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A class of functionals possessing multiple global minima

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To Professor Gheorghe Moroşanu, with friendship, on his 70th birthday.

Abstract. We get a new multiplicity result for gradient systems. Here is a very particular corollary: Let $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ be a smooth bounded domain and let $\Phi : \mathbf{R}^2 \to \mathbf{R}$ be a C^1 function, with $\Phi(0,0) = 0$, such that

$$\sup_{(u,v)\in\mathbf{R}^2} \frac{|\Phi_u(u,v)| + |\Phi_v(u,v)|}{1 + |u|^p + |v|^p} < +\infty$$

where $p > 0$, with $p = \frac{2}{n-2}$ when $n > 2$.
Then, for every convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $L^2(\Omega) \times L^2(\Omega)$, there exists $(\alpha, \beta) \in S$ such that the problem

$$-\Delta u = (\alpha(x)\cos(\Phi(u,v)) - \beta(x)\sin(\Phi(u,v)))\Phi_u(u,v) \text{ in } \Omega$$

$$-\Delta v = (\alpha(x)\cos(\Phi(u,v)) - \beta(x)\sin(\Phi(u,v)))\Phi_v(u,v) \text{ in } \Omega$$

$$u = v = 0 \text{ on } \partial\Omega$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega) \times H_0^1(\Omega)$ of the functional

$$(u,v) \to \frac{1}{2}\left(\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |\nabla v(x)|^2 dx\right)$$

$$-\int_\Omega (\alpha(x)\sin(\Phi(u(x),v(x))) + \beta(x)\cos(\Phi(u(x),v(x))))dx \,.$$

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1. Introduction

Let S be a topological space. A function $g : S \to \mathbb{R}$ is said to be inf-compact if, for each $r \in \mathbb{R}$, the set $g^{-1}(]-\infty, r])$ is compact.

If Y is a real interval and $f : S \times Y \to \mathbb{R}$ is a function inf-compact and lower semicontinuous in S, and concave in Y, the occurrence of the strict minimax inequality

$$\sup_Y \inf_S f < \inf_S \sup_Y f$$

implies that the interior of the set A of all $y \in Y$ for which $f(\cdot, y)$ has at least two local minima is non-empty. This fact was essentially shown in [4], giving then raise to an enormous number of subsequent applications to the multiplicity of solutions for nonlinear equations of variational nature (see [7] for an account up to 2010).

In [6] (see also [5]), we realized that, under the same assumptions as above, the occurrence of the strict minimax inequality also implies the existence of $\tilde{y} \in Y$ such that the function $f(\cdot, \tilde{y})$ has at least two global minima. It may happen that $\tilde{y}$ is unique and does not belong to the closure of A (see Example 7 of [1]).

In [8] and [12], we extended the result of [6] to the case where Y is an arbitrary convex set in a vector space. We also stress that such an extension is not possible for the result of [4]. We then started to build a network of applications of the results of [8] and [12] which touches several different topics: uniquely remotal sets in normed spaces ([8]); non-expansive operators ([9]); singular points ([10]); Kirchhoff-type problems ([11]); Lagrangian systems of relativistic oscillators ([13]); integral functional of the Calculus of Variations ([14]); non-cooperative gradient systems ([15]); variational inequalities ([16]).

The aim of this paper is to establish a further application within that network.

2. Results

The main abstract result is as follows:

Theorem 2.1. *Let X be a topological space, $(Y, \langle \cdot, \cdot \rangle)$ a real Hilbert space, $T \subseteq Y$ a convex set dense in Y and $I : X \to \mathbb{R}$, $\varphi : X \to Y$ two functions such that, for each $y \in T$, the function $x \to I(x) + \langle \varphi(x), y \rangle$ is lower semicontinuous and inf-compact. Moreover, assume that there exists a point $x_0 \in X$, with $\varphi(x_0) \neq 0$, such that*
(a) x_0 is a global minimum of both functions I and $\|\varphi(\cdot)\|$;
(b) $\inf_{x \in X} \langle \varphi(x), \varphi(x_0) \rangle < \|\varphi(x_0)\|^2$.

Then, for each convex set $S \subseteq T$ dense in Y, there exists $y^* \in S$ such that the function $x \to I(x) + \langle \varphi(x), y^* \rangle$ has at least two global minima in X.

Proof. In view of (b), we can find $\tilde{x} \in X$ and $r > 0$ such that

$$I(\tilde{x}) + \frac{r}{\|\varphi(x_0)\|} \langle \varphi(\tilde{x}), \varphi(x_0) \rangle < I(x_0) + r\|\varphi(x_0)\| \ . \tag{2.1}$$

Thanks to (a), we have

$$I(x_0) + r\|\varphi(x_0)\| = \inf_{x\in X}(I(x) + r\|\varphi(x)\|) \ . \tag{2.2}$$

The function $y \to \inf_{x\in X}(I(x) + \langle\varphi(x), y\rangle)$ is weakly upper semicontinuous, and so there exists $\tilde{y} \in B_r$ such that

$$\inf_{x\in X}(I(x) + \langle\varphi(x), \tilde{y}\rangle) = \sup_{y\in B_r}\inf_{x\in X}(I(x) + \langle\varphi(x), y\rangle) \ , \tag{2.3}$$

B_r being the closed ball in X, centered at 0, of radius r. We distinguish two cases. First, assume that $\tilde{y} \neq \frac{r\varphi(x_0)}{\|\varphi(x_0\|}$. As a consequence, taking into account that $r\|\varphi(x_0)\|$ is the maximum of the restriction to B_r of the continuous linear functional $\langle\varphi(x_0), \cdot\rangle$ (attained at the point $\frac{r\varphi(x_0)}{\|\varphi(x_0)\|}$ only), we have

$$\inf_{x\in X}(I(x) + \langle\varphi(x), \tilde{y}\rangle) \leq I(x_0) + \langle\varphi(x_0), \tilde{y}\rangle < I(x_0) + r\|\varphi(x_0)\| \ . \tag{2.4}$$

Now, assume that $\tilde{y} = \frac{r\varphi(x_0)}{\|\varphi(x_0\|}$. In this case, due to (2.1), we have

$$\inf_{x\in X}(I(x) + \langle\varphi(x), \tilde{y}\rangle) \leq I(\tilde{x}) + \langle\varphi(\tilde{x}), \tilde{y}\rangle = I(\tilde{x}) + \frac{r}{\|\varphi(x_0)\|}\langle\varphi(\tilde{x}), \varphi(x_0)\rangle$$

$$< I(x_0) + r\|\varphi(x_0)\| \ . \tag{2.5}$$

Therefore, from (2.2), (2.3), (2.4) and (2.5), it follows that

$$\sup_{y\in B_r}\inf_{x\in X}(I(x) + \langle\varphi(x), y\rangle) < \inf_{x\in X}\sup_{y\in B_r}(I(x) + \langle\varphi(x), y\rangle) \ . \tag{2.6}$$

Now, let $S \subseteq T$ be a convex set dense in Y. By continuity, we clearly have

$$\sup_{y\in B_r\cap S}\langle\varphi(x), y\rangle = \sup_{y\in B_r}\langle\varphi(x), y\rangle$$

for all $x \in X$. Therefore, in view of (2.6), we have

$$\sup_{y\in B_r\cap S}\inf_{x\in X}(I(x) + \langle\varphi(x), y\rangle) \leq \sup_{y\in B_r}\inf_{x\in X}(I(x) + \langle\varphi(x), y\rangle)$$

$$< \inf_{x\in X}\sup_{y\in B_r}(I(x) + \langle\varphi(x), y\rangle) = \inf_{x\in X}\sup_{y\in B_r\cap S}(I(x) + \langle\varphi(x), y\rangle) \ .$$

At this point, the conclusion follows directly applying Theorem 1.1 of [12] to the restriction of the function $(x, y) \to I(x) + \langle\varphi(x), y\rangle$ to $X \times (B_r \cap S)$. ☐

We now present an application of Theorem 2.1 to elliptic systems.

In the sequel, $\Omega \subseteq \mathbf{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary.

We denote by $\mathcal{A}$ the class of all functions $H : \Omega \times \mathbf{R}^2 \to \mathbf{R}$ which are measurable in Ω, C^1 in $\mathbf{R}^2$ and satisfy

$$\sup_{(x,u,v)\in\Omega\times\mathbf{R}^2}\frac{|H_u(x, u, v)| + |H_v(x, u, v)|}{1 + |u|^p + |v|^p} < +\infty$$

where $p > 0$, with $p < \frac{n+2}{n-2}$ when $n > 2$.

Given $H \in \mathcal{A}$, we are interested in the problem

$$\begin{aligned} -\Delta u &= H_u(x,u,v) \ \text{ in } \ \Omega \\ -\Delta v &= H_v(x,u,v) \ \text{ in } \ \Omega \\ u = v &= 0 \ \text{ on } \ \partial\Omega \ , \end{aligned}$$

H_u (resp. H_v) denoting the derivative of H with respect to u (resp. v).
As usual, a weak solution of this problem is any $(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u(x)\nabla\varphi(x)dx = \int_\Omega H_u(x,u(x),v(x))\varphi(x)dx \ ,$$

$$\int_\Omega \nabla v(x)\nabla\psi(x)dx = \int_\Omega H_v(x,u(x),v(x))\psi(x)dx$$

for all $\varphi, \psi \in H^1_0(\Omega)$.

Define the functional $I_H : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbf{R}$ by

$$I_H(u,v) = \frac{1}{2}\left(\int_\Omega |\nabla u(x)|^2dx + \int_\Omega |\nabla v(x)|^2dx\right) - \int_\Omega H(x,u(x),v(x))dx$$

for all $(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$.

Since $H \in \mathcal{A}$, the functional I_H is C^1 in $H^1_0(\Omega)\times H^1_0(\Omega)$ and its critical points are precisely the weak solutions of the problem. Moreover, due to the Sobolev embedding theorem, the functional $(u,v) \to \int_\Omega H(x,u(x),v(x))$ has a compact derivative and, as a consequence, it is sequentially weakly continuous in $H^1_0(\Omega) \times H^1_0(\Omega)$.

Also, we denote by λ_1 the first eigenvalue of the Dirichlet problem

$$\begin{aligned} -\Delta u &= \lambda u \ \text{ in } \ \Omega \\ u &= 0 \ \text{ on } \ \partial\Omega \ . \end{aligned}$$

Our result is as follows:

Theorem 2.2. *Let $F, G \in \mathcal{A}$, with $p = \frac{2}{n-2}$ when $n > 2$, and let $K \in \mathcal{A}$, with $K(x,0,0) = 0$ for all $x \in \Omega$, satisfy the following conditions:*
(a_1) one has

$$\lim_{s^2+t^2\to+\infty} \frac{\sup_{x\in\Omega}(|F(x,s,t)| + |G(x,s,t)|)}{s^2+t^2} = 0 \ ;$$

(a_2) there is $\eta \in \left]0, \frac{\lambda_1}{2}\right[$ such that

$$K(x,s,t) \leq \eta(s^2+t^2)$$

for all $x \in \Omega$, $s,t \in \mathbf{R}$;
(a_3) one has

$$\text{meas}(\{x \in \Omega : 0 < |F(x,0,0)|^2 + |G(x,0,0)|^2\}) > 0 \qquad (2.7)$$

and

$$|F(x,0,0)|^2 + |G(x,0,0)|^2 \leq |F(x,s,t)|^2 + |G(x,s,t)|^2 \qquad (2.8)$$

for all $x \in \Omega$, $s,t \in \mathbf{R}$;

(a_4) *one has*

$$\mathrm{meas}(\{x \in \Omega : \inf_{(s,t)\in\mathbf{R}^2}(F(x,0,0)F(x,s,t) + G(x,0,0)G(x,s,t)) < |F(x,0,0)|^2 + |G(x,0,0)|^2\}) > 0 \ .$$

Then, for every convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $L^2(\Omega) \times L^2(\Omega)$, there exists $(\alpha, \beta) \in S$ such that the problem

$$\begin{cases} -\Delta u = \alpha(x)F_u(x,u,v) + \beta(x)G_u(x,u,v) + K_u(x,u,v) \ \text{ in } \Omega \\ -\Delta v = \alpha(x)F_v(x,u,v) + \beta(x)G_v(x,u,v) + K_v(x,u,v) \ \text{ in } \Omega \\ u = v = 0 \ \text{ on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega) \times H_0^1(\Omega)$ of the functional

$$(u,v) \to \frac{1}{2}\left(\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |\nabla v(x)|^2 dx\right)$$
$$- \int_\Omega (\alpha(x)F(x,u(x),v(x)) + \beta(x)G(x,u(x),v(x)) + K(x,u(x),v(x)))dx \ .$$

Proof. We are going to apply Theorem 2.1, with the following choices: X is the space $H_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the weak topology induced by the scalar product

$$\langle (u,v),(w,\omega)\rangle_X = \int_\Omega (\nabla u(x)\nabla w(x) + \nabla v(x)\nabla \omega(x))dx \ ;$$

Y is the space $L^2(\Omega) \times L^2(\Omega)$ with the scalar product

$$\langle (f,g),(h,k)\rangle_Y = \int_\Omega (f(x)h(x) + g(x)k(x))dx \ ;$$

T is $L^\infty(\Omega) \times L^\infty(\Omega)$; I is the function defined by

$$I(u,v) = \frac{1}{2}\left(\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |\nabla v(x)|^2 dx\right) - \int_\Omega K(x,u(x),v(x))dx$$

for all $(u,v) \in X$; φ is the function defined by

$$\varphi(u,v) = (F(\cdot,u(\cdot),v(\cdot)), G(\cdot,u(\cdot),v(\cdot)))$$

for all $(u,v) \in X$; x_0 is the zero of X. Let us show that the assumptions of Theorem 2.1 are satisfied. First, from (2.7) and (2.8) it clearly follows, respectively, that

$$\|\varphi(0,0)\|_Y^2 = \int_\Omega (|F(x,0,0)|^2 + |G(x,0,0|^2)dx > 0$$

and that

$$\|\varphi(0,0)\|_Y^2 \le \|\varphi(u,v)\|_Y^2$$

for all $(u,v) \in X$. Moreover, from (a_2), thanks to the Poincaré inequality, we get

$$\int_\Omega K(x,u(x),v(x))dx \le \eta\int_\Omega (|u(x)|^2 + |v(x)|^2)dx \le \frac{\eta}{\lambda_1}\int_\Omega (|\nabla u(x)|^2 + |\nabla v(x)|^2)dx \tag{2.9}$$

for all $(u,v) \in X$. In particular, since $K(x,0,0) = 0$ in Ω and $\frac{\eta}{\lambda_1} < \frac{1}{2}$, from (2.9) we infer that $(0,0)$ is a global minimum of I in X. So, condition (a) is satisfied. Now, let us verify condition (b). To this end, set

$$P(x,s,t) = F(x,0,0)F(x,s,t) + G(x,0,0)G(x,s,t) - |F(x,0,0)|^2 - |G(x,0,0)|^2$$

for all $(x,s,t) \in \Omega \times \mathbf{R}^2$ and

$$D = \left\{ x \in \Omega : \inf_{(s,t)\in \mathbf{R}^2} P(x,s,t) < 0 \right\} .$$

By (a_4), D has a positive measure. In view of the Scorza-Dragoni theorem, there exists a compact set $C \subset D$, with positive measure, such that the restriction of P to $C \times \mathbf{R}^2$ is continuous. Fix a point $\tilde{x} \in C$ such that the intersection of C and any ball centered at $\tilde{x}$ has a positive measure. Choose $\tilde{s}, \tilde{t} \in \mathbf{R} \setminus \{0\}$ so that $P(\tilde{x}, \tilde{s}, \tilde{t}) < 0$. By continuity, there is $r > 0$ such that

$$P(x, \tilde{s}, \tilde{t}) < 0$$

for all $x \in C \cap B(\tilde{x}, r)$. Set

$$\gamma = \sup_{(x,s,t)\in\Omega\times[-|\tilde{s}|,|\tilde{s}|]\times[-|\tilde{t}|,|\tilde{t}|]} |P(x,t,s)| .$$

Since $F, G \in \mathcal{A}$, γ is finite. Now, choose an open set A such that

$$C \cap B(\tilde{x}, r) \subset A \subset \Omega$$

and

$$\text{meas}(A \setminus (C \cap B(\tilde{x}, r))) < -\frac{\int_{C\cap B(\tilde{x},r)} P(x,\tilde{s},\tilde{t})dx}{\gamma} . \qquad (2.10)$$

Finally, choose two functions $\tilde{u}, \tilde{v} \in H_0^1(\Omega)$ such that

$$\tilde{u}(x) = \tilde{s} \ , \ \ \tilde{v}(x) = \tilde{t}$$

for all $x \in C \cap B(\tilde{x}, r)$,

$$\tilde{u}(x) = \tilde{v}(x) = 0$$

for all $x \in \Omega \setminus A$ and

$$|\tilde{u}(x)| \leq |\tilde{s}| \ , \ \ |\tilde{v}(x)| \leq |\tilde{t}|$$

for all $x \in \Omega$. Then, taking (2.10) into account, we have

$$\langle \varphi(\tilde{u},\tilde{v}), \varphi(0,0)\rangle_Y - \|\varphi(0,0)\|_Y^2 = \int_\Omega P(x,\tilde{u}(x),\tilde{v}(x))dx$$

$$= \int_{C\cap B(\tilde{x},r)} P(x,\tilde{s},\tilde{t})dx + \int_{A\setminus(C\cap B(\tilde{x},r))} P(x,\tilde{u}(x),\tilde{v}(x))dx$$

$$< \int_{C\cap B(\tilde{x},r)} P(x,\tilde{s},\tilde{t})dx + \gamma \text{meas}(A \setminus (C \cap B(\tilde{x}, r)) < 0 .$$

This shows that (b) is satisfied. Finally, fix $\alpha, \beta \in L^\infty(\Omega)$. Clearly, the function

$$(x,s,t) \to \alpha(x)F(x,s,t) + \beta(x)F(x,s,t) + K(x,s,t)$$

belongs to $\mathcal{A}$, and so the functional

$$(u, v) \to I(u, v) + \langle \varphi(u, v), (\alpha, \beta) \rangle_Y$$

is sequentially weakly lower semicontinuous in X. Let us show that it is coercive. Set

$$\theta = \max \left\{ \|\alpha\|_{L^\infty(\Omega)}, \|\beta\|_{L^\infty(\Omega)} \right\}$$

and fix $\epsilon > 0$ so that

$$\epsilon < \frac{1}{\theta} \left(\frac{\lambda_1}{2} - \eta \right) . \tag{2.11}$$

By (a_1), there is $c_\epsilon > 0$ such that

$$|F(x, s, t)| + |G(x, s, t)| \leq \epsilon(|s|^2 + |t|^2) + c_\epsilon$$

for all $(x, s, t) \in \Omega \times \mathbf{R}^2$. Then, for each $u, v \in H_0^1(\Omega)$, recalling (2.9), we have

$$\begin{aligned}
&I(u, v) + \langle \varphi(u, v), (\alpha, \beta) \rangle_Y \\
&\geq \left(\frac{1}{2} - \frac{\eta}{\lambda_1} \right) \int_\Omega (|\nabla u(x)|^2 + |\nabla v(x)|^2) dx \\
&\quad - \int_\Omega |\alpha(x) F(x, u(x), v(x)) + \beta(x) G(x, u(x), v(x))| dx \\
&\geq \left(\frac{1}{2} - \frac{\eta}{\lambda_1} \right) \int_\Omega (|\nabla u(x)|^2 + |\nabla v(x)|^2) dx - \theta\epsilon \int_\Omega (|u(x)|^2 + |v(x)|^2) dx - \theta c_\epsilon \mathrm{meas}(\Omega) \\
&\geq \left(\frac{1}{2} - \frac{\eta}{\lambda_1} - \frac{\theta\epsilon}{\lambda_1} \right) \int_\Omega (|\nabla u(x)|^2 + |\nabla v(x)|^2) dx - \theta c_\epsilon \mathrm{meas}(\Omega) .
\end{aligned}$$

Notice that, in view of (2.11), we have $\frac{1}{2} - \frac{\eta}{\lambda_1} - \frac{\theta\epsilon}{\lambda_1} > 0$, and so

$$\lim_{\|(u,v)\|_X \to +\infty} (I(u, v) + \langle \varphi(u, v), (\alpha, \beta) \rangle_Y) = +\infty ,$$

as claimed.
In particular, this also implies that the functional $(u, v) \to I(u, v) + \langle \varphi(u, v), (\alpha, \beta) \rangle_Y$ is weakly lower semicontinuous, by the Eberlein-Smulyan theorem. Thus, the assumptions of Theorem 2.1 are satisfied. Therefore, for each convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $H_0^1(\Omega) \times H_0^1(\Omega)$, there exists $(\alpha, \beta) \in S$, such that the functional

$$\begin{aligned}
(u, v) \to \frac{1}{2} &\left(\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |\nabla v(x)|^2 dx \right) \\
&- \int_\Omega (\alpha(x) F(x, u(x), v(x)) + \beta(x) G(x, u(x), v(x)) + K(x, u(x), v(x))) dx
\end{aligned}$$

has at least two global minima in $H_0^1(\Omega) \times H_0^1(\Omega)$. Finally, by Example 38.25 of [17], the same functional satisfies the Palais-Smale condition, and so it admits at least three critical points, in view of Corollary 1 of [3]. The proof is complete. □

Remark 2.3. We are not aware of known results close enough to Theorem 2.2 in order to do a proper comparison. This sentence also applies to the case of single equations, that is to say when F, G, K depend on x and s only. For an account on elliptic systems, we refer to [2].

Among the various corollaries of Theorem 2.2, we wish to stress the following ones:

Corollary 2.4. *Let $K \in \mathcal{A}$, with $K(x,0,0) = 0$ for all $x \in \Omega$, satisfy condition (a_2). Moreover, let $\Phi : \mathbf{R}^2 \to \mathbf{R}$ be a non-constant C^1 function, with $\Phi(0,0) = 0$, belonging to $\mathcal{A}$, with $p = \frac{2}{n-2}$ when $n > 2$.*

Then, for every convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $L^2(\Omega) \times L^2(\Omega)$, there exists $(\alpha, \beta) \in S$ such that the problem

$$-\Delta u = (\alpha(x)\cos(\Phi(u,v)) - \beta(x)\sin(\Phi(u,v)))\Phi_u(u,v) + K_u(x,u,v) \text{ in } \Omega$$
$$-\Delta v = (\alpha(x)\cos(\Phi(u,v)) - \beta(x)\sin(\Phi(u,v)))\Phi_v(u,v) + K_v(x,u,v) \text{ in } \Omega$$
$$u = v = 0 \text{ on } \partial\Omega$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega) \times H_0^1(\Omega)$ of the functional

$$(u,v) \to \frac{1}{2}\left(\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |\nabla v(x)|^2 dx\right)$$
$$- \int_\Omega (\alpha(x)\sin(\Phi(u(x),v(x))) + \beta(x)\cos(\Phi(u(x),v(x))) + K(x,u(x),v(x)))dx \ .$$

Proof. It suffices to apply Theorem 2.2 to the functions $F, G : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$F(s,t) = \sin(\Phi(s,t)) \ ,$$
$$G(s,t) = \cos(\Phi(s,t))$$

for all $(s,t) \in \mathbf{R}^2$. □

Corollary 2.5. *Let $F, G : \mathbf{R} \to \mathbf{R}$ belong to $\mathcal{A}$, with $p = \frac{2}{n-2}$ when $n > 2$. Moreover, assume that F, G are twice differentiable at 0 and that*

$$\lim_{|s|\to+\infty} \frac{|F(s)| + |G(s)|}{s^2} = 0 \ ,$$
$$0 < |F(0)|^2 + |G(0)|^2 = \inf_{s\in\mathbf{R}} (|F(s)|^2 + |G(s)|^2) \ ,$$
$$F''(0)F(0) + G''(0)G(0) < 0 \ . \tag{2.12}$$

Then, for every convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $L^2(\Omega) \times L^2(\Omega)$, there exists $(\alpha, \beta) \in S$ such that the problem

$$-\Delta u = \alpha(x)F'(u) + \beta(x)G'(u) \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ of the functional

$$u \to \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega (\alpha(x)F(u(x)) + \beta(x)G(u(x)))dx \ .$$

Proof. We apply Theorem 2.2 taking $K = 0$. Since 0 is a global minimum of the function $|F(\cdot)|^2 + |G(\cdot)|^2$, we have

$$F'(0)F(0) + G'(0)G(0) = 0$$

and so, in view of (2.12), 0 is a strict local maximum for the function

$$F(\cdot)F(0) + G(\cdot)G(0).$$

Hence, (a_4) is satisfied and Theorem 2.2 gives the conclusion. □

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