Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 4, 931–941
DOI: 10.24193/subbmath.2023.4.19

Multiplicity results for nonhomogenous elliptic equation involving the generalized Paneitz-Branson operator

Kamel Tahri

Dedicated to the Memory of the Professor **Tahar Mourid**

Abstract. Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary ∂M, we consider the multiplicity result of solutions of the following nonhomogenous fourth order elliptic equation involving the generalized Paneitz-Branson operator,

$$P_g(u) = f(x)\,|u|^{2^\sharp - 2}\,u + h(x).$$

Under some conditions and using critical points theory, we prove the existence of two distinct solutions of the above equation. At the end, we give a geometric example when the equation has negative and positive solutions.

Mathematics Subject Classification (2010): 58J05, 58E99.

Keywords: Riemannian manifold, multiplicity result, nonhomogenous, Paneitz-Branson operator, critical points theory.

1. Introduction and statement of the main result

Let (M, g) be an $n\,(n \geq 5)$ $-$dimensional compact Riemannian manifold without boundary ∂M. In this decade, there has been extensive analyze of the relationship between the conformally covariant operators which satisfy some invariance properties under conformal change of metric on M and their associated partial differential equations. However, in 1983, Paneitz in [10] has introduced a conformally convariant differential operator on $4-$dimensional Riemannian manifolds. Branson in [4] has generalized the definition to $n-$dimensional Riemannian manifolds.

Received 08 December 2020; Accepted 22 September 2021.

Moreover, for any Riemannian metric g on M, there exists a local differential operator called Paneitz-Branson operator defined by:

$$P_g^n : \quad C^\infty(M) \to C^\infty(M)$$

such that for all $u \in C^\infty(M)$:

$$P_g^n(u) := \Delta_g^2(u) + \operatorname{div}_g\left[(a_n S_g g - b_n Ric_g)^\sharp\, du\right] + \frac{(n-4)}{2} Q_g^n u,$$

where $\Delta_g := -\operatorname{div}_g(\nabla_g)$ is the Laplace-Beltrami operator and

$$a_n := \frac{(n-2)^2 + 4}{2(n-2)(n-1)}, b_n := \frac{4}{(n-2)},$$

the symbol stands for the musical isomorphism (index are raised with the metric), and

$$Q_g^n := \frac{2}{n-4} P_g^n(1).$$

This operator has a pertinent geometric behavior in the sense that: if $\tilde{g} := \varphi^{\frac{4}{n-4}} g$ is a conformal metric to g, then for all $\varphi \in C^\infty(M)$,

$$P_g^n(\varphi u) = \varphi^{\frac{n+4}{n-4}}.P_{\tilde{g}}^n(u).$$

Taking account $u = 1$, we find that

$$P_g^n(\varphi) = \frac{(n-4)}{2} Q_{\tilde{g}}^n \varphi^{2^\sharp - 1},$$

such that $2^\sharp = \frac{2n}{n-4}$. We are then naturally led to study extensions to the Paneitz-Branson operator with general coefficients as an operator of the form:

$$P_g(u) := \Delta_g^2(u) + \operatorname{div}_g\left(A^\sharp du\right) + Bu,$$

where $A \in \Lambda^\infty_{(2,0)}(M)$ a smooth symmetric $(2,0)$ $-$tensor field, and $B \in C^\infty(M)$.

In this paper, we consider the multiplicity results of solutions of the following nonhomogenous fourth order elliptic equation involving the generalized Paneitz-Branson operator:

$$P_g(u) = f(x)\,|u|^{2^\sharp - 2} u + h(x), \tag{1.1}$$

where f is a C^∞-function on M with $f > 0$ and h belongs to $L^m(M)$ such that

$$m := \frac{2^\sharp}{2^\sharp - 1} = \frac{2n}{n+4}.$$

The main goal of this paper is to establish the existence and multiplicity of solutions throughout the Ekeland's Variational Principle in [8] and the Mountain-Pass Theorem in [1] in the critical theory. This article is organized as follows: in Section 2, we present some essential mathematical materials. In section 3, we recall some auxiliary lemmas which are important for main theorem result. And in section 4, we give the proof of the main result and at the end, we give a geometric application on Einsteinian Riemannian compact manifold. We prove the following theorem:

Theorem 1.1. *Let (M, g) be an $n\,(n \geq 5)$ $-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^{∞}-function on M with $f > 0$ and $h \in L^{m}(M)$ such that $h \neq 0$ satisfying $\|h\|_{m} < m_{o}$ and supposing that the operator $P_{g}(u)$ is coercive. Then, the equation (1.1) has at least two nontrivial solutions $v, w \in H_{2}^{2}(M)$ satisfying:*

$$J(v) < 0 < J(w).$$

2. Preliminaries

We let $H_{2}^{2}(M)$ be the standard Sobolev space consisting of the functions in $L^{2}(M)$ whose derivatives up the second order are in $L^{2}(M)$. The Sobolev embedding theorem asserts that $H_{2}^{2}(M)$ is continuously embedded in $L^{m}(M)$ $1 < m \leq 2^{\sharp}$, with the property of this embedding is compact when $m < 2^{\sharp}$. We know from the work [9] that K_0 is the sharp and the best constant of the embedding $H_{2}^{2}(\mathbb{R}^{n})$ in $L^{\frac{2n}{n-4}}(\mathbb{R}^{n})$ by

$$K_0 := \frac{16}{n\left(n^{2}-4\right)(n-4)\left(w_{n}\right)^{\frac{4}{n}}},$$

where w_n is the volume of the unit $n-$sphere (S^{n}, h). Moreover, the Euclidian Sobolev embedding has obtained by the extremal functions

$$u_{\lambda}(x) := \eta\left(\frac{\lambda}{1+\lambda^{2}\left|x-x_{o}\right|^{2}}\right)^{\frac{n-4}{2}}$$

where $\lambda > 0, \eta \in \mathbb{R}^{*}$ and $x_{o} \in \mathbb{R}^{n}$.

3. Auxiliary and useful lemmas

Throughout this section, we consider the energy functional J, for each $u \in H_{2}^{2}(M)$,

$$J(u) = \frac{1}{2}\int_{M} P_{g}(u).u d\mu(g) - \int_{M} h(x).u d\mu(g) - \frac{1}{2_{k}^{\sharp}}\int_{M} f(x).\left|u\right|^{2_{k}^{\sharp}} d\mu(g)$$

Define:

$$\Phi(u) := \langle \nabla J(u), u\rangle$$

$$\Phi(u) = \int_{M} P_{g}(u).u d\mu(g) - \int_{M} h(x).u d\mu(g) - \int_{M} f(x).\left|u\right|^{2^{\sharp}} d\mu(g)$$

and

$$\langle \nabla \Phi(u), u\rangle = 2\int_{M} P_{g}(u).u d\mu(g) - \int_{M} h(x).u d\mu(g) - 2^{\sharp}\int_{M} f(x).\left|u\right|^{2^{\sharp}} d\mu(g)$$

It is well known that the solutions of (1.1) can be seen as critical points of the functional $J(u)$. We assume in what follows that P_g is coercive, in the since that there exists $\Lambda > 0$ such that for all $u \in H_{2}^{2}(M)$:

$$\int_{M} P_{g}(u).u d\mu(g) \geq \Lambda \int_{M} u^{2} d\mu(g).$$

Now, we use the following Sobolev inequalities proved in [7].

Lemma 3.1. *Let (M,g) be an $n\,(n\geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Then for any $\epsilon>0$, there exists $A_\epsilon\in\mathbb{R}$ such that for all $u\in H_2^2(M)$:*

$$\left(\int_M |u|^{2^\sharp}\,d\mu(g)\right)^{\frac{2}{2^\sharp}} \leq (K_0+\epsilon)\int_M \left[(\Delta_g u)^2+(\nabla_g u)^2\right]d\mu(g)+A_\epsilon\int_M u^2 d\mu(g).$$

The main tool to prove our result is the Montain-Pass Theorem of Ambrossetti-Rabinowitz given by the following theorem:

Theorem 3.2. *Let $J\in C^1\left(H_2^2(M);\mathbb{R}\right)$ satisfies $(P.S)_c$ condition. We suppose:*
(1). *There exist $\alpha>0$, $\rho>0$ such that*

$$J(u)\mid_{\partial B(0;\beta)}\geq J(0)+\alpha$$

Where

$$B_\rho=\left\{u\in H_2^2(M):\|u\|_{H_2^2(M)}\leq\rho\right\}$$

(2). *There is an $e\in H_2^2(M)$ and $\|\,e\,\|_{H_2^2(M)}>\rho$ such that:*

$$J(e)\leq J(0)$$

Then, $J(.)$ has a critical value c which can be characterized as

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0;1]}J(\gamma(t))$$

Where

$$\Gamma:=\left\{\gamma\in C([0;1];H_2^2(M)):\gamma(0)=0 \text{ and } \gamma(1)=e\right\}.$$

Then there is a sequence $(u_m)_m$ in $H_2^2(M)$ such that:

$$\begin{cases} J(u_m)\to c \quad in \quad \mathbb{R} \\ \nabla J(u_m)\to 0 \quad in \quad \left(H_2^2(M)\right)^*. \end{cases}$$

Now, to prove theorem 1, we need the following version of Ekeland Principle which is the key for the existence of solution with bounded below functional J.

Lemma 3.3. *(Ekeland Principle-weak form) Let (X,d) be a complete metric space. Let $J:X\to\mathbb{R}\cup\{+\infty\}$ be lower semicontinuous and bounded below. Then given any $\epsilon>0$ there exists $u_\epsilon\in X$ such that*

$$J(u_\epsilon)\leq\inf_X J+\epsilon,$$

and

$$J(u_\epsilon)<J(u)+\epsilon d(u,u_\epsilon),\ \text{for all } u\in X \text{ and } u\neq u_\epsilon.$$

First, we have the following lemma whose proof is easy and can be found in [8].

Lemma 3.4. *The quantity $\|u\|_{P_g}:=\left(\int_M P_g(u).ud\mu(g)\right)^{\frac{1}{2}}$ is an equivalent norm of the usual one of $H_2^2(M)$ if only if the operator P_g is coercive.*

Our working norm as follow: for all $u\in H_2^2(M)$:

$$\|u\|_{P_g}:=\left(\int_M P_g(u).ud\mu(g)\right)^{\frac{1}{2}}.$$

Lemma 3.5. *Let (M,g) be an $n\,(n\geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^{∞}-function on M with $f>0$ and $h\in L^{q}(M)$ such that $h\neq 0$, then there exists some constants α,ρ and $m_o>0$ such that $J(u)\geq\alpha>0$ with $\|u\|_{P_g}=\rho$ for all $u\in H_2^2(M)$ and h satisfying $\|h\|_q<m_o$.*

Proof. Let $u\in H_2^2(M)$:

$$J(u)=\frac{1}{2}\|u\|_{P_g}^2-\frac{1}{2^\sharp}\int_M f(x).\,|u|^{2^\sharp}\,d\mu(g)-\int_M h(x).ud\mu(g)$$

Using Hölder inequality, we have:

$$J(u)\geq\frac{1}{2}\|u\|_{P_g}^2-\frac{1}{2^\sharp}\max_{x\in M}f(x)\,\|u\|_{2^\sharp}^{2^\sharp}-\|h\|_q\,.\,\|u\|_{2^\sharp}$$

Using Sobolev inequality, we deduce:

$$J(u)\geq\frac{1}{2}\|u\|_{P_g}^2-\frac{1}{2^\sharp}\max_{x\in M}f(x).\,(\max\left((K_0+\epsilon)\,,A_\epsilon\right))^{\frac{2^\sharp}{2}}\,.\,\|u\|_{H_2^2(M)}^{2^\sharp}$$

$$-\|h\|_q\,.\,(\max\left((K_0+\epsilon)\,,A_\epsilon\right))^{\frac{1}{2}}\,.\,\|u\|_{H_2^2(M)}$$

Again the coercivity of P_g implies that there is $\Lambda>0$, such that:

$$J(u)\geq\frac{1}{2}\|u\|_{P_g}^2-\frac{1}{2^\sharp}\max_{x\in M}f(x).\left(\frac{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}{\Lambda}\right)^{\frac{2^\sharp}{2}}.\,\|u\|_{P_g}^{2^\sharp}$$

$$-\|h\|_q\,.\left(\frac{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}{\Lambda}\right)^{\frac{1}{2}}.\,\|u\|_{P_g}$$

Thus,

$$J(u)\geq\left[\frac{1}{2}\|u\|_{P_g}-\frac{1}{2^\sharp}\max_{x\in M}f(x).\left(\frac{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}{\Lambda}\right)^{\frac{2^\sharp}{2}}.\,\|u\|_{P_g}^{2^\sharp-1}\right.$$

$$\left.-\|h\|_q\,.\left(\frac{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}{\Lambda}\right)^{\frac{1}{2}}\right].\,\|u\|_{P_g}$$

Setting for $t\geq 0$:

$$F(t):=\frac{1}{2}t-\frac{1}{2^\sharp}\max_{x\in M}f(x).\left(\frac{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}{\Lambda}\right)^{\frac{2^\sharp}{2}}.t^{2^\sharp-1}.$$

By continuity argument of the function $F(.)$, we see that

$$\max_{t\geq 0}F(t)=F(\rho)>0\text{ where }\rho^{2^\sharp-2}:=\frac{1}{2.\,(2^\sharp-1)}\left(\frac{\Lambda}{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}\right)^{\frac{2^\sharp}{2}}.\qquad(3.1)$$

Then, it follows from (3.1) that if $\|h\|_q<m_o$ such that

$$m_o:=\left(\frac{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}{\Lambda}\right)^{-\frac{1}{2}}.F(\rho).$$

Then, there exists $\alpha > 0$ such that

$$J(u)|_{\|u\|_{P_g}=\rho} \geq \alpha > 0. \qquad \square$$

Lemma 3.6. *Let (M,g) be an $n\,(n \geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$. Then there exists a function $v \in H_2^2(M)$ with $\|v\|_{P_g} > \rho$ such that $J(v) < 0$, where ρ is given by the previous lemma.*

Proof. Let $v \in H_2^2(M)$, for any $t > 0$ we have:

$$J(t.v) = \frac{t^2}{2}\|v\|_{P_g}^2 - \frac{t^{2^\sharp}}{2^\sharp}\int_M f(x).\,|v|^{2^\sharp}\,d\mu(g) - t\int_M h(x).vd\mu(g).$$

Since $2^\sharp > 2$, so we deduce that,

$$\lim_{t\to+\infty} J(t.v) = -\infty.$$

Consequently, there exists a point $v \in H_2^2(M)$ with $\|u\|_{P_g} > \rho$ such that $J(v) < 0$. $\square$

Lemma 3.7. *Let (M,g) be an $n\,(n \geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$. Assume $(u_m)_m$ is $(P.S)_c$ sequence with*

$$c < \frac{k}{n.K_0^{\frac{n}{4}}.\,(\max f(x))^{\frac{2}{2^\sharp}}}$$

Then, $(u_m)_m$ is bounded in $H_2^2(M)$.

Proof. Consider a sequence $(u_m)_m$ which satisfies

$$\begin{gathered} J(u_m) \to c \\ \nabla J(u_m) \to 0. \end{gathered}$$

We obtain,

$$J(u_m) - \frac{1}{2^\sharp}\langle \nabla J(u_m), u_m\rangle = \frac{2^\sharp - 2}{2.2^\sharp}\|u_m\|_{P_g}^2 - \frac{2^\sharp - 1}{2^\sharp}\int_M h(x).u_m d\mu(g) = c + o\,(1)$$

Using Holder and Sobolev's inequalities and by the coercivity of P_g implies that there is $\Lambda > 0$, such that:

$$c + o\,(1) \geq \frac{2^\sharp - 2}{2.2^\sharp}\|u_m\|_{P_g}^2 - \frac{2^\sharp - 1}{2^\sharp}\|h\|_q.\left(\frac{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}{\Lambda}\right)^{\frac{1}{2}}\|u_m\|_{P_g}.$$

If $\|u_m\|_{P_g} > 1$, then

$$c + o\,(1) \geq \left[\frac{2^\sharp - 2}{2.2^\sharp} - \frac{2^\sharp - 1}{2^\sharp}\|h\|_q.\left(\frac{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}{\Lambda}\right)^{\frac{1}{2}}\right].\|u_m\|_{P_g}.$$

And since,

$$\|h\|_q < m_o := \frac{2^\sharp - 2}{2.\,(2^\sharp - 1)}\left(\frac{\max\left((K_0+\epsilon)\,,A_\epsilon\right)}{\Lambda}\right)^{-\frac{1}{2}}$$

Then the sequence $(u_m)_m$ is bounded in $H_2^2(M)$. □

Lemma 3.8. *Let (M, g) be an $n\,(n \geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$. Assume $(u_m)_m$ is a bounded Palais-Smale sequence at level c of J with*

$$c < \frac{2}{n.K^{\frac{n}{2k}}(n,k).\,(\max f(x))^{\frac{2}{2^\sharp}}}$$

Then, $(u_m)_m$ has a strongly convergent sub-sequence in $H_2^2(M)$.

Proof. Using the previous lemma, let $(u_m)_m$ be a bounded $(P.S)_c$ in $H_2^2(M)$ and from the reflexivity of $H_2^2(M)$ and the compact embedding theorem, up to a subsequence noted $(u_m)_m$ there exists $u \in H_2^2(M)$ such that
(1). $u_m \to u$ weakly in $H_2^2(M)$.
(2). $u_m \to u$ strongly in $L^p(M)$ for $1 < p < 2^\sharp$.
(3). $u_m \to u$ a.e in M.
Then we deduce that:

$$\begin{aligned}\left|\int_M h(x)\,(u_m - u)\,d\mu(g)\right| &\leq \left(\int_M |h(x)|^2\,d\mu(g)\right)^{\frac{1}{2}}.\left(\int_M (u_m-u)^2\,d\mu(g)\right)^{\frac{1}{2}}\\ &\leq \|h\|_2 \,.\, \|u_m - u\|_2 = o(1).\end{aligned}$$

After these preliminaries, we can prove that $w_m := u_m - u$ converges to 0 strongly in $H_2^2(M)$.
Using Brézis-Lieb Lemma in [5], we obtain

$$\|u_m\|_{P_g}^2 - \|u\|_{P_g}^2 = \|w_m\|_{P_g}^2 + o(1)$$

and

$$\int_M f(x)\left(|u_m|^{2^\sharp} - |u|^{2^\sharp}\right) d\mu(g) = \int_M f(x)\,|w_m|^{2^\sharp}\,d\mu(g) + o(1).$$

Then,

$$J(u_m) - J(u) = \frac{1}{2}\,\|w_m\|_{P_g}^2 - \frac{1}{2^\sharp}\int_M f(x)\,|w_m|^{2^\sharp}\,d\mu(g) + o(1).$$

We obtain

$$\langle \nabla J(u_m) - \nabla J(u), (u_m - u)\rangle = \|w_m\|_{P_g}^2 - \int_M f(x)\,|w_m|^{2^\sharp}\,d\mu(g) = o(1).$$

That is to say

$$\|w_m\|_{P_g}^2 = \int_M f(x)\,|w_m|^{2^\sharp}\,d\mu(g) + o(1). \tag{3.2}$$

Put

$$\ell := \limsup_m \|w_m\|_{P_g}.$$

Using Sobolev's inequality, we have for all $w_m \in H_2^2(M)$:

$$\int_M f(x) \left|w_m\right|^{2^\sharp} d\mu(g) \leq \max_{x\in M} f(x). \int_M \left|w_m\right|^{2^\sharp} d\mu(g) = \max_{x\in M} f(x). \left\|w_m\right\|_{2^\sharp}^{2^\sharp}$$
$$\leq \max_{x\in M} f(x). \left[\max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{2^\sharp}{2}}. \left\|w_m\right\|_{H_2^2(M)}^{2^\sharp}.$$

Taking account that $P_g : C^\infty(M) \to C^\infty(M)$ is coercive, there exists a constant $\Lambda > 0$ such that:

$$\int_M f(x) \left|w_m\right|^{2^\sharp} d\mu(g) \leq \max_{x\in M} f(x). \left[\Lambda. \max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{2^\sharp}{2}}. \left\|w_m\right\|_{P_g}^{2^\sharp}. \tag{3.3}$$

Consequently, we obtain from (3.2) and (3.3) that:

$$\left\|w_m\right\|_{P_g}^2 \leq \max_{x\in M} f(x). \left[\Lambda. \max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{2^\sharp}{2}}. \left\|w_m\right\|_{P_g}^{2^\sharp}.$$

Letting $n \to +\infty$, we get

$$\ell \leq \max_{x\in M} f(x). \left[\Lambda. \max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{2^\sharp}{2}}. \ell^{2^\sharp}$$

Then,

$$\ell = 0 \quad \text{or..} \ell \geq \frac{1}{\left[\max_{x\in M} f(x)\right]^{\frac{n-2k}{n+2k}}. \left[\Lambda. \max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{n}{n+2k}}}.$$

We deduce that: $\ell = 0$ and then $w_n \to 0$ strongly in $H_2^2(M)$.
i.e. $w_n := u_n - u \to 0$ in $H_2^2(M)$. □

4. Main result

The following theorem is our main result.

Theorem 4.1. *Let (M, g) be an $n\,(n \geq 5)$ $-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$ and supposing that the operator $u \to P_g(u)$ is coercive. Then, the equation (1.1) has at least two nontrivial solutions $v, w \in H_2^2(M)$ satisfying:*

$$J(v) < 0 < J(w).$$

The proof is based on The Mountain-Pass Theorem and Ekeland's Variational Principle.

Proof. We prove this theorem, by the following two steps:
Step 1: There exists $w \in H_2^2(M)$ satisfies

$$J(w) > 0 \text{ and } \nabla J(w) = 0.$$

Using Lemmas 2 and 3 and The Mountain-Pass Theorem, there exists a sequence $(u_m)_m \in H_2^2(M)$ satisfying:

$$J(u_m) \to c^+ \text{ and } \nabla J(u_m) = 0.$$

Then, it follows from Lemmas 3 and 4 that there exists $w \in H_2^2(M)$ such that $J(w) = c > 0$ and $\nabla J(w) = 0$ if $\|h\|_q < m_o$.
Consequently, w is a weak solution of the equation (1.1).
Step 2: There exists $v \in H_2^2(M)$ such that: $J(v) < 0$ and $\nabla J(v) = 0$. Since $h \in L^q(M)$ such that $h \neq 0$, we can choose a function $\varphi \in H_2^2(M)$ such that:

$$\int_M h(x).\varphi(x) d\mu(g) > 0$$

Letting $t > 0$, we have:

$$J(t.\varphi) = \frac{t^2}{2} \|\varphi\|_{P_g}^2 - \frac{t^{2^\sharp}}{2^\sharp} \int_M f(x). |\varphi|^{2^\sharp} d\mu(g) - t \int_M h(x).\varphi(x) d\mu(g)$$

Then for $t > 0$ small enough, we get $J(t.\varphi) < 0$.
Put

$$c^- = \inf_{u \in B_\rho} J(u)$$

Where

$$B_\rho := \left\{ u \in H_2^2(M) : \|u\|_{P_g} \leq \rho \right\}$$

It seems that:

$$c^- = \inf_{u \in B_\rho} J(u) < 0$$

Now, applying Ekeland's Variational Principle, there exists a $(P.S)_{c^-}$ sequence $(v_m)_m \in \bar{B}_\rho$ satisfying:

$$J(v_m) \to c^- \text{ and } \nabla J(v_m) = 0$$

Using Lemmas 2-7, we obtain a sub-sequence of $(v_m)_m$ which converges strongly to $v \in H_2^2(M)$.
Consequently, w is a weak solution of the equation (1.1). □

5. Geometric application of the main theorem

Remark 5.1. When (M, g) is Einstein, the geometric Paneitz-Branson operator has constant coefficient and reduces as:

$$P_g^n(u) := \Delta_g^2(u) + c_n \Delta_g(u) + d_n u,$$

where

$$c_n := \tfrac{n^2-2n-4}{2n(n-1)} S_g \text{ and } \quad d_n := \tfrac{(n-4)(n^2-4)}{16n(n-1)^2} S_g^2.$$

In particular, when $(M, g) = (S^n, h)$ is the unit $n-$sphere,

$$P_h^n(u) := \Delta_g^2(u) + c_n \Delta_g(u) + d_n u,$$

where

$$c_n := \tfrac{n^2-2n-4}{2} \text{ and } \quad d_n := \tfrac{(n-4)n(n^2-4)}{16}.$$

Notice that

$$(c_n)^2 - 4d_n = \frac{S_g^2}{n^2 (n-1)^2}.$$

Since $(c_n)^2 - 4d_n \geq 0$, then

$$P_g^n(u) = \left(\Delta_g + a^+\right) \circ \left(\Delta_g + a^-\right) u,$$

with

$$a^{\pm} := \frac{c_n \pm \sqrt{(c_n)^2 - 4d_n}}{2}.$$

Remark 5.2. If $S_g > 0$, then P_g^n is coercive.

In this part we consider the elliptic equation with the condition taken above:

$$P_g^n(u) = f(x) |u|^{2^\sharp - 2} u + h(x), \tag{5.1}$$

Then we have the following result:

Theorem 5.3. *Let (M, g) be an $n\,(n \geq 5)-$dimensional compact Eisteinian Riemannian manifold without boundary ∂M with its scalar curvature $S_g > 0$. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h > 0$ satisfying $\|h\|_q < m_o$ Then, the equation (5.1) has at least two nontrivial solutions $v, w \in H_2^2(M)$ satisfying:*

$$J(u^-) < 0 < J(u^+),$$

where

$$u^- := \min(u, o) \quad u^+ := \max(u, 0).$$

Proof. Define the two functionals in $H_2^2(M)$ by

$$J^+(u) = \frac{1}{2}\|u\|^2_{P_g^n} - \int_M h(x).ud\mu(g) - \frac{1}{2_k^\sharp}\int_M f(x).\left(u^+\right)^{2_k^\sharp} d\mu(g)$$

and

$$J^-(u) = \frac{1}{2}\|u\|^2_{P_g^n} - \int_M h(x).ud\mu(g) - \frac{1}{2_k^\sharp}\int_M f(x).\left(u^-\right)^{2_k^\sharp} d\mu(g)$$

where

$$u^- := \min(u, o) \quad u^+ := \max(u, 0).$$

Applying the coercitivity of P_g^n on Eisteinian manifold (M, g) and using the same technique that relies on Mountain Pass Theorem for the energies J^- and J^+ for solving the elliptic equation

$$P_g^n(u) = f(x) |u|^{2^\sharp - 2} u + h(x).$$

Since (M, g) has a positive scalar curvature S_g, we have

$$\left(\Delta_g + a^+\right) \circ \left(\Delta_g + a^-\right) u = f(x) |u|^{2^\sharp - 2} u + h(x),$$

with

$$a^{\pm} := \frac{c_n \pm \sqrt{(c_n)^2 - 4d_n}}{2}.$$

Applying the strong maximum principle two times to show that $u^-, u^+ \in H_2^2(M)$ are two nontrivial solutions satisfying:

$$J(u^-) < 0 < J(u^+). \qquad \square$$

References

[1] Ambrosetti, A., Rabinowitz, P., *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **11**(1973), 349-381.

[2] Benalili, M., Tahri, K., *Nonlinear elliptic fourth order equations existence and multiplicity results*, NoDEA Nonlinear Differential Equations Appl., **18**(2011), 539-556.

[3] Bernis, F., Garcia-Azorero, J., Peral, I., *Existence and multiplicity of non trivial solutions in semilinear critical problems of fourth order*, Adv. Partial Differ. Equ. (Basel), (1996), 219-240.

[4] Branson, T.P., Ørsted, B., *Explicit functional determinants in four dimensions*, Proc. Amer. Math. Soc., **113**(1991), no. 3, 669-682.

[5] Brézis, H., Lieb, E.A., *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc., **88**(1983), 486-490.

[6] Caraffa, D., *Equations elliptiques du quatrième ordre avec exposants critiques sur les variétés riemanniennes compactes*, J. Math. Pures Appl., **80**(2001), no. 9, 941-960.

[7] Djadli, Z., Hebey, E., Ledoux, M., *Paneitz-type operators and applications*, Duke Math. J., **104**(2000), no. 1, 129-169.

[8] Ekeland, I., *On the variational principle*, J. Math. Anal. Appl., **47**(1974), 324-353.

[9] Lions, P.L., *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoam., **1**(1985), 145-201.

[10] Paneitz, S.M., *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, SIGMA Symmetry Integrability Geom. Methods Appl., **4**(2008), Paper 036, 3.

[11] Tahri, K., *Les Equations Elliptiques du Quatrième Ordre sur une Variété Riemannienne*, Abou Bekr Belkaid University, Tlemcen, Algeria, 1-209, (Uni Website: http://dspace.univ-tlemcen.dz), June 2015.

Kamel Tahri
High School of Management or Abou Bekr Belkaid University,
Tlemcen, Algeria
e-mail: `Tahri_kamel@yahoo.fr`