Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 4, 859–872
DOI: 10.24193/subbmath.2023.4.13

Nonlinear elliptic equations by topological degree in Musielak-Orlicz-Sobolev spaces

Mustapha Ait Hammou and Badr Lahmi

Abstract. We prove by using the topological degree theory the existence of at least one weak solution for the nonlinear elliptic equation

$$-\text{ div } a_1(x, \nabla u) + a_0(x, u) = f(x, u, \nabla u)$$

with homogeneous Dirichlet boundary condition in Musielak-Orlicz-Sobolev spaces.

Mathematics Subject Classification (2010): 35J60, 35D30, 47J05, 47H11.

Keywords: Nonlinear elliptic equation, Musielak-Orlicz-Sobolev space, topological degree.

1. Introduction

Recently, there has been an increasing interest in the study of elliptic and parabolic mathematical problems in Musielak-Orlicz-Sobolev spaces. This setting includes and generalizes variable exponent, anisotropic and classical Orlicz settings.

The interest brought to the study of such differential equations comes for example from applications to non-Newtonian fluids (see [12, 13] for a wide expository) and other physics phenomena. We refer to some results on existence of solutions for Leray-Lions problems studied in variable exponent Sobolev (see, e.g., [3, 19, 23]) or Orlicz-Sobolev spaces (see, e.g., [1, 10]).

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$. let us suppose that the boundary of Ω denoted $\partial\Omega$ is $\mathcal{C}^1$. We consider a class of nonlinear Dirichlet problems of the form:

$$\begin{cases} -\text{ div } a_1(x, \nabla u) + a_0(x, u) = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Received 26 December 2020; Accepted 03 June 2021.

The right-hand side f is a Carathéodory function which depend on the solution u and on its gradient ∇u satisfying a growth condition and where
$a_1 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying Leray-Lions-like conditions which generate an operator of the monotone type $-\operatorname{div} a_1(x, \nabla u) + a_0(x, u)$ defined on $W_0^1 L_\Phi(\Omega)$ with values in its dual $(W_0^1 L_\Phi(\Omega))'$. Here Φ is a Musielak-Orlicz function satisfying Some sufficient conditions, namely Δ_2-condition which assure the reflexivity of such spaces.

The authors in [7] studied the problem (1.1) and proved the existence of weak solutions by using a linear functional analysis and sub-supersolution methods. In the case when $a_0 = 0$, the authors in [20] obtained the existence of weak solutions for (1.1). Bisedes, for $a_0 \neq 0$ verifying suitable conditions, the author in [8] proved the existence of weak solution with homogeneous Neumann or Dirichlet boundary condition by a sub-supersolution method.

The aim of this paper is to prove the existence result that is found in [7] by using a different approach opening new perspectives: we apply the degree theory in [4, 16] to give a result about existence of nonzero solutions of operator equations of the abstract Hammerstein equation in reflexive Banach spaces X

$$u + STu = 0, \quad u \in X,$$

where $S : X' \to X$ and $S : X \to X'$ two mappings [4, 16].

The approach considered here require the reflexivity of the spaces. For that, we suppose that the Musielak-Orlicz functions satisfy suitable conditions (see condition (E) below). The principal prototype that we have in mind is the Φ-Laplacian equation, i.e.

$$\text{-div}\left(\frac{a(x, \nabla u)}{|\nabla u|} \cdot \nabla u\right) = f(x, u, \nabla u)$$

The Musielak-Orlicz setting generalize both Sobolev with variable exponent and Orlicz spaces. Typical examples of equations involving the Musielak-Orlicz setting include models of electrorheological fluids [22], elasticity [17], non-Newtonian fluids [11], the theory of potential [14] and harmonic analysis [6].

The plan of paper is as follows: in section 2, some fundamental properties concerning the Musielak and Musielak-Orlicz-Sobolev spaces spaces are given. Section 3 deals with the properties and the existence of the topological degree for some classes of operators. In section 4, we give some auxiliary results and the main result and its proof.

2. Musielak and Musielak-Orlicz-Sobolev spaces

Standard references on Musielak-Orlicz-Sobolev spaces and their properties include [15, 21, 9] and references therein.

Definition 2.1. Let Ω be an open subset of $\mathbb{R}^N$. A function $M : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a Musielak-Orlicz function if

1. $M(x,\cdot)$ is an N-function, i.e. convex, nondecreasing, continuous, $M(x,0)=0, M(x,t)>0$ $(\forall t>0)$,

$$\lim_{t\to 0^+}\sup_{x\in\Omega}\frac{M(x,t)}{t}=0 \text{ and } \lim_{t\to +\infty}\inf_{x\in\Omega}\frac{M(x,t)}{t}=+\infty,$$

2. $M(\cdot,t)$ is a measurable function.

For each $x\in\Omega$, the inverse of function $M(x,\cdot)$ is denoted by $M_x^{-1}(x,\cdot)$ or for simplicity $M^{-1}(x,\cdot)$ and then $M^{-1}(x,\Phi(x,s))=s$ and $M(x,M^{-1}(x,s))=s$ for all $s\geq 0$.

Remark 2.2. M admits the representation

$$M(x,t)=\int_0^t m(x,s)\,ds, \text{ for all } t\geq 0,$$

where $m(x,\cdot)$ is the right-hand derivative of $M(x,\cdot)$ for a fixed $x\in\Omega$. We recall that for every x in Ω, the function $m(x,\cdot)$ is a right-continuous and nondecreasing verifying for all $s\geq 0$: $m(x,0)=0$, $m(x,s)>0$ for $s>0$, $\lim_{s\to+\infty}\inf_{x\in\Omega} m(x,s)=+\infty$ and $M(x,s)\leq sm(x,s)\leq M(x,2s)$.

The complementary function $\overline{M}$ to a Musielak-Orlicz function M is defined as follows:

$$\overline{M}(x,r)=\sup_{s\geq 0}\big(sr-M(x,s)\big), \qquad \text{for } x\in\Omega, r\geq 0.$$

Note that $\overline{M}$ is a Musielak-Orlicz function which admits a similar representation where $\overline{m}$ is defined as above or by

$$\overline{m}(x,s)=\sup\{\delta;\ m(x,\delta)\leq s\}.$$

We recall *Young's inequality*

$$r\cdot s\leq M(x,s)+\overline{M}(x,r), \quad \forall r,s\in\mathbb{R}^+, x\in\Omega,$$

Note that when $\overline{M}$ satisfy the Δ_2-condition, a variant of Young's inequality holds, i.e.,

$$r\cdot s\leq \varepsilon M(x,s)+c(\varepsilon)\overline{M}(x,r), \quad \forall r,s\in\mathbb{R}^+, x\in\Omega,$$

where $\varepsilon\in]0,1[$ and $c(\varepsilon)$ a constant depending of ε.
For $u:\Omega\to\mathbb{R}$ measurable function, we define the modular $\varrho_{M,\Omega}$ or ϱ_M induced by the positive Musielak-Orlicz function M as

$$\varrho_M(u)=\int_\Omega M(x,|u(x)|)\,dx$$

Let us consider the *Musielak-Orlicz class*

$$K_M(\Omega)=\{u:\Omega\to\mathbb{R} \text{ measurable};\ \varrho_M(u)<\infty\}.$$

The *Orlicz space* $L_M(\Omega)$ is defined as the linear hull of $K_M(\Omega)$ and it is a Banach space with respect to the *Luxemburg norm*

$$\|u\|_M=\inf\left\{k>0;\ \int_\Omega M\left(x,\frac{|u(x)|}{k}\right)\leq 1\right\}.$$

Or the equivalent norm called *Orlicz norm*

$$\|u\|_{(M)} = \sup\left\{\left|\int_\Omega u(x)v(x)\,dx\right|; v \in K_{\overline{M}}(\Omega), \varrho_{\overline{M}}(v) \le 1\right\}.$$

One has a *Hölder's type inequality*: if $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\left|\int_\Omega u(x)v(x)\,dx\right| \le 2\|u\|_M \|v\|_{\overline{M}}.$$

The closure in $L_M(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. It is a separable space and $(E_{\overline{M}}(\Omega))' = L_M(\Omega)$. Generally $K_M(\Omega) \subset L_M(\Omega)$ but we can obtain $E_M(\Omega) = L_M(\Omega) = K_M(\Omega)$ if and only if M satisfies the Δ_2-condition, i.e. there is a constant $k > 1$ independent of $x \in \Omega$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$M(x, 2s) \le kM(x, s) + h(x), \qquad \text{for all } s \ge 0, \text{ a.e. } x \in \Omega.$$

Note also that under this condition, the space $L_M(\Omega)$ is reflexive.
Let M and P two Musielak-Orlicz functions, $M \preceq P$ means that M is weaker than P, i.e. there is two positive constants k_1 and k_2 and a nonnegative function $H \in L^1(\Omega)$ such that

$$M(x, s) \le k_1 P(x, k_2 s) + H(x), \qquad \text{for all } s \ge 0, \text{ a.e. } x \in \Omega.$$

Remark 2.3. [21, 15]
Let M and P two Musielak-Orlicz functions such that $M \preceq P$. Then $\overline{P} \preceq \overline{M}$, $L_P(\Omega) \hookrightarrow L_M(\Omega)$ and $L_{\overline{M}}(\Omega) \hookrightarrow L_{\overline{P}}(\Omega)$.

We say that the sequence $(u_n)_n \subset L_M(\Omega)$ converges to $u \in L_M(\Omega)$ in the modular sense if there exists $\lambda > 0$ such that

$$\varrho_M\left(\frac{u_n - u}{\lambda}\right) \to 0, \qquad \text{when } n \to +\infty.$$

In any Musielak-Orlicz space, norm convergence implies the modular convergence and the modular convergence implies the weak convergence.

Proposition 2.4. [21, 15, 8] *Let M be a Museilak-Orlicz function satisfy Δ_2-condition. Let $u \in L_M(\Omega)$ and $(u_n)_n \subset L_M(\Omega)$. Then the following assertions hold.*

1. $\int_\Omega M(x, u_n)\,dx > 1$ (*resp* $= 1; < 1$) $\Leftrightarrow \|u\|_M > 1$ (*resp* $= 1; < 1$),
2. $\int_\Omega M(x, u_n)\,dx \underset{n\to\infty}{\to} 0$ $(resp = 1; +\infty) \Leftrightarrow \|u_n\|_M \underset{n\to\infty}{\to} 0$ $(resp = 1; +\infty)$,
3. $u_n \underset{n\to\infty}{\to} u$ *in* $L_M(\Omega) \Rightarrow \int_\Omega M(x, u_n)\,dx \underset{n\to\infty}{\to} \int_\Omega M(x, u)\,dx$,
4. $\|u\|_M \le \varrho_M(u) + 1$,
5. $m(\cdot, u(\cdot)) \in L_{\overline{M}}(\Omega)$ *(the function m is defined in remark 2.2).*

The Musielak-Orlicz-Sobolev space $W^1L_M(\Omega)$ is the space of all $u \in L_M(\Omega)$ whose distributional derivatives $D^\alpha u$ are in $L_M(\Omega)$ for any α, with $|\alpha| \leq 1$. Let

$$\varrho_{1,M} = \sum_{|\alpha|\leq 1} \varrho_M(D^\alpha u)$$

the convex modular on $W^1L_M(\Omega)$. The space $W^1L_M(\Omega)$ equipped with the norm

$$\|u\|_{1,(M)} := \|u\|_{W^1L_M(\Omega)} = \inf\left\{\lambda > 0; \varrho_{1,M}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

or the equivalent norm

$$\|u\|_{1,M} := \|u\|_M + \|\nabla u\|_M.$$

This space is a Banach space if and only if there is a constant c such that $\inf_{x\in\Omega} M(x,1) > c$ (see [21]). The space $W_0^1L_M(\Omega)$ is defined as the norm-closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. Moreover if this condition is satisfied, then $W^1L_M(\Omega)$ and $W_0^1L_M(\Omega)$ are separable Banach spaces and $W_0^1L_M(\Omega) \hookrightarrow W^1L_M(\Omega) \hookrightarrow W^{1,1}(\Omega)$.
We say that the sequence $(u_n)_n \subset L_M(\Omega)$ converges to $u \in W^1L_M(\Omega)$ in the modular sense if there exists $\lambda > 0$ such that

$$\varrho_{1,M}\left(\frac{u_n - u}{\lambda}\right) \to 0, \qquad \text{when } n \to +\infty.$$

Suppose also that

$$\lim_{t\to 0}\int_t^1 \frac{M_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau < \infty, \quad \lim_{t\to\infty}\int_1^t \frac{M_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau = \infty. \tag{2.1}$$

With (2.1) satisfied, we define the *Sobolev conjugate* M_* of M as the reciprocal function of F with respect to t where

$$F(x,t) = \int_0^t \frac{M_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau, t \geq 0.$$

Proposition 2.5. [2] *If the Musielak-Orlicz function M satisfies* (2.1)*, then*

$$W_0^1L_M(\Omega) \hookrightarrow L_{\overline{M}}(\Omega).$$

Moreover, if Ω_0 is a bounded subdomain of Ω, then the imbeddings

$$W_0^1L_M(\Omega) \hookrightarrow\hookrightarrow L_P(\Omega_0)$$

exist and are compact for any Musielak-Orlicz function P increasing essentially more slowly than $\overline{M}$ near infinity (see proof of Theorem 4. in [2] *for more informations).*

We have the following result:

Lemma 2.6. *Let Ω be a bounded domain in $\mathbb{R}^N$. Let ν a Musielak-Orlicz function locally integrable satisfy Δ_2-condition such that $\inf_{x\in\Omega} \nu(x,1) = c_1 > 0$. If $(u_n)_n \subset L_\nu(\Omega)$ with $u_n \to u$ in $L_\nu(\Omega)$, then there exists $\widetilde{w} \in L_\nu(\Omega)$ and a subsequence $(u_{n_k})_{n_k}$ such that:*

$$|u_{n_k}(x)| \leq \widetilde{w}(x), \quad \textit{and} \quad u_{n_k}(x) \to u(x) \textit{ a.e. in } \Omega.$$

Proof. Let $(u_n) \subset L_\nu(\Omega)$ such that $u_n \to u$ in $L_\nu(\Omega)$, we can suppose that

$$\|(u_n - u)\|_M \leq \frac{1}{2},$$

then by proposition 2.4

$$\begin{aligned}\int_\Omega \nu(x, 2(u_n(x) - u(x))\,dx &\leq 2\|(u_n - u)\|_M \int_\Omega \nu(x, \frac{u_n(x) - u(x)}{\|(u_n - u)\|_M})\,dx.\\ &\leq 2\|(u_n - u)\|_M.\end{aligned}$$

Therefore $\|\varrho_\nu(u_n - u)\|_{L^1(\Omega)} \to 0$ as $n \to \infty$. On other hand, since Ω has a finite measure, the continous embedding $L_\nu(\Omega) \hookrightarrow L^1(\Omega)$ hold (by using the generalized Hölder's inequality) then $u_n \to u$ in $L^1(\Omega)$. We deduce that there exists $w \in L^1(\Omega)$ and a subsequence $(u_{n_k})_{n_k}$ such that $u_{n_k}(x) \to u(x)$ a.e. in Ω and $\nu(x, u_{n_k}(x) - u(x)) \leq w(x)$ a.e. in Ω. Since ν_x^{-1} is a nondecreasing function, we obtain

$$| u_{n_k}(x) | \leq | u(x) | + \nu_x^{-1}(x, w(x)).$$

Let $\widetilde{w}(x) = | u(x) | + \nu^{-1}(x, w(x))$, then

$$\int_\Omega \nu(x, \widetilde{w}(x))\,dx \leq \frac{1}{2}\int_\Omega \nu(x, 2 \mid u(x) \mid) + \int_\Omega w(x)\,dx.$$

Thus $\widetilde{w} \in K_\nu(\Omega) = L_\nu(\Omega)$. □

2.1. Functional setting

Let Φ and Ψ are two Musielak-Orlicz functions defined on $\Omega \times \mathbb{R}^+$.
We say that Φ and Ψ satisfy the condition (E) if:

- E_1. Φ, Ψ, $\overline{\Phi}$ and $\overline{\Psi}$ are locally integrable, uniformly convex and satisfy Δ_2-condition,
- E_2. Φ satisfy the condition (2.1),
- E_3. $\Phi \preceq \Psi$ and the embedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$ is compact,
- E_4. Φ satisfies the following coerciveness condition: there is a function ζ defined on $(0; +\infty)$ such that $\lim_{s\to+\infty} \zeta(s) = +\infty$ and $\Phi(x, ts) \geq \zeta(s)s\Phi(x, t)$ for $x \in \Omega$, $s > 0$ and $t \in \mathbb{R}^+$.
- E_5. there is a constant c_1 such that $\inf_{x\in\Omega} \Phi(x, 1) = c_1 > 0$ and for every $t_0 > 0$ there exists $c_2 = c_2(t_0)$ such that $\inf_{x\in\Omega} \dfrac{\overline{\Phi}(x, t)}{t} = c_2 > 0$ for every $t \geq t_0$.

Note that under the condition (E), the spaces $L_\Phi(\Omega)$, $L_\Psi(\Omega)$, $W_0^1 L_\Phi(\Omega)$ and $W^1 L_\Phi(\Omega)$ are separable reflexive Banach spaces [21].

3. Topological degree

Degree theory has been developed as a tool for checking the solution existence of nonlinear equations. A number of degree theories for various combinations of nonlinear operators have been developed by various authors. References that contain the theory of topological degree and historical information on the development of this theory

include [4, 5, 16] and references therein.
Let X and Y be two real Banach spaces and Γ a nonempty subset of X.
An operator $F : X \to Y$ is said to be bounded if it takes any bounded set into a bounded set.
F is said to be demicontinuous if for each $u \in \Gamma$ and any sequence $\{u_n\}$ in Γ, $u_n \to u$ imply that $F(u_n) \rightharpoonup F(u)$.
F is said to be compact if is continuous and the image of any bounded set is relatively compact. Let X be a real reflexive Banach space with dual X'.
We say that an operator $F : \Gamma \subset X \to X'$ satisfies condition (S_+) if for any sequence (u_n) in Γ with $u_n \rightharpoonup u$ and $\limsup\langle Fu_n, u_n - u\rangle \leq 0$ we have $u_n \to u$.
F is said to be quasimonotone if for any sequence (u_n) in Γ with $u_n \rightharpoonup u$, we have $\limsup\langle Fu_n, u_n - u\rangle \geq 0$.
For any operator $F : \Gamma \subset X \to X$ and any bounded operator
$T : \Gamma_1 \subset X \to X'$ such that $\Gamma \subset \Gamma_1$, we say that F satisfies condition $(S_+)_T$ if for any sequence(u_n) in Γ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$.

For any $\Gamma \subset X$, we consider the following classes of operators:

$$\mathcal{F}_1(\Gamma) := \{F : \Gamma \to X' \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)\},$$
$$\mathcal{F}_T(\Gamma) := \{F : \Gamma \to X \mid F \text{ is demicontinuous and satisfies condition } (S_+)_T\}.$$

For any $\Omega \subset D_F$, where D_F denotes the domain of F, and any $T \in F_1(\Omega)$, let

$$\mathcal{F}(X) := \{F \in \mathcal{F}_T(\bar{G}) \mid G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G})\},$$

where $\mathcal{O}$ denotes the collection of all bounded open set in X. Here, $T \in \mathcal{F}_1(\bar{G})$ is called an essential inner map to F.

Lemma 3.1. [16, Lemma 2.3][4, Lemma 2.2] *Suppose that $T \in \mathcal{F}_1(\bar{G})$ is continuous and $S : D_S \subset X' \to X$ is demicontinuous such that $T(\bar{G}) \subset D_s$, where G is a bounded open set in a real reflexive Banach space X. Then the following statement are true:*

- **(i).** *If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\bar{G})$, where I denotes the identity operator.*
- **(ii).** *If S satisfies condition (S_+), then $SoT \in \mathcal{F}_T(\bar{G})$*

As in [16] and in [4], we introduce a suitable topological degree for the class $\mathcal{F}(X)$:

Theorem 3.2. *Let*

$$M = \{(F, G, h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G}), F \in \mathcal{F}_T(\bar{G}), h \notin F(\partial G)\}.$$

There exists a unique degree function $d : \mathcal{M} \to \mathbb{Z}$ that satisfies the following properties:

1. *(Existence) if $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in G,*
2. *(Additivity) Let $F \in \mathcal{F}_T(\bar{G})$. If G_1 and G_2 are two disjoint open subset of G such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h),$$

3. *(Homotopy invariance) Suppose that* $H : [0,1] \times \bar{G} \to X$ *is an admissible affine homotopy with a common continuous essential inner map and* $h : [0,1] \to X$ *is a continuous path in* X *such that* $h(t) \notin H(t, \partial G)$ *for all* $t \in [0,1]$ *,then the value of* $d(H(t,.), G, h(t))$ *is constant for all* $t \in [0,1]$*,*
4. *(Normalization) For any* $h \in G$*, we have*
$$d(I, G, h) = 1,$$
5. *(Boundary dependence) If* $F, S \in \mathcal{F}_T(\bar{G})$ *coincide on* ∂G *and* $h \notin F(\partial G)$*, then*
$$d(F, G, h) = d(S, G, h).$$

4. Main result

4.1. Basic assumptions and technical lemmas

Let Φ and Ψ satisfying the condition (E) and $a_1 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ Carathéodory functions which satisfies the growth, the coercivity and the monotony conditions: for a.e. $x \in \Omega$, for every ξ, $\xi' \in \mathbb{R}^N$ and $t, t' \in \mathbb{R}$ there is two positive constants C an C', a nonnegative function g in $L_{\overline{\Phi}}(\Omega)$ and a nonnegative function h in $L^1(\Omega)$ such that

$$|a_1(x,\xi)| \leq C\overline{\Phi}^{-1}(x, \Phi(x, |\xi|)) + g(x), \tag{4.1}$$

$$a_1(x,\xi).\xi \geq C'\Phi(x, |\xi|) - h(x), \tag{4.2}$$

$$\big(a_1(x,\xi) - a_1(x,\xi')\big) \cdot \big(\xi - \xi'\big) > 0, \qquad \xi \neq \xi', \tag{4.3}$$

and

$$|a_0(x,t)| \leq C\overline{\Phi}^{-1}(x, \Phi(x, |t|)) + g(x), \tag{4.4}$$

$$a_0(x,t)t \geq C'\Phi(x, |t|) - h(x), \tag{4.5}$$

$$\big(a_0(x,t) - a_0(x,t')\big)\big(t - t'\big) > 0, \qquad t \neq t', \tag{4.6}$$

$f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function verifying the following growth condition: there is a function q in $L_{\overline{\Phi}}(\Omega)$ and two positives constants α and β such that

$$|f(x,t,\xi)| \leq q(x) + \alpha\overline{\Phi}^{-1}\Phi(x, |t|) + \beta\overline{\Phi}^{-1}\Phi(x, |\xi|) \tag{4.7}$$

for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.
The Nemytsky operator F defined by f is given by

$$F(u)(x) = f(x, u(x), \nabla u(x)), \qquad x \in \Omega.$$

Lemma 4.1. *Let* Φ *a Musielak-Orlicz function such that both* Φ *and* $\overline{\Phi}$ *satisfy the* Δ_2*-condition. Assume* (4.7)*. Then* $F(W_0^1 L_\Phi(\Omega)) \subset L_{\overline{\Psi}}(\Omega)$ *and moreover,* F *is continuous from* $W_0^1 L_\Phi(\Omega)$ *into* $L_{\overline{\Psi}}(\Omega)$ *and maps bounded sets into bounded sets.*

Proof. Let $u \in W_0^1 L_\Phi(\Omega)$. For $\lambda > max(3\alpha; 3\beta)$ we have

$$\begin{aligned} &\int_\Omega \overline{\Phi}(x, \frac{F(u)(x)}{\lambda})\, dx \\ &= \int_\Omega \overline{\Phi}(x, \frac{f(x, u(x), \nabla u(x))}{\lambda})\, dx \\ &\leq \int_\Omega \overline{\Phi}(x, \frac{1}{\lambda}[q(x) + \alpha\overline{\Phi}^{-1}\Phi(x, |u(x)|) + \beta\overline{\Phi}^{-1}(x, \Phi(x, |\xi|)])\, dx \\ &\leq \int_\Omega \frac{1}{3}\overline{\Phi}(x, \frac{3q(x)}{\lambda}) + \frac{1}{3}\Phi(x, |u(x)|) + \frac{1}{3}\Phi(x, |\nabla u(x)|)\, dx \\ &< +\infty. \end{aligned} \tag{4.8}$$

By condition (E) we have $\Phi \prec \Psi$ then $\overline{\Psi} \prec \overline{\Phi}$ and by consequent there is $\lambda' > 0$ such that

$$\int_\Omega \overline{\Psi}(x, \frac{F(u)(x)}{\lambda'})\, dx < +\infty.$$

For the continuity of F, let us consider a sequence $(u_n)_n \subset W_0^1 L_\Phi(\Omega)$ such that $\|u_n - u\|_{1,\Phi} \to 0$ as $n \to +\infty$ in $W^1 L_\Phi(\Omega)$(we mean by $\|.\|_{1,\Phi}$ the norm of $W_0^1 L_\Phi(\Omega)$ defined as the norm-closure of $\mathcal{D}(\Omega)$). Then $\|u_n - u\|_\Phi \to 0$ and $\|\nabla u_n - \nabla u\|_\Phi \to 0$ as $n \to +\infty$. Applying Lemma 2.6 we can find $w \in L_\Phi(\Omega)$ and extract a subsequence of $(u_n)_n$ still denoted $(u_n)_n$ such that

$$\begin{aligned} &|u_n(x)| \leq w(x), \quad u_n(x) \to u(x) \quad \text{a.e. in } \Omega, \\ &|\nabla u_n(x)| \leq w(x), \quad \nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega. \end{aligned} \tag{4.9}$$

Since f is a Carathéodory function, we obtain that

$$f(x, u_n, \nabla u_n) \to f(x, u, \nabla u) \quad \text{a.e. in } \Omega \quad \text{as } n \to +\infty$$

therefore,

$$\overline{\Phi}(x, F(u_n)(x) - F(u)(x)) \to 0 \quad \text{a.e. in } \Omega \quad \text{as } n \to +\infty.$$

By using (4.7), (4.9) and a similar argument to that in (4.8), there is a positive constant such that

$$\begin{aligned} &\int_\Omega \overline{\Phi}(x, F(u_n)(x) - F(u)(x))\, dx \\ &\leq c\int_\Omega \overline{\Phi}(x, q(x)) + \Phi(x, w(x)) + \Phi(x, |u(x)|) + \Phi(x, |\nabla u(x)|)\, dx \end{aligned}$$

The right term of this inequality belongs to $L^1(\Omega)$, then by applying Lebesgue's dominated convergence theorm it follows that

$$\lim_{n\to+\infty} \int_\Omega \overline{\Phi}(x, F(u_n)(x) - F(u)(x))\, dx = 0$$

which implies by the continuous embedding $L_{\overline{\Phi}} \hookrightarrow L_{\overline{\Psi}}$ that

$$\lim_{n\to+\infty} \int_\Omega \overline{\Psi}(x, F(u_n)(x) - F(u)(x))\, dx = 0$$

therefore the subsequence $F(u_n)$ converges to $F(u)$ in $L_{\overline{\Psi}}(\Omega)$ for the modular convergence. By applying proposition 2.4 we deduce that the sequence $F(u_n)$ converges in norm to $F(u)$ in $L_{\overline{\Psi}}(\Omega)$. The limit $F(u)$ is independent of the subsequence, by consequent this convergence hold true for the sequence $(u_n)_n$. Thus F is continuous from $W_0^1L_\Phi(\Omega)$ into $L_{\overline{\Psi}}(\Omega)$.
The functions Ψ and $\overline{\Psi}$ satisfy Δ_2-condition, then modular boundedness is equivalent to the norm boundedness. Using arguments similar to those above, F maps bounded sets of $W_0^1L_\Phi(\Omega)$ into bounded sets of $L_{\overline{\Psi}}(\Omega)$. □

Define A_1 and $A_0 : W_0^1L_\Phi(\Omega) \to (W_0^1L_\Phi(\Omega))'$ respectively for all $u, v \in W_0^1L_\Phi(\Omega)$ by

$$\langle A_1u, v\rangle = \int_\Omega a_1(x, \nabla u)v\,dx,$$

$$\langle A_0u, v\rangle = \int_\Omega a_0(x, u)v\,dx.$$

By the same way like in the proof of Theorem 2.2. an Theorem 2.3. in [8] we can proof the following lemma

Lemma 4.2. *Under the assumptions* (E), (4.1),(4.2), (4.3), (4.4),(4.5) *and* (4.6) *the mapping* $A := A_1 + A_0$ *is bounded, continuous and strictly monotone homeomorphism of type* (S^+).

Lemma 4.3. *Suppose that the assupmtions* (E),(4.2) *and* (4.5) *hold. Then* A *is coercive, i.e.,*

$$\frac{\langle Au, u\rangle}{\|u\|_{1,\Phi}} \to +\infty \ \textit{as}\ \|u\|_{1,\Phi} \to +\infty.$$

Proof. Let $u \in W_0^1L_\Phi(\Omega)$ $(u \neq 0)$ such that Φ verify the coerciveness condition (see condition (E) below), by using (4.2) and (4.5) we have

$$\begin{aligned}
\langle Au, u\rangle &= \int_\Omega a_1(x, \nabla u)\cdot\nabla u + a_0(x, u)u\,dx \\
&\geq 2C'\Big(\int_\Omega \Phi(x, |\nabla u|) + \Phi(x, |u|) - h(x)\,dx\Big) \\
&\geq 2C'\Big(\int_\Omega \Phi(x, \frac{\|u\|_{1,\Phi}|\nabla u|}{\|u\|_{1,\Phi}}) + \Phi(x, \frac{\|u\|_{1,\Phi}|u|}{\|u\|_{1,\Phi}})\,dx\Big) - 2\|h\|_{L^1(\Omega)} \\
&\geq 2C'\zeta(\|u\|_{1,\Phi})\|u\|_{1,\Phi}\Big(\int_\Omega \Phi(x, \frac{|\nabla u|}{\|u\|_{1,\Phi}}) + \Phi(x, \frac{|u|}{\|u\|_{1,\Phi}})\,dx\Big) - 2\|h\|_{L^1(\Omega)}
\end{aligned}$$

We have

$$\left\|\frac{|\nabla u|}{\|u\|_{1,\Phi}}\right\|_{1,\Phi} \leq 1, \quad \left\|\frac{|u|}{\|u\|_{1,\Phi}}\right\|_{1,\Phi} \leq 1$$

and

$$\lim_{\|u\|_{1,\Phi}\to+\infty} \zeta(\|u\|_{1,\Phi}) = +\infty,$$

therefore $\dfrac{\langle Au, u\rangle}{\|u\|_{1,\Phi}} \to +\infty$ as $\|u\|_{1,\Phi} \to +\infty$. □

By applying Minty-Browder theorem (or Lemma 4.3 and Lemma 5.2. in [4]), we deduce that the inverse operator $T : (W_0^1 L_\Phi(\Omega))' \to W_0^1 L_\Phi(\Omega)$ of A is also bounded, continuous and of type (S^+). On other hand, by the condition (E), the embedding $I : W_0^1 L_\Phi(\Omega) \to L_\Psi(\Omega)$ is compact, by consequent the adjoint operator $I^* :\to L_{\overline{\Psi}}(\Omega) \to (W_0^1 L_\Phi(\Omega))'$ is also compact. On other hand, the continuity and boundedness of Nemytsky operator F proved in Lemma 4.1 implies that the composition $S := -I^* oF$ is compact. Consequently we have the following lemma

Lemma 4.4. *The mapping $S : W_0^1 L_\Phi(\Omega) \to (W_0^1 L_\Phi(\Omega))'$ is continuous and compact, in particular it is quasimonotone.*

4.2. Existence result

Let us give a definition of a weak solution of problem (1.1):

Definition 4.5. A function u is called weak solution for (1.1) if $u \in W_0^1 L_\Phi(\Omega)$, $F(u) \in L_{\overline{\Psi}}(\Omega)$ and

$$\int_\Omega a_1(x, \nabla u) v\, dx + \int_\Omega a_0(x, u) v\, dx = \int_\Omega f(x, u, \nabla u) v\, dx, \text{ for all } v \in W_0^1 L_\Phi(\Omega). \tag{4.10}$$

Theorem 4.6. *Let Φ and Ψ satisfy the condition (E). Suppose that the assumptions* (4.1)–(4.7) *hold true. Then there exists at least one weak solution of problem* (1.1).

Proof. The weak formulation (4.10) is equivalent to the abstract Hammerstein equation

$$(I + SoT)v = 0, \qquad \text{and} \qquad u = Tv. \tag{4.11}$$

and T are the maps defined in Lemme 4.2 and Lemma 4.4. To solve equation 4.11, We can proceed with degree theoretic arguments, it suffices to prove the boundedness of solution set of the homotopy equation

$$v + tSoTv = 0, \quad v \in (W_0^1 L_\Phi(\Omega))', \quad t \in [0, 1],$$

Let

$$B = \{v \in (W_0^1 L_\Phi(\Omega))'; v + tSoTv = 0, \quad v \in X, \quad \text{ for some } t \in [0, 1]\}$$

let $v \in B$ and $u \in W_0^1 L_\Phi(\Omega)$ such that $Tv = u$, we have for some t in $[0, 1]$

$$\begin{aligned}\langle v, Tv\rangle &= \langle Au, u\rangle \\ &= -t\langle SoTv, Tv\rangle \\ &= t\int_\Omega f(x, u, \nabla u) u\, dx \\ &\le \int_\Omega |f(x, u, \nabla u)||u|\, dx.\end{aligned}$$

As in the proof of Lemma 4.3, there two positive constants C and $\widetilde{C}$ such that

$$\langle Au, u\rangle \ge C\zeta(\|u\|_{1,\Phi})\|u\|_{1,\Phi} - \widetilde{C} \tag{4.12}$$

Let $\lambda > max(3\alpha; 3\beta)$. Since Φ satisfy the Δ_2-condition, then by using proposition 2.3 in [7], there is a function $\gamma \in L^1(\Omega)$ and a constant c such that

$$\Phi(x, \lambda|u(x)|) \le c\Phi(x, |u(x)|) + \gamma(x)$$

which implies, by using the young's inequality, that

$$\begin{aligned} |f(x,u,\nabla u)||u| &\leq \overline{\Phi}(x, \frac{|f(x,u,\nabla u)|}{\lambda}) + \Phi(x,\lambda|u|) \\ &\leq \frac{1}{3}\overline{\Phi}(x, \frac{3q(x)}{\lambda}) + \frac{1}{3}\Phi(x,|u(x)|) \\ &+ \frac{1}{3}\Phi(x,|\nabla u(x)|) + c\Phi(x,|u(x)|) + \gamma(x). \end{aligned}$$

by combining (4.12) and (4.13) we can find two constants C' and $\widetilde{C'}$ such that

$$\|u\|_{1,\Phi}(\zeta(\|u\|_{1,\Phi}) - C') \leq \widetilde{C'}$$

which implies that $u = Tv$ remain bounded in $W_0^1L_\Phi(\Omega)$, consequently, there exists $R > 0$ such that

$$\|v\|_{(W_0^1L_\Phi(\Omega))'} \leq R \quad \forall v \in B.$$

We deduce that for all $t \in [0,1]$,

$$v + tSoTv \neq 0, \quad \forall v \in \partial B_R(0).$$

According to Lemma 3.1, the Hammersein operator $I + SoT$ belongs to the class $\mathcal{F}_T(\overline{B_R(0)})$.
Let us consider the homotopy $\mathcal{H} : [0,1] \times \overline{B_R(0)} \to (W_0^1L_\Phi(\Omega))'$ defined by

$$\mathcal{H}(t,v) = v + tSoTv.$$

By invariance and normalisation properties of the degree d of the class $\mathcal{F}_T$ (see Theorem 3.2) we deduce that

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1.$$

By Theorem 3.2 we conclude that there is at least one $\overline{v} \in B_R(0)$ verifying

$$\overline{v} + SoT\overline{v} = 0.$$

Thus $\overline{u} = T\overline{v}$ is a weak solution of problem (1.1). □

Example 4.7. Let $x \in \Omega$ and $t \in \mathbb{R}^+$. Set

$$\Phi(x,t) = \Psi(x,t) = \frac{1}{p(x)}t^{p(x)},$$

then $\varphi(x,t) = t^{p(x)-1}$ where $p : \Omega \to \mathbb{R}$ is a measurable function such that

$$2 \leq p^- \leq p(x) \leq p^+ < N.$$

Put

$$a_1(x,\xi) = \varphi(x,|\xi|)\frac{\xi}{|\xi|} = |\xi|^{p(x)-2}\xi, \quad a_0(x,t) = \varphi(x,|t|) = |t|^{p(x)-1}$$

and

$$f(x,t,\xi) = \alpha|t|^{p(x)-2}t + \beta|\xi|^{p(x)-1}$$

for $x \in \Omega, t \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ where α and β are two positives constants. So, the problem (1.1) becomes

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-1} = \alpha|u|^{p(x)-2}u + \beta|\nabla u|^{p(x)-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

where $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplace operator.

- It is clear that the assumptions (4.1)–(4.7) are verified.
- E_1 and E_5 are verified as in example 3.1 in [7].
- We have
$$\lim_{t\to 0}\int_t^1 \frac{\phi_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau = \frac{p(x)^{\frac{1}{p(x)}}}{\frac{1}{p(x)}-\frac{1}{N}} < \frac{p^{+\frac{1}{p^-}}}{\frac{1}{p^+}-\frac{1}{N}} < \infty$$
and
$$\lim_{t\to\infty}\int_1^t \frac{\phi_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau = \lim_{t\to\infty}\frac{p(x)^{\frac{1}{p(x)}}}{\frac{1}{p(x)}-\frac{1}{N}}(t^{\frac{1}{p(x)}-\frac{1}{N}}-1) = \infty$$
because $p^+ < N$, then E_2 is verified.
- Since Φ satisfies the Δ_2-condition, then there is a constant $k > 1$ independent of $x \in \Omega$ and a nonnegative function $h \in L^1(\Omega)$ such that
$$\Phi(x,s) \leq k\Phi(x,\frac{1}{2}s) + h(x) = k\Psi(x,\frac{1}{2}s) + h(x)$$
for all $s \geq 0$, a.e. $x \in \Omega$. Therefore $\Phi \preceq \Psi$.
Furthermore we have $W_0^1L_\Phi(\Omega) = W_0^{1,p(x)}(\Omega)$ and $L^{p(x)}(\Omega) = L_\Psi(\Omega)$. Since $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ with compact embedding (see [18]), then we have the compact embedding $W_0^1L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$. So E_3 is verified.
- Finally, E_4 is verified for $\zeta(s) = s^{p(x)-1}$.

We deduce that the problem (4.13) admits at least one weak solution.

References

[1] Benkirane, A., Elmahi, A., *An existence for a strongly nonlinear elliptic problem in Orlicz spaces,* Nonlinear Anal., **36**(1999), 11-24.

[2] Benkirane, A., Sidi El Vally, M., *An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces,* Bull. Belg. Math. Soc. Simon Stevin, **20**(2013), 57-75.

[3] Bendahmane, M., Wittbold, P., *Renormalized solutions for nonlinear elliptic equations with variable exponents and L^1 data,* Nonlinear Anal., **70**(2009), no. 2, 567-583.

[4] Berkovits, J., *Extension of the Leray-Schauder degree for abstract Hammerstein type mappings,* J. Differential Equations, **234**(2007), 289-310.

[5] Berkovits, J., Mustonen, V., *On topological degree for mappings of monotone type,* Nonlinear Anal., **10**(1986), 1373-1383.

[6] Diening, L., *Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces,* Bull. Sci. Math., **129**(2005), no. 8, 657-700.

[7] Dong, G., Fang, X., *Differential equations of divergence form in separable Musielak-Orlicz-Sobolev spaces,* Bound. Value Probl., (2016), 2016:106.

[8] Fan, X., *Differential equations of divergence form in Musielak-Sobolev spaces and a sub-supersolution method,* J. Math. Anal. Appl., **386**(2012), 593-604.

[9] Fan, X.L., Guan, C.X., *Uniform convexity of Musielak-Orlicz-Sobolev spaces and applications,* Nonlinear Anal., **73**(2010), 163-175.

[10] Gossez, J.-P., *A strongly non-linear elliptic problem in Orlicz-Sobolev spaces,* In: Proc. Sympos. Pure Math., **45**(1986), 455-462.

[11] Gwiazda, P., Świerczewska-Gwiazda, A., *On steady non-newtonian fluids with growth conditions in generalized Orlicz spaces,* Topol. Methods Nonlinear Anal., **32**(2008), no. 1, 103-113.

[12] Gwiazda, P., Świerczewska-Gwiazda, A., *On non-newtonian fluids with the property of rapid thickening under different stimulus,* Math. Models Methods Appl. Sci., **18**(2008), 1073-1092.

[13] Gwiazda, P., Świerczewska-Gwiazda, A., Wróblewska, A., *Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids,* Math. Methods Appl.Sci., **33**(2010), 125-137.

[14] Harjulehto, P., Hästö, P., *Riesz potential in generalized Orlicz spaces,* Forum Math., **29**(2017), no. 1, 229-244.

[15] Hudzik, H., *On generalized Orlicz-Sobolev space,* Funct. Approx. Comment. Math., **4**(1976), 37-51.

[16] Kim, I.S., Hong, S.J., *A topological degree for operators of generalized* (S_+) *type*, Fixed Point Theory Appl., (2015), 2015:194.

[17] Klawe, F.Z., *Thermo-visco-elasticity for models with growth conditions in Orlicz spaces,* Topol. Methods Nonlinear Anal., **47**(2016), no. 2, 457-497.

[18] Kováčik, O., Rákosník, J., *On spaces* $L^{p(x)}$ *and* $W^{1,p(x)}$, Czechoslovak Math. J., **41** (1991), 592-618.

[19] Le, V.K., *On a sub-supersolution method for variational inequalities with Leray-Lions operators in variable exponent spaces,* Nonlinear Anal., **71**(2009), 3305-3321.

[20] Liu, D., Zhao, P., *Solutions for a quasilinear elliptic equation in Musielak-Sobolev spaces,* Nonlinear Anal. Real World Appl., **26**(2015), 315-329.

[21] Musielak, J., *Orlicz Spaces and Modular Spaces,* Lecture Notes in Math., vol. 1034, Springer-Verlag, Berlin, 1983.

[22] Růžička, M., *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math., Vol. 1748, Springer-Verlag, Berlin, 2000.

[23] Youssfi, A., Azroul, E., Lahmi, B., *Nonlinear parabolic equations with nonstandard growth,* Appl. Anal., **95**(2016), no. 12, 2766-2778.

Mustapha Ait Hammou
Sidi Mohamed Ben Abdellah University,
Faculty of Sciences Dhar El Mahraz,
Department of Mathematics, Laboratory LAMA,
Fez, Morocco
e-mail: `mustapha.aithammou@usmba.ac.ma`

Badr Lahmi
Moulay Ismail University, Faculty of Sciences,
Department of Mathematics, Laboratory LMI,
Meknes, Morocco
e-mail: `lahmi.badr@gmail.com`