Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 2, 197–204
DOI: 10.24193/subbmath.2017.2.06

Schwarzian derivative and Janowski convexity

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Abstract. Sufficient conditions relating the Schwarzian derivative to the Janowski convexity of a normalized analytic function f are obtained. As a consequence, sufficient conditions are determined for the function f to be Janowski convex and convex of order α. Also, some equivalent sharp inequalities are proved for f to be Janowski convex.

Mathematics Subject Classification (2010): 30C45, 30C80.

Keywords: Schwarzian derivative, Janowski convexity, subordination.

1. Introduction and main results

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}$. An analytic function f is subordinate to an analytic function g, written as $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. For $-1 \leq B < A \leq 1$, let $\mathcal{P}[A, B]$ be the class consisting of normalized analytic functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

The class $K[A, B]$ of Janowski convex functions [2] consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}[A, B].$$

For $0 \leq \alpha < 1$, $K[1 - 2\alpha, -1] \equiv K(\alpha)$ is the usual class of convex functions of order α. For $f \in \mathcal{S}$, the Schwarzian derivative of f is defined as

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The Schwarzian derivative has the property that it is invariant with respect to Möbius transformations, that is, $S(Mof,z) \equiv S(f,z)$ for any Mobius transformation $M(z)$, and $S(M,z) \equiv 0$ if and only if $M(z)$ is a Möbius transformation. There are several sufficient conditions relating the Schwarzian derivative to the univalency of f (see [5] and [6]). Miller and Mocanu in [3] determined sufficient conditions relating the Schwarzian derivative to the convexity of f. In this paper, we find the sufficient conditions for Janowski convexity of f. Also, Harmelin in [1] derived sharp bounds for $|(1-|z|)^2 f''(z)/f'(z) - 2\bar{z}|$ and for $(1-|z|^2)^2|S_f(z)|$, obtaining the refinement of Nehari's result [7] for convex functions of order α. Here, we further extend this result for the class $K[A,B]$ of Janowski convex functions. Our first result gives a general condition for a function to be Janowski convex.

Theorem 1.1. *Let* $\Phi : \mathbb{C}^2 \to \mathbb{C}$ *satisfy* $\operatorname{Re}\, \Phi\left(\dfrac{(1+A)\rho i + (1-A)}{(1+B)\rho i + (1-B)}, \tau + i\eta\right) \leq 0$ *when* ρ, τ, $\eta \in \mathbb{R}$ *and*

$$2\tau(\rho^2(1+B)^2 + (1-B)^2)^2 + (A^2-B^2)(\rho^2(1+B) - (1-B))^2 - 4\rho^2(A^2-B^2) \leq 0. \tag{1.1}$$

Let $f \in \mathcal{A}$ *with* $f'(z) \neq 0$ *and* $(A-B)f'(z) - (1+B)zf''(z) \neq 0$. *If*

$$\operatorname{Re}\Phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2 S_f(z)\right) > 0, \tag{1.2}$$

where $z \in \mathbb{D}$, *then* $f \in K[A,B]$.

Remark 1.2. For $A = 1$ and $B = -1$, Theorem 1.1 reduces to [4, Theorem 4.6 b.].

Remark 1.3. The following functions satisfies the condition (1.1) of Theorem 1.1.

(1) $\Phi_1(u,v) = (A+B)(u-1)^2 + 2(A-B)v$,
(2) $\Phi_2(u,v) = 2(A-B)v - (A+B)(\operatorname{Im} u)^2$.

Thus, we have the following.

Corollary 1.4. *Let* $f \in \mathcal{A}$ *with* $f'(z) \neq 0$ *and* $(A-B)f'(z) - (1+B)zf''(z) \neq 0$. *Then each of the following is a sufficient condition for* f *to be in* $K[A,B]$.

(1) $\operatorname{Re}\left((A+B)\left(\dfrac{zf''(z)}{f'(z)}\right)^2 + 2(A-B)z^2 S_f(z)\right) > 0,$

(2) $\operatorname{Re}\left(2(A-B)z^2 S_f(z) - (A+B)\left(\operatorname{Im}\left(1 + \dfrac{zf''(z)}{f'(z)}\right)\right)^2\right) > 0.$

For $A = 1 - 2\alpha$, $B = -1$, Theorem 1.1 gives the following sufficient condition for a function $f \in \mathcal{A}$ to be convex of order α.

Corollary 1.5. *Let* $\Phi : \mathbb{C}^2 \to \mathbb{C}$ *satisfy* $\operatorname{Re}\, \Phi((1-\alpha)\rho i + \alpha, \tau + i\eta) \leq 0$ *when* ρ, τ, $\eta \in \mathbb{R}$, $0 \leq \alpha < 1$ *and*

$$2\tau - \alpha(1-\alpha)(1-\rho^2) \leq 0. \tag{1.3}$$

Let $f \in \mathcal{A}$ *with* $f'(z) \neq 0$. *If*

$$\operatorname{Re}\Phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2 S_f(z)\right) > 0, \quad \textit{where} \quad z \in \mathbb{D},$$

then $f \in K(\alpha)$.

Remark 1.6. The following functions satisfies the condition (1.3) of Corollary 1.5.

(1) $\Phi_1(u,v) = 2v - \alpha$,
(2) $\Phi_2(u,v) = 2v + u^2 - \alpha$,
(3) $\Phi_3(u,v) = 2v(1-\alpha) - \alpha(u-1)^2$.

Corollary 1.7. *Let* $f \in \mathcal{A}$ *with* $f'(z) \neq 0$. *Then each of the following is a sufficient condition for* f *to be in* $K(\alpha)$.

(1) $\operatorname{Re}\left(2z^2 S_f(z) - \alpha\right) > 0$,
(2) $\operatorname{Re}\left(2z^2 S_f(z) + \left(1 + \dfrac{zf''(z)}{f'(z)}\right)^2 - \alpha\right) > 0$,
(3) $\operatorname{Re}\left(2(1-\alpha)z^2 S_f(z) - \alpha\left(\dfrac{zf''(z)}{f'(z)}\right)^2\right) > 0$.

The next theorem gives necessary and sufficient conditions for a function $f \in \mathcal{A}$ to be Janowski convex.

Theorem 1.8. *Let* $f \in \mathcal{A}$. *The following statements are equivalent:*

(1) $f \in K[A,B]$.
(2) $\left|2B\overline{z} + \dfrac{2(1-B^2r^2) - (1-r^2)|A+B|}{A-B}\dfrac{f''(z)}{f'(z)}\right|^2 \leq 2(2 - (1-r^2)|A+B|)$.
(3) $\left|1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{1-ABr^2}{1-B^2r^2}\right| \leq \dfrac{(A-B)r}{1-B^2r^2}$.
(4) $\dfrac{2(1-B^2r^2)(1-r^2) - (1-r^2)^2|A+B|}{A-B}|S_f(z)|$
$+ \dfrac{1}{2}\left|2B\overline{z} + \dfrac{2(1-B^2r^2) - (1-r^2)|A+B|}{A-B}\dfrac{f''(z)}{f'(z)}\right|^2 \leq 2 - (1-r^2)|A+B|$,
where $|z| = r < 1$.

Moreover, the inequalities (3) *and* (4) *are sharp.*

Inequalities (3) and (4) gives the following coefficient bounds.

Corollary 1.9. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K[A,B]$. *Then*

$$|a_2| \leq \frac{A-B}{2}, \qquad |a_3| \leq \frac{1}{6}(A-B)(A-B+1).$$

Moreover, the bounds are sharp.

2. Proofs of main theorems

We will use the following lemma.

Lemma 2.1. [4] *Let* $\Omega \subset \mathbb{C}$ *and* $\Psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ *satisfy* $\Psi(i\rho, \sigma; z) \notin \Omega$ *whenever* $z \in \mathbb{D}$, ρ *real and* $\sigma \leq -(1+\rho^2)/2$. *If* p *is analytic in* $\mathbb{D}$ *with* $p(0) = 1$, *and* $\Psi(p(z), zp'(z); z) \in \Omega$ *for* $z \in \mathbb{D}$, *then* $\operatorname{Re} p(z) > 0$ *in* $\mathbb{D}$.

Proof of Theorem 1.1. Let $p : \mathbb{D} \to \mathbb{C}$ be defined as

$$p(z) = \frac{(A-B)f'(z) + (1-B)zf''(z)}{(A-B)f'(z) - (1+B)zf''(z)}. \tag{2.1}$$

Then p is analytic and $p(0) = 1$. Also, a calculation using equation (2.1) shows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{(1+A)p(z) + (1-A)}{(1+B)p(z) + (1-B)}$$

and

$$z^2 S_f(z) = \frac{(A-B)(4zp'(z) - (A+B+2)p^2(z) + 2(A+B)p(z) + 2 - B - A)}{2((1+B)p(z) + (1-B))^2}.$$

We define a transformation from $\mathbb{C}^2 \to \mathbb{C}^2$ as

$$u = \frac{(1+A)r + (1-A)}{(1+B)r + (1-B)}$$
$$v = \frac{(A-B)(4s - (A+B+2)r^2 + 2(A+B)r + 2 - B - A)}{2((1+B)r + (1-B))^2}.$$

Let $\Psi(r, s) = \Phi(u, v)$
$= \Phi\left(\frac{(1+A)r + (1-A)}{(1+B)r + (1-B)}, \frac{(A-B)(4s - (A+B+2)r^2 + 2(A+B)r + 2 - B - A)}{2((1+B)r + (1-B))^2}\right).$
Then

$$\Psi(p(z), zp'(z)) = \Phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2 S_f(z)\right).$$

Hence, according to (1.2), we have $\operatorname{Re} \Psi(p(z), zp'(z)) > 0$. We will use Lemma 2.1 to prove that $\operatorname{Re} p(z) > 0$.
Taking $r = i\rho$ and $s = \sigma$, we obtain

$$u = \frac{(1+A)\rho i + (1-A)}{(1+B)\rho i + (1-B)}$$
$$v = \frac{(A-B)(4\sigma + (A+B+2)\rho^2 + 2(A+B)\rho i + 2 - B - A)}{2((1+B)\rho i + (1-B))^2}.$$

The condition $\sigma \le -(1+\rho^2)/2$ is equivalent to

$$\frac{2\tau((1-B)^2 + \rho^2(1+B)^2)^2 - 4\rho^2(1-B^2)(A^2-B^2)}{(A-B)((1-B)^2 - \rho^2(1+B)^2)} + (A+B)(1-\rho^2) \le 0,$$

where τ is real part of v. On simplification, we have

$$\rho^4(2\tau(1+B)^4 + (A^2-B^2)(1+B)^2) + \rho^2(4\tau(1-B^2)^2 - 2(A^2-B^2)(3-B^2)) + 2\tau(1-B)^4 + (A^2-B^2)(1-B)^2 \le 0,$$

which is equivalent to

$$2\tau(\rho^2(1+B)^2 + (1-B)^2)^2 + (A^2-B^2)(\rho^2(1+B) - (1-B))^2 - 4\rho^2(A^2-B^2) \le 0.$$

Hence Re $\Phi\left(\dfrac{(1+A)\rho i+(1-A)}{(1+B)\rho i+(1-B)}, \tau+i\eta\right)=\operatorname{Re}\Phi(u,v)\leq 0$ using (1.1). This gives Re $\Psi(\rho i,\sigma)\leq 0$ whenever $\sigma\leq -(1+\rho^2)/2$.
From Lemma 2.1, we get Re $p(z)>0$ or equivalently

$$\frac{(A-B)f'+(1-B)zf''}{(A-B)f'-(1+B)zf''}\prec\frac{1+z}{1-z}.$$

By definition of subordination, there exists an analytic map $w:\mathbb{D}\to\mathbb{D}$ with $w(0)=0$ and

$$\frac{(A-B)f'+(1-B)zf''}{(A-B)f'-(1+B)zf''}=\frac{1+w(z)}{1-w(z)}.$$

A simple computation gives

$$1+\frac{zf''(z)}{f'(z)}=\frac{1+Aw(z)}{1+Bw(z)},$$

and hence

$$1+\frac{zf''(z)}{f'(z)}\prec\frac{1+Az}{1+Bz},\quad\text{or}\quad f\in K[A,B].$$ □

Proof of Theorem 1.8. Clearly (1) ⇔ (3). We show that (1) ⇒ (4) ⇒ (2) ⇒ (1).
Let $f\in K[A,B]$. Then there exists an analytic function $w:\mathbb{D}\to\mathbb{D}$ with $|w(z)|\leq|z|$ such that

$$1+\frac{zf''(z)}{f'(z)}=\frac{1+Aw(z)}{1+Bw(z)}.$$

This gives

$$\frac{f''(z)}{f'(z)}=\frac{(A-B)\phi(z)}{1+Bz\phi(z)}\quad\text{or}\quad\phi(z)=\frac{f''(z)/f'(z)}{(A-B)-Bzf''(z)/f'(z)},$$

where $\phi(z)=w(z)/z$ is analytic and satisfies $|\phi(z)|\leq 1$ in $\mathbb{D}$. A simple computation gives

$$\phi'(z)=\frac{2(A-B)S_f(z)+\left(\dfrac{f''(z)}{f'(z)}\right)^2(A+B)}{2((A-B)-Bzf''(z)/f'(z))^2}.$$

But $|\phi'(z)|\leq(1-|\phi(z)|^2)/(1-|z|^2)$ by the invariant form of Schwarz lemma, so we get

$$\frac{(1-|z|^2)}{2}\frac{\left|2(A-B)S_f(z)+\left(\dfrac{f''(z)}{f'(z)}\right)^2(A+B)\right|}{|(A-B)-Bzf''(z)/f'(z)|^2}\leq 1-\left|\frac{f''(z)/f'(z)}{(A-B)-Bzf''(z)/f'(z)}\right|^2.$$

This gives

$$\begin{aligned}&(1-|z|^2)(A-B)|S_f(z)|-\frac{(1-|z|^2)}{2}\left|\left(\frac{f''(z)}{f'(z)}\right)^2(A+B)\right|\\&\leq(A-B)^2-(1-B^2|z|^2)\left|\frac{f''(z)}{f'(z)}\right|^2-2B(A-B)\operatorname{Re}\frac{zf''(z)}{f'(z)}.\end{aligned}$$

After simplification, we have the desired inequality given by (4).
Clearly (4) $\Rightarrow$ (2). We show that (2) $\Rightarrow$ (1). Opening the square in (2) yields

$$\left((1-B^2r^2)-\frac{(1-r^2)}{2}|A+B|\right)\left|\frac{f''(z)}{f'(z)}\right|^2+2B(A-B)\operatorname{Re}\frac{zf''(z)}{f'(z)}\le (A-B)^2. \quad (2.2)$$

Adding and subtracting $(1-B^2)r^2\left|\frac{f''(z)}{f'(z)}\right|^2$ in the left hand side of (2.2), we get

$$(1-B^2)r^2\left|\frac{f''(z)}{f'(z)}\right|^2+\frac{(1-r^2)}{2}(2-|A+B|)\left|\frac{f''(z)}{f'(z)}\right|^2+2B(A-B)\operatorname{Re}\frac{zf''(z)}{f'(z)}\le (A-B)^2.$$

Since $\frac{(1-r^2)}{2}(2-|A+B|)\left|\frac{f''(z)}{f'(z)}\right|^2\ge 0$ for all $z\in\mathbb{D}$, we get

$$(1-B^2)r^2\left|\frac{f''(z)}{f'(z)}\right|^2+2B(A-B)\operatorname{Re}\frac{zf''(z)}{f'(z)}\le (A-B)^2. \quad (2.3)$$

Now, if $B\neq -1$, the above equation gives

$$r^2\left|\frac{f''(z)}{f'(z)}\right|^2+\frac{2B(A-B)}{1-B^2}\operatorname{Re}\frac{zf''(z)}{f'(z)}\le\frac{(A-B)^2}{1-B^2}.$$

Upon simplification, we have

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{1-AB}{1-B^2}\right|\le\frac{A-B}{1-B^2},$$

which means $f\in K[A,B]$. For $B=-1$, inequality (2.3) reduces to

$$-2(A+1)\operatorname{Re}\frac{zf''(z)}{f'(z)}\le (A+1)^2.$$

This gives

$$1+\operatorname{Re}\frac{zf''(z)}{f'(z)}\ge\frac{1-A}{2},$$

which means $f\in K[A,-1]$.

To verify the sharpness for inequality (3), let $1+\frac{zf''(z)}{f'(z)}=\frac{1+Az}{1+Bz}$. We show that $\left|\frac{1+Az}{1+Bz}-\frac{1-ABr^2}{1-B^2r^2}\right|=\frac{(A-B)r}{1-B^2r^2}$. Let $w=\frac{1+Az}{1+Bz}$. Then

$$|w|^2-\frac{2\operatorname{Re}w(1-ABr^2)}{1-B^2r^2}=\frac{A^2r^2-1}{1-B^2r^2}.$$

Adding $\left(\frac{1-ABr^2}{1-B^2r^2}\right)^2$ both sides, we have the desired equality.
To verify the sharpness of inequality (4), we substitute

$$\frac{f''(z)}{f'(z)}=\frac{A-B}{1+Bz}\quad\text{and}\quad S_f(z)=\frac{-(A^2-B^2)}{2(1+Bz)^2}$$

in the left hand side of inequality (4). We show that

$$\frac{2(1-B^2r^2)(1-r^2)-(1-r^2)^2|A+B|}{|1+Bz|^2}|A+B| + \left|2B\overline{z}+\frac{2(1-B^2r^2)-(1-r^2)|A+B|}{1+Bz}\right|^2 - 4 + 2(1-r^2)|A+B| = 0. \qquad (2.4)$$

Simplifying the left hand side of equation (2.4), we get

$$\begin{aligned}
&\frac{4(1-B^2r^2)^2-2(1-B^2r^2)(1-r^2)|A+B|}{|1+Bz|^2} - 4(1-B^2r^2)+2(1-r^2)|A+B| \\
&\quad + 4B(2(1-B^2r^2)-(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right), \\
&= (2(1-B^2r^2)-(1-r^2)|A+B|)\left(\frac{2(1-B^2r^2)}{|1+Bz|^2}-2\right) + 4B(2(1-B^2r^2) \\
&\quad -(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right), \\
&= (2(1-B^2r^2)-(1-r^2)|A+B|)\left(\frac{-4B(\operatorname{Re} z+Br^2)}{|1+Bz|^2}\right) + 4B(2(1-B^2r^2) \\
&\quad -(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right), \\
&= -4B(2(1-B^2r^2)-(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right) + 4B(2(1-B^2r^2) \\
&\quad -(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right), \\
&= 0.
\end{aligned}$$

This completes the proof. □

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