Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 4, 743–754
DOI: 10.24193/subbmath.2023.4.04

Radius problems for certain classes of analytic functions

Yao Liang Chung, Maisarah Haji Mohd and Shamani Supramaniam

Abstract. Radius constants for functions in three classes of analytic functions to be a starlike function of order α, parabolic starlike function, starlike function associated with lemniscate of Bernoulli, exponential function, cardioid, sine function, lune, a particular rational function, and reverse lemniscate are obtained. One of these classes are characterized by the condition $\operatorname{Re} g/(ze^z) > 0$. The other two classes are defined by using the function g and they consist respectively of functions f satisfying $\operatorname{Re} f/g > 0$ and $|f/g - 1| < 1$.

Mathematics Subject Classification (2010): 30C45.

Keywords: Starlike function, radius of starlikeness, exponential function.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Let $\mathcal{P}$ be the class of functions with positive real part consisting of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ satisfying $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$. For $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ be the subclasses of $\mathcal{S}$ consisting of starlike functions of order α. Analytically, we have $f \in \mathcal{S}^*(\alpha)$ if and only if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$. For $\alpha = 0$, we have $\mathcal{S}^*(0) := \mathcal{S}^*$ which is the starlike functions. For analytic functions f and g on $\mathbb{D}$, we say that f is subordinate to g, denoted $f \prec g$, if there exists a Schwarz function ω in $\mathbb{D}$ such that $f(z) = g(\omega(z)), z \in \mathbb{D}$. Several subclasses of starlike functions defined by subordination were discussed in the literature. We shall be interested in the following classes:

Received 29 December 2020; Accepted 23 March 2021.

- $\mathcal{S}_L^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z},\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_p^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_e^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_c^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_{\sin}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sin z,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_m^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2},\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_R^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{z}{k} \left(\frac{k+z}{k-z} \right),\ k = \sqrt{2}+1,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_{RL}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}},\ z \in \mathbb{D} \right\}$.

For more information on the subclasses, refer [1, 2, 4, 6, 10, 11, 12, 13, 17, 18].

The radius problems is an important area of study in geometric function theory (see [1, 9]). Let F and G be two subclasses of $\mathcal{A}$. If for every $f \in F, r^{-1}f(rz) \in G$ for $r \leq r_0$, and r_0 is the largest number for which this holds, then r_0 is the G radius (or the radius of the property connected to G) in F. For example, the radius of starlikeness for the class $\mathcal{S}$ is $\tanh(\pi/4)$. Recently, Asha and Ravichandran [14] consider some analytic functions and obtained the radii for these functions to belong to various subclasses of starlike functions. See also [3, 5, 7, 8]. Motivated by the aforementioned works, three subclasses of analytic functions are introduced below:

$$\begin{aligned} E_1 &= \{f \in \mathcal{A} :\ f/g \in \mathcal{P} \text{ for some } g \in \mathcal{A} \text{ with } g/(ze^z) \in \mathcal{P}\}, \\ E_2 &= \{f \in \mathcal{A} :\ |f/g - 1| < 1 \text{ for some } g \in \mathcal{A} \text{ with } g/(ze^z) \in \mathcal{P}\}, \\ E_3 &= \{f \in \mathcal{A} :\ f/(ze^z) \in \mathcal{P}\}. \end{aligned}$$

The main objective of the paper is to compute radius constants of the above functions for several subclasses of $\mathcal{A}$ such as starlike functions of order α, parabolic starlike functions, starlike functions associated with lemniscate of Bernoulli, exponential function, cardioid, sine function, lune, a particular rational function, and reverse lemniscate.

2. Main results

Our first theorem gives several radius results for the class E_1. Recall that E_1 is defined by

$$E_1 = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{f(z)}{g(z)} > 0 \text{ for some } g \in \mathcal{A} \text{ with } \operatorname{Re} \frac{g(z)}{ze^z} > 0,\ z \in \mathbb{D} \right\}.$$

The function $f_1 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_1(z) = \left(\frac{1+z}{1-z} \right)^2 ze^z \tag{2.1}$$

belongs to E_1 and acts as an extremal function.

Theorem 2.1. *For the class E_1, the following results hold:*

(i) For $0 \leq \alpha < 1$, the $\mathcal{S}^*_\alpha$ radius is the smallest positive real root of the equation

$$r^3 - \alpha r^2 - 5r + \alpha = 0.$$

(ii) The $\mathcal{S}^*_L$-radius is the smallest positive real root of the equation

$$r^3 + (1-\sqrt{2})r^2 - 5r + \sqrt{2} - 1 = 0, \; i.e. \; R_{\mathcal{S}^*_L} \approx 0.0824.$$

(iii) The $\mathcal{S}^*_p$-radius is the smallest positive real root of the equation

$$2r^3 - r^2 - 10r + 1 = 0 \; i.e. \; R_{\mathcal{S}^*_p} \approx 0.09921.$$

(iv) The $\mathcal{S}^*_e$-radius is the smallest positive root of the equation

$$er^3 + (1-e)r^2 - 5er + e - 1 = 0 \; i.e. \; R_{\mathcal{S}^*_e} \approx 0.1248.$$

(v) The $\mathcal{S}^*_c$-radius is the smallest positive root of the equation

$$3r^3 - 2r^2 - 15r + 2 = 0 \; i.e. \; R_{\mathcal{S}^*_c} \approx 0.13148.$$

(vi) The $\mathcal{S}^*_{\sin}$-radius is the smallest positive root of the equation

$$r^3 - r^2 \sin 1 - 5r + \sin 1 = 0 \; i.e. \; R_{\mathcal{S}^*_{\sin}} \approx 0.1646.$$

(vii) The $\mathcal{S}^*_m$-radius is the smallest positive root of the equation

$$r^3 - r^2(2-\sqrt{2}) - 5r + 2 - \sqrt{2} = 0 \; i.e. \; R_{\mathcal{S}^*_m} \approx 0.1159.$$

(viii) The $\mathcal{S}^*_R$-radius is the smallest positive root of the equation

$$r^3 - r^2(2-2\sqrt{2}) - 5r + 3 - 2\sqrt{2} = 0 \; i.e. \; R_{\mathcal{S}^*_R} \approx 0.0345.$$

(ix) The $\mathcal{S}^*_{RL}$-radius is $R_{\mathcal{S}^*_{RL}}$ which is root of the equation

$$\frac{(5r-r^3)^2}{(1-r^2)^2} = (1-(\sqrt{2}-(1+r^2)/(1-r^2))^2)^{1/2} - (1-(\sqrt{2}-(1+r^2)/(1-r^2))^2).$$

Proof. Let $f \in E_1$ and $g : \mathbb{D} \to \mathbb{C}$ be chosen such that

$$\operatorname{Re} \frac{f(z)}{g(z)} > 0 \quad \text{and} \quad \operatorname{Re} \frac{g(z)}{ze^z} > 0 \quad \text{for all } z \in \mathbb{D}. \tag{2.2}$$

Define the functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ by

$$p_1(z) = \frac{f(z)}{g(z)} \quad \text{and} \quad p_2(z) = \frac{g(z)}{ze^z}. \tag{2.3}$$

By equations (2.2) and (2.3), we have p_1 and p_2 are in $\mathcal{P}$. Also, equation (2.3) yields

$$f(z) = ze^z p_1(z) p_2(z).$$

Further computations then yields

$$\frac{zf'(z)}{f(z)} = 1 + z + \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)}. \tag{2.4}$$

For $p \in \mathcal{P}(\alpha) := \{p \in \mathcal{P} : \mathrm{Re}\,(p(z)) > \alpha,\ z \in \mathbb{D}\}$, by [15, Lemma 2], we have

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\alpha)r}{(1-r)(1+(1-2\alpha)r)},\ |z| \leq r. \tag{2.5}$$

By using (2.4) and setting $\alpha = 0$ in (2.5), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{5r - r^3}{1-r^2}. \tag{2.6}$$

Hence, by (2.6), we have

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geq \frac{1-5r-r^2+r^3}{1-r^2} \geq 0.$$

Thus the function $f \in E_1$ is starlike in $|z| \leq 0.1939$. Hence, all the radius estimate here will be less than 0.1939.

(i) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1},\ 0 \leq r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*(\alpha)}$ be the smallest positive root of the equation $m(r) = \alpha$. From (2.6), it follows that

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geq \frac{1-5r-r^2+r^3}{1-r^2} = m(r) \geq m(\varrho) = \alpha.$$

This shows that $R_{\mathcal{S}^*(\alpha)}$ is at least ϱ. At $z = R_{\mathcal{S}^*(\alpha)} = \varrho$, the function f_1 defined in (2.1) satisfies

$$\mathrm{Re}\,\frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = \alpha.$$

Thus the radius is sharp.

(ii) The function $m(r) = (5r-r^3)(1-r^2)^{-1}+1,\ 0 \leq r < 1$ is an increasing function. Let $\varrho = R_{\mathcal{S}_L^*}$ be the root of the equation $m(r) = \sqrt{2}$. For $0 < r \leq R_{\mathcal{S}_L^*}$, we have $m(r) \leq \sqrt{2}$. That is,

$$\frac{5r-r^3}{1-r^2} + 1 \leq \sqrt{2} = m(\varrho).$$

For the class E_1, the centre of the disc in (2.6) is 1. Using [1, Lemma 2.2], the disc obtained in (2.6) is contained in the region bounded by lemniscate. For the function f_1 defined in (2.1), at $z = R_{\mathcal{S}_L^*} = -\rho$,

$$\left|\left(\frac{zf_1'(z)}{f_1(z)}\right)^2 - 1\right| = \left|\left(\frac{1+5\rho-\rho^2+\rho^3}{1-\rho^2}\right)^2 - 1\right| = \left|(\sqrt{2})^2 - 1\right| = 1.$$

(iii) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1},\ 0 \leq r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_p^*}$ be the root of the equation $m(r) = 1/2$. For $0 < r \leq R_{\mathcal{S}_p^*}$, we have $m(r) \geq 1/2$. That is,

$$\frac{5r-r^3}{1-r^2} \leq \frac{1}{2} = m(\rho).$$

Using [16, Lemma 1], we see that the disc obtained in (2.6) is contained in the region bounded by parabola. For the function f_1 defined in (2.1), at $z = R_{\mathcal{S}_p^*} = \rho$,

$$\operatorname{Re}\frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = \frac{1}{2} = \left|\frac{zf_1'(z)}{f_1(z)} - 1\right|.$$

(iv) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1}$, $0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_e^*}$ be the root of the equation $m(r) = 1/e$. For $0 < r \le R_{\mathcal{S}_e^*}$, we have $m(r) \ge 1/e$. That is,

$$\frac{5r-r^3}{1-r^2} \le 1 - \frac{1}{e}.$$

Using [11, Lemma 2.2], the disc obtained in (2.6) is contained in the region bounded by exponential function. For the function f_1 defined in (2.1), at $z = R_{\mathcal{S}_e^*} = \rho$,

$$\left|\log\frac{zf_1'(z)}{f_1(z)}\right| = \left|\log\frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2}\right| = 1.$$

(v) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1}$, $0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_c^*}$ be the root of the equation $m(r) = 1/3$. For $0 < r \le R_{\mathcal{S}_c^*}$, we have $m(r) \ge 1/3$. That is,

$$\frac{5r-r^3}{1-r^2} \le 1 - \frac{1}{3}.$$

Using [17, Lemma 2.5], the disc obtained in (2.6) is contained in the region bounded by the cardioid. For the function f_1 defined in (2.1), at $z = R_{\mathcal{S}_c^*} = \rho$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = \frac{1}{3} = h_c(-1),$$

where $h_c(z) = 1+(4/3)z+(2/3)z^2$ is the superordinate function in the class $\mathcal{S}_c^*$.

(vi) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1}$, $0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_{sin}^*}$ be the root of the equation $m(r) = 1 - \sin 1$. For $0 < r \le R_{\mathcal{S}_{sin}^*}$, we have $m(r) \ge 1 - \sin 1$. That is,

$$\frac{5r-r^3}{1-r^2} \le \sin 1.$$

Using [2, Lemma 3.3], the disc obtained in (2.6) is contained in the region Ω_s bounded by the sine function. For the function f_1 defined in (2.1), at $z = -R_{\mathcal{S}_{sin}^*} = -\rho$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = 1 + \sin 1 = h_s(1),$$

where $h_s(z) = 1 + \sin z$ is the superordinate function in the class $\mathcal{S}_{sin}^*$.

(vii) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_m}$ be the root of the equation $m(r) = \sqrt{2}-1$. For $0 < r \le R_{\mathcal{S}^*_m}$, we have $m(r) \ge \sqrt{2}-1$. That is,

$$\frac{5r-r^3}{1-r^2} \le 2-\sqrt{2}.$$

Using [4, Lemma 2.1], the disc obtained in (2.6) is contained in the region bounded by the intersection of disk $\{w : |w-1| < \sqrt{2}\}$ and $\{w : |w+1| < \sqrt{2}\}$. For the function f_1 defined in (2.1), at $z = -R_{\mathcal{S}^*_m} = -\rho$,

$$\left|\left(\frac{zf_1'(z)}{f_1(z)}\right)^2 - 1\right| = \left|\left(\frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2}\right)^2 - 1\right| = 2\left|\frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2}\right|.$$

(viii) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_R}$ be the root of the equation $m(r) = 2(\sqrt{2}-1)$. For $0 < r \le R_{\mathcal{S}^*_R}$, we have $m(r) \ge 2(\sqrt{2}-1)$. That is,

$$\frac{5r-r^3}{1-r^2} \le 1-2(\sqrt{2}-1).$$

Using [6, Lemma 2.2], the disc obtained in (2.6) is contained in the region bounded by the rational function. For the function f_1 defined in (2.1), at $z = -R_{\mathcal{S}^*_R} = -\rho$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = 2(\sqrt{2}-1) = h_R(-1)$$

where $h_R(z) = 1+(zk+z^2)/(k^2-kz)$, $k = 1+\sqrt{2}$ is the superordinate function in the class $\mathcal{S}^*_R$.

(ix) The function $m(r) = ((5r-r^3)(1-r^2)^{-1})+1,\ 0 \le r < 1$ is an increasing function. Let $\varrho = R_{\mathcal{S}^*_{RL}}$ be the root of the equation

$$m(r) = ((1-(\sqrt{2}-1)^2)^{1/2} - (1-(\sqrt{2}-1)^2))^{1/2}.$$

Using [10, Lemma 3.2], the disc obtained in (2.6) is contained in the region

$$\{w : |(w-\sqrt{2})^2 - 1| < 1\}.$$

For the function f_1 defined in (2.1), at $z = -R_{\mathcal{S}^*_{RL}} = -\rho$,

$$\left|\left(\frac{zf_1'(z)}{f_1(z)}\right)^2 - 1\right| = \left|\left(\frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} - \sqrt{2}\right)^2 - 1\right| = 1. \qquad \square$$

Recall that the class E_2 was defined by

$$E_2 = \left\{ f \in \mathcal{A} : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ for some } g \in \mathcal{A} \text{ with } \operatorname{Re} \frac{g(z)}{ze^z} > 0,\ z \in \mathbb{D} \right\}.$$

The function f_2 defined by

$$f_2(z) = \frac{(1+z)^2}{1-z} ze^z \tag{2.7}$$

belongs to the class E_2 and is an extremal function.

Theorem 2.2. *For the class E_2, the following results hold:*

(i) For $0 \le \alpha < 1$, the $\mathcal{S}^*_\alpha$-radius is the smallest positive real root of the equation

$$r^3 - (\alpha+1)r^2 - 4r + \alpha = 0.$$

(ii) The $\mathcal{S}^*_L$-radius is the smallest positive root of the equation

$$r^3 + r^2(2-\sqrt{2}) - 4r + \sqrt{2} - 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_L} \approx 0.1055.$$

(iii) The $\mathcal{S}^*_p$-radius is the smallest positive root of the equation

$$2r^3 - 3r^2 - 8r + 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_p} \approx 0.1200.$$

(iv) The $\mathcal{S}^*_e$-radius is the smallest positive root of the equation

$$er^3 + r^2(1-2e) - 4er + e - 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_e} \approx 0.1497.$$

(v) The $\mathcal{S}^*_C$-radius is the smallest positive root of the equation

$$3r^3 - 5r^2 - 12r + 2 = 0 \text{ i.e. } R_{\mathcal{S}^*_C} \approx 0.1573.$$

(vi) The $\mathcal{S}^*_{\sin}$-radius the smallest positive root of the equation

$$r^3 - r^2 \sin 1 - 5r + \sin 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_{\sin}} \approx 0.00349.$$

(vii) The $\mathcal{S}^*_m$-radius is the smallest positive root of the equation

$$r^3 - r^2(3-\sqrt{2}) - 4r + 2 - \sqrt{2} = 0 \text{ i.e. } R_{\mathcal{S}^*_m} \approx 0.1394.$$

(viii) The $\mathcal{S}^*_R$-radius is the smallest positive root of the equation

$$r^3 - r^2(3-2\sqrt{2}) - 4r + 3 - 2\sqrt{2} = 0 \text{ i.e. } R_{\mathcal{S}^*_R} \approx 0.0428.$$

(ix) The $\mathcal{S}^*_{RL}$-radius is $R_{\mathcal{S}^*_{RL}}$ which is root of the equation

$$\frac{(r^2+4r-r^3)^2}{(1-r^2)^2} = ((1-(\sqrt{2}-(1+r^2)/(1-r^2)^2))^{1/2} - (1-(\sqrt{2}-(1+r^2)/(1-r^2))^2).$$

Proof. Let $f \in E_2$ and $g : \mathbb{D} \to \mathbb{C}$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 \quad \text{and} \quad \operatorname{Re}\frac{g(z)}{ze^z} > 0. \tag{2.8}$$

Using the fact $|w-1| < 1$ if and only if $\operatorname{Re}(1/w) > 1/2$, it follows that $\operatorname{Re}(g(z)/f(z)) > 1/2$. Define the functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ by

$$p_1(z) = \frac{g(z)}{ze^z} \quad \text{and} \quad p_2(z) = \frac{g(z)}{f(z)}. \tag{2.9}$$

By (2.8) and (2.9), we have $p_1 \in \mathcal{P}$ and $p_2 \in \mathcal{P}(1/2)$. Also, from (2.9), we have

$$f(z) = \frac{ze^z p_1(z)}{p_2(z)}.$$

and eventually

$$\frac{zf'(z)}{f(z)} = 1 + z + \frac{zp_1'(z)}{p_1(z)} - \frac{zp_2'(z)}{p_2(z)}.$$

Hence,

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{r^2+4r-r^3}{1-r^2} \tag{2.10}$$

and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-4r-2r^2+r^3}{1-r^2} \geq 0.$$

Thus the function $f \in E_2$ is starlike in $|z| \leq 0.2271$. Hence, all the radius estimate here will be less than 0.2271.

(i) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \leq r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*(\alpha)}$ is the smallest positive root of the equation $m(r) = \alpha$. From (2.10), it follows that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-4r-2r^2+r^3}{1-r^2} = m(r) \geq m(\varrho) = \alpha.$$

This shows that $R_{\mathcal{S}^*(\alpha)}$ is at least ϱ. At $z = R_{\mathcal{S}^*(\alpha)} = \varrho$, the function f_2 defined in (2.7) satisfies

$$\operatorname{Re}\frac{zf_2'(z)}{f_2(z)} = \frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2} = \alpha.$$

Thus the radius is sharp.

(ii) The function $m(r) = (4r+r^2-r^3)(1-r^2)^{-1}+1,\ 0 \leq r < 1$ is an increasing function. Let $\varrho = R_{\mathcal{S}^*_L}$ be the root of the equation $m(r) = \sqrt{2}$. For $0 < r \leq R_{\mathcal{S}^*_L}$, we have $m(r) \leq \sqrt{2}$. That is,

$$\frac{4r+r^2-r^3}{1-r^2}+1 \leq \sqrt{2} = m(\varrho).$$

For the class E_2, the centre of the disc in (2.10) is 1. Using [1, Lemma 2.2], the disc obtained in (2.10) is contained in the region bounded by lemniscate. For the function f_2 defined in (2.7), at $z = R_{\mathcal{S}^*_L} = -\rho$,

$$\left|\left(\frac{zf_2'(z)}{f_2(z)}\right)^2-1\right| = \left|\left(\frac{1+4\rho-2\rho^2+\rho^3}{1-\rho^2}\right)^2-1\right| = |(\sqrt{2})^2-1| = 1.$$

(iii) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \leq r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_p}$ be the root of the equation $m(r) = 1/2$. For $0 < r \leq R_{\mathcal{S}^*_p}$, we have $m(r) \geq 1/2$. That is,

$$\frac{4r+r^2-r^3}{1-r^2} \leq \frac{1}{2} = m(\rho).$$

Using [16, Lemma 1], we see that the disc obtained in (2.10) is contained in the region bounded by parabola. For the function f_2 defined in (2.7), at $z = R_{\mathcal{S}^*_p} = \rho$,

$$\operatorname{Re}\frac{zf_2'(z)}{f_2(z)} = \frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2} = \frac{1}{2} = \left|\frac{zf_2'(z)}{f_2(z)}-1\right|.$$

(iv) The function $m(r) = (1 - 4r - 2r^2 + r^3)(1 - r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_e^*}$ be the root of the equation $m(r) = 1/e$. For $0 < r \le R_{\mathcal{S}_e^*}$, we have $m(r) \ge 1/e$. That is,

$$\frac{4r + r^2 - r^3}{1 - r^2} \le 1 - \frac{1}{e}.$$

Using [11, Lemma 2.2], it follow that the disc obtained in (2.10) is contained in the region bounded by exponential function. For the function f_2 defined in (2.7), at $z = R_{\mathcal{S}_e^*} = \rho$,

$$\left|\log \frac{zf_2'(z)}{f_2(z)}\right| = \left|\log \frac{1 - 4\rho - 2\rho^2 + \rho^3}{1 - \rho^2}\right| = 1.$$

(v) The function $m(r) = (1 - 4r - 2r^2 + r^3)(1 - r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_c^*}$ be the root of the equation $m(r) = 1/3$. For $0 < r \le R_{\mathcal{S}_c^*}$, we have $m(r) \ge 1/3$. That is,

$$\frac{4r + r^2 - r^3}{1 - r^2} \le 1 - \frac{1}{3}.$$

Using [17, Lemma 2.5], we see that the disc obtained in (2.10) is contained in the region bounded by the cardioid. For the function f_2 defined in (2.7), at $z = R_{\mathcal{S}_c^*} = \rho$,

$$\frac{zf_2'(z)}{f_2(z)} = \frac{1 - 4\rho - 2\rho^2 + \rho^3}{1 - \rho^2} = \frac{1}{3} = h_c(-1),$$

where $h_c(z) = 1 + (4/3)z + (2/3)z^2$ is the superordinate function in the class $\mathcal{S}_c^*$.

(vi) The function $m(r) = (1 - 4r - 2r^2 + r^3)(1 - r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_{sin}^*}$ be the root of the equation $m(r) = 1 - \sin 1$. For $0 < r \le R_{\mathcal{S}_{sin}^*}$, we have $m(r) \ge 1 - \sin 1$. That is,

$$\frac{4r + r^2 - r^3}{1 - r^2} \le \sin 1.$$

Using [2, Lemma 3.3], the disc obtained in (2.10) is contained in the region Ω_s bounded by the sine function. For the function f_2 defined in (2.7), at $z = -R_{\mathcal{S}_{sin}^*} = -\rho$,

$$\frac{zf_2'(z)}{f_2(z)} = \frac{1 - 4\rho - 2\rho^2 + \rho^3}{1 - \rho^2} = 1 + \sin 1 = h_s(1),$$

where $h_s(z) = 1 + \sin z$ is the superordinate function in the class $\mathcal{S}_{sin}^*$.

(vii) The function $m(r) = (1 - 4r - 2r^2 + r^3)(1 - r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_m^*}$ be the root of the equation $m(r) = \sqrt{2} - 1$. For $0 < r \le R_{\mathcal{S}_m^*}$, we have $m(r) \ge \sqrt{2} - 1$. That is,

$$\frac{4r + r^2 - r^3}{1 - r^2} \le 2 - \sqrt{2}.$$

Using [4, Lemma 2.1], the disc obtained in (2.10) is contained in the region bounded by the intersection of disks $\{w : |w-1| < \sqrt{2}\}$ and $\{w : |w+1| < \sqrt{2}\}$. For the function f_2 defined in (2.7), at $z = -R_{\mathcal{S}^*_m} = -\rho$,

$$\left|\left(\frac{zf_2'(z)}{f_2(z)}\right)^2 - 1\right| = \left|\left(\frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2}\right)^2 - 1\right| = 2\left|\frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2}\right|.$$

(viii) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_R}$ be the root of the equation $m(r) = 2(\sqrt{2}-1)$. For $0 < r \le R_{\mathcal{S}^*_R}$, we have $m(r) \ge 2(\sqrt{2}-1)$. That is,

$$\frac{4r+r^2-r^3}{1-r^2} \le 1 - 2(\sqrt{2}-1).$$

Using [6, Lemma 2.2], the disc obtained in (2.10) is contained in the region bounded by the rational function. For the function f_2 defined in (2.7), at $z = -R_{\mathcal{S}^*_R} = -\rho$,

$$\left|\frac{zf_2'(z)}{f_2(z)}\right| = \left|\frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2}\right| = 2(\sqrt{2}-1) = h_R(-1),$$

where $h_R(z) = 1 + (zk+z^2)/(k^2-kz)$, $k = 1+\sqrt{2}$ is the superordinate function in the class $\mathcal{S}^*_R$.

(ix) The function $m(r) = ((4r+r^2-r^3)(1-r^2)^{-1}) + 1,\ 0 \le r < 1$ is an increasing function. Let $\varrho = R_{\mathcal{S}^*_{RL}}$ be the root of the equation

$$m(r) = ((1-(\sqrt{2}-1)^2)^{1/2} - (1-(\sqrt{2}-1)^2))^{1/2}.$$

Using [10, Lemma 3.2], the disc obtained in (2.10) is contained in the region $\{w : |(w-\sqrt{2})^2 - 1| < 1\}$. For the function f_2 defined in (2.7), at $z = -R_{\mathcal{S}^*_{RL}} = -\rho$,

$$\left|\left(\frac{zf_2'(z)}{f_2(z)}\right)^2 - 1\right| = \left|\left(\frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2} - \sqrt{2}\right)^2 - 1\right| = 1. \qquad \square$$

Recall that the class E_3 is defined by

$$E_3 = \left\{f : \mathcal{A} : \operatorname{Re}\frac{f(z)}{ze^z} > 0,\ z \in \mathbb{D}\right\}.$$

An extremal function in the class E_3 is

$$f(z) = \frac{ze^z(1+z)}{1-z}.$$

For this class E_3, we have the following result:

Theorem 2.3. *For the class E_3, the following results hold:*

(i) For $0 \le \alpha < 1$, the $\mathcal{S}^*_\alpha$-radius is the smallest positive real root of the equation

$$r^3 + (\alpha-1)r^2 - 4r + \alpha = 0.$$

(ii) The $\mathcal{S}^*_L$-radius is the smallest positive root of the equation

$$r^3 + r^2(1-\sqrt{2}) - 3r + \sqrt{2} - 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_L} \approx 0.1363.$$

(iii) The $\mathcal{S}_p^*$-radius is the smallest positive root of the equation

$$2r^3 - r^2 - 6r + 1 = 0 \text{ i.e. } R_{\mathcal{S}_p^*} \approx 0.1637.$$

(iv) The $\mathcal{S}_e^*$-radius is the smallest positive root of the equation

$$er^3 + r^2(1-e) - 3er + e - 1 = 0 \text{ i.e. } R_{\mathcal{S}_e^*} \approx 0.2047.$$

(v) The $\mathcal{S}_C^*$-radius is the smallest positive root of the equation

$$3r^3 - 2r^2 - 9r + 2 = 0 \text{ i.e. } R_{\mathcal{S}_C^*} \approx 0.2153.$$

(vi) The $\mathcal{S}_{\sin}^*$-radius the smallest positive root of the equation

$$r^3 - r^2 \sin 1 - 3r + \sin 1 = 0 \text{ i.e. } R_{\mathcal{S}_{\sin}^*} \approx 0.005817.$$

(vii) The $\mathcal{S}_m^*$-radius is the smallest positive root of the equation

$$r^3 - r^2(2-\sqrt{2}) - 3r + 2 - \sqrt{2} = 0 \text{ i.e. } R_{\mathcal{S}_m^*} \approx 0.1905.$$

(viii) The $\mathcal{S}_R^*$-radius is the smallest positive root of the equation

$$r^3 - r^2(2-2\sqrt{2}) - 3r + 3 - 2\sqrt{2} = 0 \text{ i.e. } R_{\mathcal{S}_R^*} \approx 0.0428.$$

(ix) The $\mathcal{S}_{RL}^*$-radius is $R_{\mathcal{S}_{RL}^*}$ which is root of the equation

$$\frac{(3r-r^3)^2}{(1-r^2)^2} = ((1-(\sqrt{2}-(1+r^2)/(1-r^2)^2))^{1/2} - (1-(\sqrt{2}-(1+r^2)/(1-r^2))^2).$$

Proof. We can conclude the hypothesis appropriately adopting the similar technique as in the previous proof. □

Acknowledgment. The research of the first and second authors is supported by the USM RUI grant 1001/PMATHS/8011015.

References

[1] Ali, R.M., Jain, N.K., Ravichandran, V., *Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane*, Appl. Math. Comput., **218**(2012), no. 11, 6557-6565.

[2] Cho, N.E., Kumar, V., Sivaprasad Kumar, S., Ravichandran, V., *Radius problems for starlike functions associated with the sine function*, Bull. Iranian Math. Soc., **45**(2019), no. 1, 213-232.

[3] El-Faqeer, A.S.A., Mohd, M.H., Ravichandran, V., Supramaniam, S., *Starlikeness of certain analytic functions*, preprint.

[4] Gandhi, S., Ravichandran, V., *Starlike functions associated with a lune*, Asian-Eur. J. Math., **10**(2017), no. 4, 1750064, 12 pp.

[5] Kanaga, R., Ravichandran, V., *Starlikeness for certain close-to-star functions*, Hacett. J. Math. Stat., appeared online (https://dergipark.org.tr/en/download/article-file/1003494).

[6] Kumar, S., Ravichandran, V., *A subclass of starlike functions associated with a rational function*, Southeast Asian Bull. Math., **40**(2016), no. 2, 199-212.

[7] Lecko, A., Ravichandran, V., Sebastian, A., *Starlikeness of certain non-univalent function*, preprint.

[8] Lee, S.K., Khatter, K., Ravichandran, V., *Radius of starlikeness for classes of analytic functions*, Bull. Malays. Math. Sci. Soc., **43**(2020), no. 6, 4469-4493.

[9] Madaan, V., Kumar, A., Ravichandran, V., *Radii of starlikeness and convexity of some entire functions*, Bull. Malays. Math. Sci. Soc., **43**(2020), no. 6, 4335-4359.

[10] Mendiratta, R., Nagpal, S., Ravichandran, V., *A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli*, Internat. J. Math., **25**(2014), no. 9, 1450090, 17 pp.

[11] Mendiratta, R., Nagpal, S., Ravichandran, V., *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc., **38**(2015), no. 1, 365-386.

[12] Raina, R.K., Sokół, J., *Some properties related to a certain class of starlike functions*, C.R. Math. Acad. Sci. Paris, **353**(2015), no. 11, 973-978.

[13] Rønning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118**(1993), no. 1, 189-196.

[14] Sebastian, A., Ravichandran, V., *Radius of starlikeness of certain analytic functions*, Math. Slovaca, **71**(2021), no. 1, 83-104.

[15] Shah, G.M., *On the univalence of some analytic functions*, Pacific J. Math., **43**(1972), 239-250.

[16] Shanmugam, T.N., Ravichandran, V., *Certain properties of uniformly convex functions*, in: Computational methods and function theory, 1994 (Penang), 319-324, Ser. Approx. Decompos., 5, World Sci. Publ., River Edge, NJ.

[17] Sharma, K., Jain, N.K., Ravichandran, V., *Starlike functions associated with a cardioid*, Afr. Mat., **27**(2016), no. 5-6, 923-939.

[18] Sokół, J., Stankiewicz, J., *Radius of convexity of some subclasses of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat., **19**(1996), 101-105.

Yao Liang Chung
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 Penang, Malaysia
e-mail: chungyaoliang@gmail.com

Maisarah Haji Mohd
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 Penang, Malaysia
e-mail: maisarah_hjmohd@usm.my

Shamani Supramaniam
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 Penang, Malaysia
e-mail: shamani@usm.my