Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 4, 873–884
DOI: 10.24193/subbmath.2023.4.14

Existence of solutions for a biharmonic equation with gradient term

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Abstract. In this paper, we mainly study the existence of radial solutions for a class of biharmonic equation with a convection term, involving two real parameters λ and ρ. We mainly use a combination of the fixed point index theory and the Banach contraction theorem to prove that there are $\lambda_0 > 0$ and $\rho_0 > 0$ such the equation admits at least one radial solution for all $(\lambda, \rho) \in [-\lambda_0, \infty[\times [0, \rho_0]$.

Mathematics Subject Classification (2010): 35K55, 35K65.

Keywords: Radial solution, biharmonic equation, index theory, existence.

1. Introduction and the main result

In the present paper, we mainly investigate the existence of radial solutions for the following biharmonic problems

$$(P_{\lambda,\rho}) \quad \begin{cases} \Delta(\Delta u) + \lambda|\nabla u|^q = \rho f(u) & \text{in} \quad B_1 \\ u = 0, \quad \Delta u = 0 & \text{in} \quad \partial B_1, \end{cases}$$

where $B_1 = \{x \in \mathbb{R}^N : |x| \leq 1\}$ is the unit ball in $\mathbb{R}^N$ $(N \geq 2)$, $(\lambda, \rho) \in \mathbb{R} \times \mathbb{R}^+$, $q \geq 1$ and $f \in C^1(\mathbb{R},]0, \infty[)$. Fourth-order equations are derived as models of different engineering and physical phenomena, such as the motion of fluid, static deflection of an elastic plate in a fluid [2, 4], epitaxial growth of nanoscale thin films [10, 14] and traveling waves in suspension bridges [5, 12]. Due to their several applications, both quasilinear and semilinear biharmonic equations have attracted much attention and many papers appeared in the literature studying existence and the multiplicity

Received 13 November 2020; Accepted 10 January 2021.
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of solutions, see for instance [9, 15, 14, 6, 7, 11] and the references therein. In [11], L. Kong studied the following boundary value problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \rho g(x)f(u) + h(x) & \text{in} \quad B_1 \\ u = \Delta u = 0 & \text{in} \quad \partial B_1, \end{cases} \tag{1.1}$$

and by Schauder's fixed point, introduced some sufficient conditions for existence of radial solutions. In particular, Guo et al. [7] considered the above problem with $h = 0$, and by using the fixed point index theory and the upper-lower solutions method, proved that for some $\rho^* > 0$, problem (1.1) has no positive radial solution if $\rho > \rho^*$; while if $\rho < \rho^*$, (1.1) has at least two positive radial solutions. Motivated by the above results, especially [7, 11], the purpose of this work is to prove the existence of radial solutions for the biharmonic problem $(P_{\lambda,\rho})$ by combining the fixed point index theory and the Banach contraction theorem. By changing the variable $u(x) = u(|x|)$, $r = |x|$, we transform problem $(P_{\lambda,\rho})$ to the following problem

$$\begin{cases} \mathcal{L}(\mathcal{L}(u)) + \lambda|u'|^q = \rho f(u) \text{ in } (0,1) \\ u(1) = \mathcal{L}(u)(1) = 0, \end{cases} \tag{1.2}$$

where $\mathcal{L}$ denotes the polar form of the Laplacian operator given by

$$\mathcal{L} := \frac{1}{r^{N-1}}\frac{d}{dr}\left(r^{N-1}\frac{d}{dr}\right).$$

We notice that any solution u of the ordinary equation (1.2), $u(|x|)$ is a radial solution of problem $(P_{\lambda,\rho})$. Similar to in [7, Pages 4-5] with $p = q = 2$, we see that problem (1.2) has an integral formulation given by

$$u(t) = \int_0^1\int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|u'(s)|^q + \rho f(u(s)\right) dsdt, \tag{1.3}$$

where, for $0 \le t, s \le 1$,

$$K(t,s) := \begin{cases} \frac{1}{N-2}s^{N-1}\left(\max\{t,s\}^{2-N} - 1\right), & if N > 2, \\ -s\ln(\max\{t,s\}), & if\, N = 2. \end{cases}$$

Define operators T and $\breve{T}$ in $C^1([0,1])$ as follows

$$T(u)(t) = T_{\lambda,\rho}(u)(t) := \int_0^1\int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|u'(s)|^q + \rho f(u(s))\right) dsdt \tag{1.4}$$

and for $(h,\beta) \in \mathbb{R}^2$,

$$\breve{T}(u)(t) = K_{\beta,h} + \int_0^1\int_0^1 K(1-t,\tau)K(\tau,1-s)\left(-\lambda|u'(s)|^q + \rho f(u(s))\right) dsdt \tag{1.5}$$

where

$$K_{\beta,h}(t) := h + \beta\int_0^1\left(k(t,s) + \frac{t}{N}\right) dt.$$

Remark 1.1. From [7] and [13] , we have

(i) $K(t,s) > 0$ for all $(t,s) \in (0,1)^2$;
(ii) $K(t,s) \le K(s,s)$ for all $(t,s) \in [0,1]^2$.
(iii) $K(t,s) \le K_\infty$, for all $(t,s) \in [0,1]^2$,

with $K_\infty := \frac{1}{e}$ if $n = 2$ and $K_\infty := (n-2)(n-1)^{-\frac{(n-1)}{n-2}}$ if $n \geq 3$.

We are now in position to present the main results.

Theorem 1.2. *Let $f : (-\infty, \infty) \longrightarrow \mathbb{R}^+$ be a nondecreasing continuous function such that* $\inf f > 0$. *Then there are $\lambda_0 > 0$ and $\rho_0 > 0$ such that problem $(P_{\lambda,\rho})$ has at least one radial solution for any $(\lambda, \rho) \in [-\lambda_0, \infty) \times [0, \rho_0]$. Moreover, for all $0 < \rho \leq \rho_0$,*

$$\lambda_\infty := \sup\{\lambda / \, (\lambda, \rho) \in S\} < \infty$$

and for all $0 > \lambda \geq -\lambda_0$,

$$\rho_\infty := \sup\{\rho / \, (\lambda, \rho) \in S\} < \infty$$

where

$$S := \{(\lambda, \rho) \in \mathbb{R}^2 / every\, \sigma, \mu \in \mathbb{R},\ \sigma \geq -\lambda,\ 0 \leq \mu \leq \rho,\ P_{\sigma,\mu}\ has\, a\, radial\ sol\}.$$

2. Preliminary results and proof of Theorem 1.2

We now introduce some basic technical lemmas that will be necessary to prove the main result. Let's start with a result introduced in [3], [7] and [1].

Lemma 2.1. *Let E be a Banach space, and P be a cone in E, and Ω be a boundary open set in E. Suppose that $T : \overline{\Omega \cap P} \to P$ is a completely continuous operator. If $Tu \neq \nu u$, for all $u \in \partial(\Omega \bigcap P)$ and all $\nu > 1$, then the fixed point index $i(T, \Omega, P) = 1$.*

Lemma 2.2. *If $g \in C[0, 1]$, we have that there exists $c_a(t) \in [a, 1]$, independent of t, such that*

$$\int_0^t \tau^{n-1} \int_a^1 K(\tau, s)|g(s)|ds d\tau = |g(c_a(t))| \int_0^t \tau^{n-1} \int_a^1 K(\tau, s) ds d\tau \qquad (2.1)$$

for all $t \geq a \geq 0$.

Proof. By Fubini's theorem we obtain

$$\int_0^t \tau^{n-1} \int_a^1 K(\tau, s)|g(s)|ds dt = \int_a^1 |g(s)| h(s, t) ds$$

where $h(s, t) := \int_0^t \tau^{n-1} K(\tau, s) d\tau$. It is easy to see that

$$\min_{[a,1]} | g | \leq \frac{\int_a^1 |g(s)| h(s, t) ds}{\int_a^1 h(s, t) ds} \leq \max_{[a,1]} | g |$$

Thus, there exists $a \leq c_a(t) \leq 1$, such that

$$\int_a^1 |g(s)| h(s, t) ds = |g(c_a(t))| \int_a^1 h(s, t) ds.$$

This completes the proof. □

Let us stress that in addition to the properties of function K presented in Remark 1.1, we will give another property in the following lemma.

Lemma 2.3. *Function $K(t,s)$ verifies the following assertion*

$$\int_0^1 K(1-t,s)ds = \frac{2t-1}{2N} + \int_0^1 K(t,s)ds$$

for all $t \in [0,1]$ and $N \geq 2$.

Proof. Let

$$\varphi(t) = \int_0^1 K(1-t,s)ds.$$

Then

$$\varphi(t) = \int_0^{1-t} K(1-t,s)ds + \int_{1-t}^1 K(1-t,s)ds =: \varphi_0(t) + \varphi_1(t).$$

Note that

$$\varphi_1(t) = \int_{1-t}^1 K(1-t,s)ds = \int_{1-t}^1 K(s,s)ds,$$

thus $\varphi_1'(t) = K(1-t,1-t)$. We also have

$$\varphi_0'(t) = \frac{1-t}{N} - K(1-t,1-t).$$

Therefore $\varphi'(t) = \frac{1-t}{N}$. Similarly, we have

$$\psi'(t) = \frac{-t}{N},\ \psi(t) := \int_0^1 K(t,s)ds.$$

If we set $\phi(t) = \varphi(t) - \psi(t) - \frac{t}{N}$, we obtain $\phi'(t) = 0$ for all $t \in [0,1]$, which implies

$$\phi(t) = \phi(0) = -\int_0^1 K(0,s)ds = -\int_0^1 K(s,s)ds = \frac{1}{2N}.$$

this completes the proof of the lemma. □

Lemma 2.4. *Let $(\alpha,\beta) \in \mathbb{R}^* \times \mathbb{R}^*$. Suppose that $\widetilde{T}$ has a fixed point in $C^1([0,1])$. Then the following problem*

$$\left(P^{\alpha,\beta}\right) \quad \begin{cases} \Delta(\Delta u) + \lambda|\nabla u|^q = \rho f(u) & \textit{in} \quad B_1 \\ u = \alpha, \quad \Delta u = -\beta & \textit{in} \quad \partial B_1, \end{cases}$$

has at least one solution.

Proof. Let $\overline{u}$ be a fixed point of $\widetilde{T}$ in $C^1([0,1])$ and let $v(r) = \overline{u}(1-r)$ for all $r \in [0,1]$. By the change of variable $\tau = 1-s$, we get

$$v(r) = K_{\beta,h}(r) + \int_0^1 \int_0^1 K(r,t)K(t,\tau)\left(-\lambda|v'(\tau)|^q + \rho f(v(\tau)\right) d\tau dt$$

It follows, from Lemma 2.3, that $v(r) = K_\beta(t) + T(v)(r)$. By a straightforward computation, we have

$$\mathcal{L}(\mathcal{L}(T(v))) = -\lambda|v'|^q + f(v).$$

Since $\mathcal{L}(\mathcal{L}(\int_0^1 K(.,t)dt)) = 0$, we deduce that $\mathcal{L}(\mathcal{L}(v)) = -\lambda|v'|^q + \rho f(v)$. Furthermore, we have $v(1) = h + \frac{\beta}{2N}$, $\mathcal{L}(v)(1) = -\beta$. Therefore, by taking $h = \alpha - \frac{\beta}{2N}$, we obtain that $u(x) = v(|x|)$ is a solution of problem $(P^{\alpha,\beta})$. □

Lemma 2.5. *There are $\beta_0, \lambda_0 > 0$ and $\rho_0 > 0$ such that $\breve{T}$ has a fixed point, for all $|\lambda| \leq \lambda_0$ and all $|\rho| \leq \rho_0$, with*

$$\lambda_0 = \begin{cases} \lambda_0(\beta_0) & \text{if } |\beta| \leq \beta_0 \\ \lambda_0(\beta) & \text{if } |\beta| > \beta_0 \end{cases} \quad \text{and} \quad \rho_0 = \begin{cases} \rho_0(\beta_0) & \text{if } |\beta| \leq \beta_0 \\ \rho_0(\beta) & \text{if } |\beta| > \beta_0. \end{cases}$$

Proof. We argue as [8], to prove the above lemma. Let $c > 0$ be fixed. By the continuity of f' on $[0, 1]$, we can find $\lambda_0^{(1)}, \rho_0^{(1)}, \beta_0 > 0$ depended on c and sufficiently small such that

$$\frac{\beta_0}{N} + \left(\rho_0^{(1)} \sup_{0<|t|<\frac{\beta_0}{2N}+|h|} f(t) + \rho_0^{(1)} c \sup_{0<|t|<c+\frac{\beta_0}{2N}+|h|} |f'(t)| + \lambda_0^{(1)} c^q \right) K_\infty < c.$$

Thus for all $|\beta| \leq \beta_0$, $|\lambda| \leq \lambda_0^{(1)}$ and $|\rho| \leq \rho_0^{(1)}$, we have

$$\frac{|\beta|}{N} + \left(\rho f \left(\frac{\beta}{2N} + h \right) + |\rho| c \sup_{[0,c+\frac{\beta_0}{2N}+|h|]} |f'| + |\lambda| c_\beta^q \right) K_\infty < c. \tag{2.2}$$

Let $|\beta| > \beta_0$, there are $c_\beta, \lambda_\beta, \rho_\beta > 0$ such that for all $|\lambda| \leq \lambda_\beta$ and $\rho \leq \rho_\beta$,

$$\frac{|\beta|}{N} + \left(\rho f \left(\frac{\beta}{2N} + h \right) + \rho c_\beta \sup_{[0,c_\beta+\frac{|\beta|}{2N}+|h|]} |f'| + |\lambda| c_\beta^q \right) K_\infty < c_\beta. \tag{2.3}$$

Consider

$$E_\beta := \left\{ u \in C([0,1]) : \left\| u - \frac{\beta}{2N} - h \right\| \leq M \right\},$$

where $||u|| := \max\{|u|_\infty, |u'|_\infty\}$ and $M = c$ if $|\beta| \leq \beta_0$, $M = c_\beta$ if $|\beta| > \beta_0$. For all $u \in E_\beta$, from $\int_0^1 K(r,t)dt \leq \int_0^1 K(t,t)dt$ (see Remark 1.1) and as $\int_0^1 K(t,t)dt = \frac{1}{2N}$, we have that

$$\breve{T}(u)(r) - A \geq -\frac{|\beta|}{2N} + \int_0^1 \int_0^1 K(1-r,t)K(t,1-s)F_{\lambda,\rho}(s)dsdt \tag{2.4}$$

where

$$F_{\lambda,\rho}(s) := -\lambda|u'(s)|^q + \rho f(u(s)),\ A := \frac{\beta}{2N} + h$$

. It is easy to check that if $u \in E_\beta$, we have

$$\rho f(u) < \rho f(\frac{\beta}{2N} + h) + \rho L M,$$

with $L := \sup_{|t|<M+\frac{\max\{|\beta|,\beta_0\}}{2N}+|h|} |f'(t)|$. It follows, from $u \in E_\beta$ and (2.4), that

$$\breve{T}(u)(r) - A \geq -\frac{|\beta|}{2N} - C \int_0^1 \int_0^1 K(1-r,t)K(t,1-s)dsdt. \tag{2.5}$$

where $C = |\lambda| M^q + \rho f(A) + \rho L M$. Since

$$0 < K_\infty < 1 \, and \int_0^1 \int_0^1 K(1-r,t)K(t,1-s)dsdt \leq K_\infty^2 \tag{2.6}$$

we obtain that

$$\breve{T}(u)(r)-\frac{\beta}{2N}-h\geq-\frac{|\beta|}{2N}-\left(|\lambda|\,M^{q}+\rho f(\frac{\beta}{2N}+h)+\rho LM\right)K_{\infty}^{2}>-M. \quad (2.7)$$

From $\int_0^1 K(t,r)dr \leq \int_0^1 K(r,r)dr = \frac{1}{2N}$ and (2.6), we have

$$\breve{T}(u)(r)-\frac{\beta}{2N}-h\leq\frac{|\beta|}{N}+\left(|\lambda|\,M^{q}+\rho f(\frac{\beta}{2N}+h)+\rho LM\right)K_{\infty}^{2}<M. \quad (2.8)$$

It follows that

$$\left|\breve{T}(u)(r)-\frac{\beta}{2N}-h\right|<M,$$

for all $|\lambda| \leq \lambda_0^{(1)}$ and all $0 \leq \rho \leq \rho_0^{(1)}$. Now we are able to show that $|\breve{T}(u)'(r)| < M$. Indeed, a straightforward computations show that

$$\begin{aligned}\left|\breve{T}(u)'(r)\right| &= \left|K'_{\beta}+\int_0^1\int_0^{1-r}\left(\frac{t}{1-r}\right)^{N-1}K(t,1-s)F_{\lambda,\rho}(s)\right| \\ &\leq \left(\frac{|\beta|}{N}+|\lambda|\,M^{q}+\rho f(\frac{\beta}{2N}+h)+\rho LM\right)K_{\infty}.\end{aligned}$$

Since $0 < K_\infty < 1$, we deduce that $\left|\breve{T}(u)'(r)\right| < M$. On the other hand, for u and $v \in E_\beta$, we obtain that

$$\begin{aligned}\left|\breve{T}(u)(r)-\breve{T}(v)(r)\right| \leq{}& |\lambda|\int_0^1\int_0^1 K(1-r,t)K(t,1-s)qM^{q-1}\,|v'-u'| \\ &+\rho\sup_{|t|<M+\frac{\beta}{2N}+h}|f'(t)|\int_0^1\int_0^1 K(1-r,t)K(t,1-s)|u-v|dsdt.\end{aligned}$$

We deduce that

$$\left|\breve{T}(u)(r)-\breve{T}(v)(r)\right| \leq D\left|\int_0^1\int_0^1 K(1-r,t)K(t,1-s)dsdt\right|\,||u-v||$$

where

$$D=|\lambda|\,qM^{q-1}+\rho\sup_{|t|<M+\frac{|\beta|}{2N}+|h|}|f'(t)|\,.$$

From (2.6), there are $\lambda_0^{(2)} > 0$ and $\rho_0^{(2)} > 0$ such that, for all $|\lambda| < \lambda_0^{(2)}$ and all $0 \leq \rho < \rho_0^{(2)}$,

$$\begin{aligned}\left|\breve{T}(u)(r)-\breve{T}(v)(r)\right| &\leq K_{\infty}^{2}D_0\,||u-v|| \\ &\leq \frac{1}{2}\,||u-v||\end{aligned}$$

with

$$D_0:=|\lambda|\,qM^{q-1}+\rho\sup_{|t|<M+|\frac{\max\{\beta,\beta_0\}}{2N}|+|h|}|f'(t)|\,.$$

On the other hand, we have

$$\begin{aligned}
&\left|\breve{T}'(u)(r)-\breve{T}'(v)(r)\right| \\
&\leq \left|\int_0^1\int_0^{1-r}\left(\frac{t}{1-r}\right)^{N-1}K(t,1-s)\left(\lambda|v'(s)|^q-\lambda|u'(s)|^q\right)\right| \\
&+\left|\int_0^1\int_0^{1-r}\left(\frac{t}{1-r}\right)^{N-1}K(t,1-s)\rho\left(f(u(s))-f(v(s))\right)dsdt\right|.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\breve{T}'(u)(r)-\breve{T}'(v)(r)\right| &\leq D_0K_\infty\,||u-v|| \\
&\leq \frac{1}{2}\,||u-v||
\end{aligned}$$

for all $0\leq\rho<\rho_0^{(2)}$ and $|\lambda|<\lambda_0^{(2)}$.
Therefore, for all

$$|\lambda|<\lambda_0=\min\left\{\lambda_0^{(2)},\lambda_0^{(1)}\right\},\ 0\leq\rho<\rho_0=\min\left\{\rho_0^{(2)},\rho_0^{(1)}\right\},$$

we obtain that

$$\|\breve{T}(u)(r)-\breve{T}(v)(r)\|\leq\frac{1}{2}\,||u-v||\,.$$

According to the Banach contraction theorem, $\breve{T}$ has a fixed point in E_β. □

2.1. Proof of Theorem 1.2

Let P be a cone defined as

$$P:=\{u\in C\,[0,1]\,,\ u\geq 0\}\,.$$

The proof is done in five steps.
Step 1. Case $-\lambda_0\leq\lambda\leq 0$. Consider the following operator

$$\widetilde{T}_\beta(u)(t):=K_{\beta,0}(t)+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|u'(s)|^q+\rho f(u(s))\right)dsdt. \tag{2.9}$$

In view of Lemma 2.5, we obtain that $\breve{T}_\beta$ has a fixed point in $C\,[0,1]$. Then, by Lemma 2.5, for all $|\beta|<\beta_0$ and $|\lambda|<\lambda_0$ there exists v_β in $C\,[0,1]$ such that $\breve{T}_\beta(v_\beta)=v_\beta$. Taking $W_\beta:=-v_\beta+\frac{t\beta}{N}$, we get

$$\begin{aligned}
W_\beta(t)&=-\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|W'|^q+\rho f\left(-W_\beta+\frac{t\beta}{N}\right)\right)dsd\tau \\
&-\beta\int_0^1 K(t,s)ds \\
&=:\breve{L}(W_\beta)(t).
\end{aligned}$$

Then

$$W_\beta' = \frac{t\beta}{N} + \int_0^t \left(\frac{\tau}{t}\right)^{N-1} \int_0^1 K(\tau, s) \left(-\lambda |W'|^q + \rho f\left(\int_s^1 W_\beta' d\xi + \frac{t\beta}{N}\right)\right)$$
$$= \breve{L}(W_\beta)'(t).$$

Let $X = C[0,1]$, with norm $||u|| = |u|_\infty$ and consider

$$L(u)(t) := \begin{cases} \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s) \left(-\lambda |u|^q + \rho f(\int_s^1 u(s)ds)\right) dsdr, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

for $u \in X$. Clearly, $(X, ||.||)$ is a Banach space. On other hand, $L : P \longrightarrow P$ is completely continuous. Indeed, by Hospital's rule, we obtain that for all $u \in X$, $L(u) \in X$. It is easy to see that $L(u) \geq 0$. We deduce that $L(P) \subset P$. By Ascoli-Arzela theorem and absolute continuity of integral, we obtain that L is completely continuous. Let us consider the set $\Omega := \left\{u \in X,\ u < W_\beta^{'}\right\}$. For $u \in \partial\Omega \cap P$, we have

$$-\lambda |u|^q \leq -\lambda \left|W_\beta^{'}\right|^q \ and \int_s^1 uds < \int_s^1 W_\beta^{'} ds.$$

Using f is nondecreasing and $u \in \partial\Omega$ and by choosing $\beta > 0$, we have

$$L(u)(t) \leq L(W_\beta^{'})(t) < \breve{L}(W_\beta)^{'}(t) = W_\beta^{'}(t) = u(t)$$

Then $L(u)(t) \neq \nu u(t)$, for all $\nu > 1$. Moreover, from $f(0) > 0$, we have that $L(0)(t) \neq 0$. Then $L(u)(t) \neq \nu u(t)$, for all $u \in \partial(\Omega \bigcap P)$ and for all $\nu > 1$. It follows, from Lemma 2.3, that $i(L, \Omega, P) = 1$. Thus, there exits $u \in \Omega$ such that $L(u) = u$. Let

$$W(r) := \int_r^1 u(s)ds.$$

Then, we have

$$\begin{aligned} W(r) &= \int_r^1 u(s)ds = \int_r^1 L(u)(s)ds \\ &= \left[-\int_0^1 \int_0^1 K(t,\tau)K(\tau,s) \left(-\lambda |u|^q + \rho f(\int_s^1 u(\xi)d\xi)\right) dsdt\right]_r^1 \\ &= \int_0^1 \int_0^1 K(t,\tau)K(\tau,s) \left(-\lambda |W^{'}(s)|^q + \rho f(W(s))\right) dsdt. \end{aligned}$$

This implies that $W = T(W)$. Therefore, the function $W : B(0,1) \to \mathbb{R}$, $x \to W(|x|)$ is a solution of problem $(P_{\lambda,\rho})$, for all $-\lambda_0 \leq \lambda \leq 0$ and for all $0 < \rho \leq \rho_0$.

Step 2. Case $\lambda_0' > \lambda > 0$ (λ_0' will be defined below). By taking $\lambda = 0$ in step 1, we obtain that there exists $V_\beta \in C[0,1]$ such that

$$\begin{aligned} V_\beta &= K_{\beta,-\beta/N}(t) + \rho \int_0^1 \int_0^1 K(t,\tau)K(\tau,s) f(V_\beta) dsdt \\ &= : \breve{L}_0(V_\beta). \end{aligned}$$

Then

$$V_\beta' = \beta\frac{(1-t)}{tN} - \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s)\rho f\left(V_\beta\right) dsdr$$
$$= \left(\breve{L}_0(V_\beta)\right)'.$$

Let us consider the set

$$\Omega' := \left\{u \in X,\ u < -V_\beta{}' - \frac{\beta}{N}\right\},$$

for $\beta < 0$. Then, for $u \in \Omega' \cap P$, we have $0 < u < -V_\beta{}' - \frac{\beta}{N}$. This implies that $\|u\| \leq \left\|V_\beta'\right\|$. So, if we take

$$0 \leq \lambda \leq \lambda_0' := \rho \min\left\{\frac{\inf f(t)}{\left\|-V_\beta{}' - \frac{\beta}{N}\right\|^q}, \lambda_0\right\}$$

we obtain

$$-\lambda|u(s)|^q + \rho f\left(\int_r^1 u(s)ds\right) \geq 0.$$

Therefore, $L(\Omega' \cap P) \subset P$. Now, let $u \in \partial\Omega' \cap P$. We have

$$\begin{aligned} L(t) &:= \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s)\left(-\lambda|u(s)|^q + \rho f\left(\int_s^1 u(\xi)d\xi\right)\right) dsdr \\ &< \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s)\rho f\left(-\int_s^1 V_\beta(\xi)'d\xi - \frac{\beta}{N} + \frac{\beta s}{N}\right) dsdr \\ &< \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s)\rho f\left(V_\beta(s) + \frac{\beta s}{N}\right) dsdr \end{aligned}$$

By using $\beta < 0$ and the fact that f is nondecreasing, we get

$$L(t) < -\left(\breve{L}_0(V_\beta)\right)'(t) = -V_\beta'(t) = u(t) + \frac{\beta}{N} < u(t)$$

Then $L(t) \neq \nu u(t)$ for all $\nu > 1$ and for all $u \in \partial\Omega' \cap P$. Moreover, $L(0)(t) \neq 0$. Thus, $L(u)(t) \neq \nu u(t)$ for all $\nu > 1$ and for all $u \in \partial(\Omega' \cap P)$. Therefore, from Lemma 2.1, $i(L,\Omega', P) = 1$. Then, there exists $u \in C[0,1]$ such that $L(u) = u$. We deduce that

$$W : X \to \mathbb{R},\ t \to \int_t^1 u(s)ds$$

satisfies $T(W) = W$. So, we obtain that problem $(P_{\lambda,\rho})$ has a radial solution for all $0 \leq \lambda \leq \lambda_0'$ and for all $0 < \rho \leq \rho_0$.

Step 3. For every $(\lambda, \rho) \in [\lambda_0', \infty[\times[0, \rho_0]$, the problem $(P_{\lambda,\rho})$ has a radial solution. Indeed, let $(\lambda, \rho) \in [\lambda_0', \infty[\times[0, \rho_0]$. From Step 3, problem (P_{0,ρ_0}) has a radial solution. Then, there exists $u_0 \in C[0,1]$ such that $T_{0,\rho_0}(u_0) = u_0$. Consider the cone

$$P := \{u \in X,\ u \geq 0\}$$

and the set $\Omega := \{u \in X,\ u < u_0\}$. Then, we have

$$\Omega \cap P = \{u \in X,\ 0 \leq u < u_0\}.$$

So, $\partial(\Omega \cap P) = \{0\} \cup \{u = u_0\}$. Since f is nondecreasing, we get

$$T_{\lambda,\rho}(u)(t) < T_{0,\rho_0}(u)(t) < T_{0,\rho_0}(u_0)(t) = u_0(t) = u(t)$$

for $u \in \partial\Omega$. We also have $T_{\lambda,\rho}(0)(t) > 0$. Therefore, $T_{\lambda,\rho}(u)(t) \neq \nu u(t)$, for all $\nu \geq 1$ and for all $u \in \partial(\Omega \cap P)$. So, from Lemma 2.1, $i(T_{\lambda,\rho}, \Omega, P) = 1$. Consequently, $(P_{\lambda,\rho})$ has a least one radial solution.

Step 4. $\lambda_\infty(\rho) < \infty$ **and** $\rho_\infty(\lambda) < \infty$. Let $0 \leq \rho \leq \rho_0$. Suppose that $\lambda_\infty(\rho) = -\infty$. Then, there exits $(\lambda_n, \rho) \in S$, with $\lambda_n \to -\infty$ and let u_n be a solution radial of problem $(P_{\lambda_n,\rho})$. Then

$$u_n(t)' = -\int_0^t \left(\frac{\tau}{t}\right)^{N-1} \int_0^1 K(\tau, s)\left(-\lambda_n|u_n'(s)| + \rho f(u_n(s))\right) ds d\tau < 0 \tag{2.10}$$

since $f > 0$, we get

$$|u_n(t)'| > -\lambda_n \int_0^t \left(\frac{\tau}{t}\right)^{N-1} \int_{\frac{1}{2}}^1 K(\tau, s)|u_n(s)'|^q ds d\tau.$$

In view of Lemma 2.2, there exists $1/2 \leq c_{1/2}(t) \leq 1$ such that

$$|u_n(t)'| > -\lambda_n |u_n(c_{1/2}(t))'|^q \int_0^t \left(\frac{\tau}{t}\right)^{N-1} \int_{\frac{1}{2}}^1 K(\tau, s) ds d\tau.$$

From (2.10), we have $0 < |u_n'(1/2)| \leq |u_n'(c_{1/2}(t))|$. By taking $t = \frac{1}{2}$, we get

$$1 > -\lambda_n |u_n(1/2)'|^{q-1} \int_0^{1/2} (2\tau)^{N-1} \int_0^1 K(\tau, s) ds d\tau.$$

By (2.10) and $\epsilon_0 := \inf f > 0$, we get

$$\begin{aligned} |u_n'(1/2)| &= \int_0^{1/2} (2\tau)^{N-1} \int_0^1 K(\tau, s)\rho f(u_n(s)) ds d\tau \\ &> \epsilon_0 \int_0^{\frac{1}{2}} (2\tau)^{N-1} \int_0^1 K(\tau, s)\rho ds d\tau. \end{aligned}$$

It follows that

$$1 > -\lambda_n (\rho\epsilon_0)^{q-1} \left(\int_0^{1/2} 2\tau^{N-1} \int_{\frac{1}{2}}^1 K(\tau, s) ds d\tau\right)^q.$$

Letting $n \longrightarrow \infty$, we obtain a contradiction. On other hand, let $-\lambda_0 < \lambda \leq 0$. Suppose that $\rho_\infty(\lambda) = \infty$. Then, there exits $(\lambda, \rho_n(\lambda)) \in S$ such that $\rho_n(\lambda) \to \infty$. If we follow the same way as above, we obtain

$$1 > -\lambda (\rho_n\epsilon_0)^{q-1} \left(\int_0^{1/2} 2\tau^{N-1} \int_{\frac{1}{2}}^1 K(\tau, s) ds d\tau\right)^q. \tag{2.11}$$

Letting $n \to \infty$, we obtain a contradiction. This concludes the proof of Theorem 1.2.

Acknowledgments. The authors warmly thank the anonymous referee for her/his useful comments on the paper.

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