equation (1.1)

In what follows, in addition to the continuity of H, K and g, we suppose on K and H that:

(i) There exists $L_1 \in C([a,b] \times [a,c], \mathbb{B})$ such that:

$$|K(t,s,\xi) - K(t,s,\eta)| \le L_1(t,s)|\xi - \eta|, \text{ for all } t \in [a,b],\ s \in [a,c],\ \xi, \eta \in \mathbb{B}.$$

(ii) There exists $L_2 \in C([a,b] \times [a,b], \mathbb{B})$ such that:

$$|H(t,s,\xi) - H(t,s,\eta)| \le L_2(t,s)|\xi - \eta|, \text{ for all } t,s \in [a,b],\ \xi, \eta \in \mathbb{B}.$$

(iii) $\displaystyle\left(\int_{[a,c]\times[a,c]} \big(L_1(t,s) + L_2(t,s)\big)^2 dtds\right)^{\frac{1}{2}} < 1.$

The basic result of our paper is the following.

Theorem 4.1. *In the above conditions we have that:*

(1) *The equation* (1.1) *has in* $C([a,b], \mathbb{B})$ *a unique solution* x^*.

(2) *The operator* $\tilde{T}$ *is a Picard operator with respect to* $\overset{unif.}{\to}$. *Let* $F_{\tilde{T}} = \{(u^*, v^*)\}$.

(3) *The operator* T *is a Picard operator with respect to* $\overset{unif.}{\to}$ *and* $F_T = \{x^*\}$. *Moreover,* $x^* = R^{-1}(u^*, v^*)$.

Proof. From the remarks which were given in §3, it is sufficient to prove that the operator $\tilde{T}$ is a Picard operator with respect to the uniform convergence on $X_1 \times X_2$.

In order to apply the Fiber contraction principle, we shall prove that:

(j) $T_1 : (X_1, \overset{unif.}{\to}) \to (X_1, \overset{unif.}{\to})$ is a Picard operator;

(jj) $T_2(u, \cdot) : (X_2, \|\cdot\|_\tau) \to (X_2, \|\cdot\|_\tau)$ is a contraction.

Let us prove (j).

We consider on X_1, the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{L^2}$. By using the assumptions (i) and (ii), we have the following estimations:

$$|T_1(u_1)(t) - T_1(u_2)(t)| \le \int_a^c |K(t,s,u_1(s)) - K(t,s,u_2(s))|ds$$
$$+ \int_a^t |H(t,s,u_1(s)) - H(t,s,u_2(s))|ds$$

$$\leq \int_a^c L_1(t,s)|u_1(s)-u_2(s)|ds + \int_a^c L_2(t,s)|u_1(s)-u_2(s)|ds$$

$$\overset{\text{Hölder's inequality}}{\leq} \left(\int_a^c L_1(t,s)^2 ds\right)^{\frac{1}{2}} \left(\int_a^c |u_1(s)-u_2(s)|^2 ds\right)^{\frac{1}{2}}$$

$$+ \left(\int_a^c L_2(t,s)^2 ds\right)^{\frac{1}{2}} \left(\int_a^c |u_1(s)-u_2(s)|^2 ds\right)^{\frac{1}{2}}.$$

By taking the $\max\limits_{t\in[a,c]}$ in the above inequalities, there exists a real positive constant

$$c := \max_{t\in[a,c]} \left(\int_a^c L_1(t,s)^2 ds\right)^{\frac{1}{2}} + \max_{t\in[a,c]} \left(\int_a^c L_2(t,s)^2 ds\right)^{\frac{1}{2}}$$

such that

$$\|T_1(u_1) - T_1(u_2)\|_\infty \leq c\|u_1 - u_2\|_{L^2}, \text{ for all } u_1, u_2 \in X_1.$$

On the other hand, we have that

$$\begin{aligned}
\|T_1(u_1) - T_1(u_2)\|_{L^2} &= \left(\int_a^c |T_1(u_1)(t) - T_1(u_2)(t)|^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_a^c \left(\int_a^c (L_1(t,s)ds + L_2(t,s))^2 ds\right) \|u_1 - u_2\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\
&= \left(\int_a^c \int_a^c (L_1(t,s) + L_2(t,s))^2 \, dsdt\right)^{\frac{1}{2}} \|u_1 - u_2\|_{L^2}, \\
&\text{for all } u_1, u_2 \in X_1.
\end{aligned}$$

By using the assumption (iii), it follows that the operator T_1 is a contraction with respect to $\|\cdot\|_{L^2}$ on X_1.
The conclusion follows from the variant of Maia theorem.
Let us prove (jj).
For $t \in [c,b]$ and $M_{L_2} := \max\limits_{t,s\in[c,b]} L_2(t,s)$, we have that

$$\begin{aligned}
|T_2(u,v_1)(t) - T_2(u,v_2)(t)| &\leq \int_c^t |H(t,s,v_1(s)) - H(t,s,v_2(s))|ds \\
&\leq \int_c^t L_2(t,s)|v_1(s) - v_2(s)|ds \\
&\leq M_{L_2} \int_c^t |v_1(s) - v_2(s)|e^{-\tau(s-c)} e^{\tau(s-c)} ds \\
&\leq M_{L_2}\|v_1 - v_2\|_\tau \int_c^t e^{\tau(s-c)} ds \leq M_{L_2}\|v_1 - v_2\|_\tau \frac{e^{\tau(t-c)}}{\tau}.
\end{aligned}$$

It follows that

$$|T_2(u,v_1)(t) - T_2(u,v_2)(t)|e^{-\tau(t-c)} \leq \frac{M_{L_2}}{\tau}\|v_1 - v_2\|_\tau.$$

By taking $\max_{t\in[c,b]}$ and by choosing $\tau > M_{L_2}$, there exists a real positive constant

$$l := \frac{M_{L_2}}{\tau} < 1$$

such that

$$\|T_2(u,v_1) - T_2(u,v_2)\|_\tau \leq l\|v_1 - v_2\|_\tau, \text{ for all } v_1, v_2 \in X_2. \qquad \square$$

Remark 4.2. Let $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, $|\cdot|$ be a norm on $\mathbb{B} := \mathbb{K}^m$ ($|\cdot|_1$, $|\cdot|_2$, $|\cdot|_\infty$, $\ldots$), $a < c < b$, $K = (K_1, \ldots, K_m) \in C([a,b], \mathbb{K}^m)$ and $H = (H_1, \ldots, H_m) \in C([a,b], \mathbb{R}^m)$. In this case, the equation (1.1) takes the following form

$$\begin{cases} x_1(t) &= \int_a^c K_1(t,s,x_1(s),\ldots,x_m(s))ds \\ &\quad + \int_a^t H_1(t,s,x_1(s),\ldots,x_m(s))ds, \ t \in [a,b] \\ &\vdots \\ x_m(t) &= \int_a^c K_m(t,s,x_1(s),\ldots,x_m(s))ds \\ &\quad \int_a^t H_m(t,s,x_1(s),\ldots,x_m(s))ds, \ t \in [a,b]. \end{cases} \tag{4.1}$$

From Theorem 4.1 we have an existence and uniqueness result for the system (4.1).

In the case when $\mathbb{B}$ is a Banach space of infinite sequences with elements in $\mathbb{K}$ ($c(\mathbb{K})$, $C_p(\mathbb{K})$, $m(\mathbb{K})$, $l^p(\mathbb{K})$, $\ldots$) we have from Theorem 4.1 an existence and uniqueness result for an infinite system of Fredholm-Volterra integral equations.

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