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A spectral criterion for the existence of the stabilizing solution of a class of Riccati type differential equations with periodic coefficients

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Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th anniversary.

Abstract. In this paper, we investigate the existence and uniqueness of a stabilizing solution to a periodic backward nonlinear differential equation. This class of nonlinear equations includes as special cases many of the continuous-time Riccati equations arising both in deterministic and stochastic linear quadratic (LQ) type control problems.

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1. Introduction

Matrix Riccati differential equations have been extensively studied during the past decades by researchers both in control theory and differential equations area. First, in the work of Kalman [13] it is obtained the existence of global solution for a matrix Riccati differential equation under the controllability and observability conditions. Second, Wonham [20] established these results to the framework of stochastic control considering the so-called Riccati equations of stochastic control. Both results have been related to the so-called LQ optimisation problem.

The above earlier results impose controllability and observability conditions, which are somewhat conservative. More recent works replace these conditions by some weaker ones, as stabilizability and detectability.

In the particular case of constant coefficients, the well-known (standard) algebraic Riccati equation arise naturally and play an essential role. The solutions, whose existence is assured by Popov conditions, properties and applications are studied in many works, see e.g. [8], [15], and [18]. In the study of stochastic case, a class of modified algebraic Riccati equations appear. The result from [23] provides a necessary and sufficient condition for the existence of a mean-square stabilizing solution for this kind of equations. This assures that the corresponding stochastic LQ problem is solvable, i.e. the cost function is minimized and the stochastic closed-loop system is mean-square stable. Also, it was shown in [10] that a mean-square stabilizing solution of a modified algebraic Riccati equation, is unique (if it exists) and coincides with the maximal solution. In [19] it is considered LQ optimal stochastic control problem with random coefficients via the stochastic maximum principle.

Further, if we consider the case of periodic coefficients, the early results that provide the existence of a stabilizing solution for a matrix Riccati differential equation goes to [21]. In addition, the method developed for algebraic Riccati equation has been extended to the Riccati equations for continuous-time linear periodic systems in [17]. More recently, [14] extends the eigenvalue method to the differential Riccati equation in terms of nonlinear eigenvalues and eigenvectors. Various results concerning periodic systems are presented in [1].

The novelty to be pursued in this paper is the use of spectral theory of positive operators to deal with generalized Riccati differential equation (GRDE) in order to obtain necessary and sufficient conditions for the existence and uniqueness of the stabilizing solution. The contribution of the paper can be summarised as follows. First, regarding the set-up of the problem, i.e. the class of nonlinear backward differential equation we first present several special cases. We consider LQ optimal regulator and stochastic LQ optimal regulator. For stochastic framework the cases when the controlled system is affected by multiplicative white noise and of the perturbation model by a Markov process simultaneous, respectively, is considered. This indicates that the considered problem arises in a natural way in optimal control problems and incorporates all these cases. Discussions on backward differential equations and applications can be found in [6] and [22].

Motivated by this goal, we present the definition of stabilizing solution of the class of nonlinear continuous-time backward equation under consideration, see Definition 2.5. Further, we introduce the concept of unobservable characteristic multipliers of a pair formed by a continuous-time linear equation with periodic coefficients and an output, see Definition 3.2. In this framework it is proved the central contribution of the paper, i.e. the existence of the stabilizing solution of the GRDE under consideration, see Theorem 4.3.

Organization of the paper. First, in Subsection 1.1 we introduce useful notations. Section 2 is devoted to the formulation of the problem. It presents our framework, motivated by several special cases of LQ control problems related to our model. Section 3 is dedicated to state and describe some preliminary results, while Section 4 contains the main result of the paper, i.e. the existence of the positive semidefinite and periodic stabilizing solution of the GRDE. Finally, in Section 5, we draw conclusions.

1.1. Notations

The notations used in this work are in general the standard ones. Here we mention some notations less met. If m, n, N are fixed natural numbers, then

$$\mathcal{M}_{nm}^N \triangleq \underbrace{\mathbb{R}^{n\times m} \times \cdots \times \mathbb{R}^{n\times m}}_{N\ \ times},$$

$\mathbb{R}^{n\times m}$ being the linear space of the matrices with $n-$rows and $m-$columns. $\mathbf{B} \in \mathcal{M}_{nm}^N$ if and only if $\mathbf{B} = (B_1, B_2, \ldots, B_N)$, $B_i \in \mathbb{R}^{n\times m}$. When $n = m$ we shall write $\mathcal{M}_n^N$ instead of $\mathcal{M}_{nn}^N$. If $\mathbf{B} \in \mathcal{M}_{nm}^N$, $\mathbf{C} \in \mathcal{M}_{mp}^N$ then $\mathbf{BC}$ is defined by

$$\mathbf{BC} = (B_1C_1, B_2C_2, \ldots, B_NC_N) \in \mathcal{M}_{np}^N.$$

If $\mathbf{A} = (A_1, A_2, \ldots, A_N) \in \mathcal{M}_n^N$ is such that $det\ A_i \neq 0$, $1 \leq i \leq N$, then

$$\mathbf{A}^{-1} \triangleq (A_1^{-1}, A_2^{-1}, \ldots, A_N^{-1}) \in \mathcal{M}_n^N.$$

If $\mathbf{X} = (X_1, X_2, \ldots, X_N) \in \mathcal{M}_{mp}^N$ then

$$\mathbf{X}^T \triangleq (X_1^T, X_2^T, \ldots, X_N^T) \in \mathcal{M}_{pm}^N.$$

Here and in the sequel superscript $()^T$ stands for the transpose of a matrix or a vector. $\mathcal{S}_n \subset \mathbb{R}^{n\times n}$ stands for the linear space of symmetric matrices of size $n \times n$ and

$$\mathcal{S}_n^N \triangleq \underbrace{\mathcal{S}_n \times \mathcal{S}_n \times \cdots \times \mathcal{S}_n}_{N\ \ times}.$$

$\mathcal{S}_n^N$ has a structure of finite dimensional real Hilbert space induced by the inner product

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i=1}^N Tr\ [X_iY_i], \tag{1.1}$$

for all $\mathbf{X} = (X_1, X_2, \ldots, X_N)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_N)$ from $\mathcal{S}_n^N$. $Tr\ [\cdot]$ denotes the trace of a matrix. On $\mathcal{S}_n^N$ one should consider the order relation $\succeq$ induced by the solid, closed, normal, convex cone

$$\mathcal{S}_n^{N+} \triangleq \{\mathbf{X} \in \mathcal{S}_n^N \mid \mathbf{X} = (X_1, X_2, \ldots, X_N)\ \text{ with }\ X_i \geq 0,\ 1 \leq i \leq N\}.$$

Here $X_i \geq 0$ means that X_i is positive semidefinite matrix. If $\mathcal{X}$, $\mathcal{Y}$ are two vector spaces, then $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denotes the linear space of linear operators $T : \mathcal{X} \to \mathcal{Y}$. If $\mathcal{X} = \mathcal{Y}$ we shall write $\mathcal{B}(\mathcal{X})$ instead of $\mathcal{B}(\mathcal{X}, \mathcal{X})$.

2. A class of nonlinear backward differential equations

2.1. Model description and basic assumptions

On the Hilbert space $\mathcal{S}_n^N$ we consider backward differential equation

$$-\frac{d}{dt}\mathbf{X}(t) = \mathcal{R}(t, \mathbf{X}(t)),\ \ t \in \mathbb{R}, \tag{2.1}$$

with the unknown function $\mathbf{X}(t) = (X(t,1), X(t,2), \ldots, X(t,N)) \in \mathcal{S}_n^N$. In (2.1) the pair $(t, \mathbf{X}) \to \mathcal{R}(t, \mathbf{X}) :\ Dom\ \mathcal{R} \subset \mathbb{R} \times \mathcal{S}_n^N \to \mathcal{S}_n^N$ is described by

$$\mathcal{R}(t,\mathbf{X}) = (\mathcal{R}_1(t,\mathbf{X}), \mathcal{R}_2(t,\mathbf{X}), \ldots, \mathcal{R}_N(t,\mathbf{X}))$$

with

$$\begin{aligned}\mathcal{R}_i(t,\mathbf{X}) =& A^T(t,i)X(i) + X(i)A(t,i) + \Pi_1(t)[\mathbf{X}](i) \\ &- (X(i)B(t,i) + \Pi_2(t)[\mathbf{X}](i) + L(t,i)) \\ &\cdot (\Pi_3(t)[\mathbf{X}](i) + R(t,i))^{-1}(X(i)B(t,i) \\ &+ \Pi_2(t)[\mathbf{X}](i) + L(t,i))^T + M(t,i), \ \ 1 \le i \le N. \end{aligned} \tag{2.2}$$

Here,

$$\mathbf{X} \to \mathbf{\Pi}_1(t)[\mathbf{X}] \triangleq (\Pi_1(t)[\mathbf{X}](1), \ldots, \Pi_1(t)[\mathbf{X}](N)) : \mathcal{S}_n^N \to \mathcal{S}_n^N;$$

$$\mathbf{X} \to \mathbf{\Pi}_2(t)[\mathbf{X}] \triangleq (\Pi_2(t)[\mathbf{X}](1), \ldots, \Pi_2(t)[\mathbf{X}](N)) : \mathcal{S}_n^N \to \mathcal{M}_{nm}^N;$$

$$\mathbf{X} \to \mathbf{\Pi}_3(t)[\mathbf{X}] \triangleq (\Pi_3(t)[\mathbf{X}](1), \ldots, \Pi_3(t)[\mathbf{X}](N)) : \mathcal{S}_n^N \to \mathcal{S}_m^N;$$

are given linear operators.

$$Dom\ \mathcal{R} \triangleq \{(t,\mathbf{X}) \in \mathbb{R} \times \mathcal{S}_n^N \,|\, det\ (\Pi_3(t)[\mathbf{X}](i) + R(t,i)) \neq 0,\ 1 \le i \le N\}.$$

According to the convention of notations from Subsection 1.1, we may rewrite (2.2) in a compact form as:

$$\begin{aligned}\mathcal{R}(t,\mathbf{X}) =& \mathbf{A}^T(t)\mathbf{X} + \mathbf{X}\mathbf{A}(t) + \mathbf{\Pi}_1(t)[\mathbf{X}] - (\mathbf{X}\mathbf{B}(t) + \mathbf{\Pi}_2(t)[\mathbf{X}] + \mathbf{L}(t)) \\ &\cdot (\mathbf{\Pi}_3(t)[\mathbf{X}] + \mathbf{R}(t))^{-1}(\mathbf{X}\mathbf{B}(t) + \mathbf{\Pi}_2(t)[\mathbf{X}] + \mathbf{L}(t))^T + \mathbf{M}(t)\end{aligned} \tag{2.3}$$

for all $(t,\mathbf{X}) \in Dom\ \mathcal{R}$, where we have used the following notations:

$$\begin{aligned}&\mathbf{A}(t) = (A(t,1), A(t,2), \ldots, A(t,N)) \in \mathcal{M}_n^N, \\ &\mathbf{B}(t) = (B(t,1), B(t,2), \ldots, B(t,N)) \in \mathcal{M}_{nm}^N,\end{aligned} \tag{2.4a}$$

$$\begin{aligned}&\mathbf{M}(t) = (M(t,1), M(t,2), \ldots, M(t,N)) \in \mathcal{S}_n^N, \\ &\mathbf{L}(t) = (L(t,1), L(t,2), \ldots, L(t,N)) \in \mathcal{M}_{nm}^N, \\ &\mathbf{R}(t) = (R(t,1), R(t,2), \ldots, R(t,N)) \in \mathcal{S}_m^N.\end{aligned} \tag{2.4b}$$

Based on the operators $\Pi_k(t)[\cdot](i)$ involved in (2.2) we define:

$$\mathbf{\Pi}(t)[\mathbf{X}] = (\Pi(t)[\mathbf{X}](1), \Pi(t)[\mathbf{X}](2), \ldots, \Pi(t)[\mathbf{X}](N))$$

as

$$\Pi(t)[\mathbf{X}](i) = \begin{pmatrix} \Pi_1(t)[\mathbf{X}](i) & \Pi_2(t)[\mathbf{X}](i) \\ \Pi_2^T(t)[\mathbf{X}](i) & \Pi_3(t)[\mathbf{X}](i) \end{pmatrix} \in \mathcal{S}_{n+m}^N. \tag{2.5}$$

Hence, for each $t \in \mathbb{R}$, $\mathbf{X} \to \mathbf{\Pi}(t)[\mathbf{X}] : \mathcal{S}_n^N \to \mathcal{S}_{n+m}^N$ is a linear operator.

Remark 2.1. From (2.3) one sees that the operator $\mathcal{R}(\cdot,\cdot)$ is defined by the quadruple $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot), \mathbf{Q}(\cdot))$ where $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$ are described in (2.4a), the operator valued function $\mathbf{X} \to \mathbf{\Pi}(\cdot)[\mathbf{X}]$ is described in (2.5) and

$$\mathbf{Q}(t) = \begin{pmatrix} \mathbf{M}(t) & \mathbf{L}(t) \\ \mathbf{L}^T(t) & \mathbf{R}(t) \end{pmatrix} \in \mathcal{S}_{n+m}^N. \tag{2.6}$$

Obvious the components $Q(t,i)$ of $\mathbf{Q}(t)$ are

$$Q(t,i) = \begin{pmatrix} M(t,i) & L(t,i) \\ L^T(t,i) & R(t,i) \end{pmatrix} \in \mathcal{S}_{n+m},\ 1 \leq i \leq N.$$

The developments from this work are done under the following assumption: (**H**).

(a). $\mathbf{A}(\cdot) : \mathbb{R} \to \mathcal{M}_n^N$, $\mathbf{B}(\cdot) : \mathbb{R} \to \mathcal{M}_{nm}^N$, $\mathbf{M}(\cdot) : \mathbb{R} \to \mathcal{S}_n^N$, $\mathbf{L}(\cdot) : \mathbb{R} \to \mathcal{M}_{nm}^N$, $\mathbf{R}(\cdot) : \mathbb{R} \to \mathcal{S}_n^N$ are continuous functions which are periodic of period θ.

(b). $t \to \mathbf{\Pi}(t)[\cdot] : \mathbb{R} \to \mathcal{B}(\mathcal{S}_n^N, \mathcal{S}_{n+m}^N)$ is a continuous operator valued function which is periodic with period θ.

(c). for each $t \in \mathbb{R}$, $\mathbf{X} \to \mathbf{\Pi}(t)[\mathbf{X}] : \mathcal{S}_n^N \to \mathcal{S}_{n+m}^N$ is a positive operator, i.e. $\mathbf{\Pi}(t)[\mathbf{X}] \succeq 0$ whenever $\mathbf{X} \succeq 0$.

(d).

$$R(t,i) > 0 \tag{2.7a}$$

$$M(t,i) - L(t,i)R^{-1}(t,i)L^T(t,i) \geq 0 \tag{2.7b}$$

for all $t \in \mathbb{R}$, $1 \leq i \leq N$.

2.2. Several relevant special cases of the differential equation (2.1)

In this subsection we display several special cases of the backward differential equation (2.1) arising in a natural way, in some optimal control problems in both deterministic and stochastic framework.

A. The LQ optimal regulator.
We consider the optimal control problem described by the controlled system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0 \end{aligned} \tag{2.8}$$

the performance criterion

$$J(x_0, u) = \int_{t_0}^{\infty} (x^T(t)M(t)x(t) + 2x^T(t)L(t)u(t) + u^T(t)R(t)u(t))dt \tag{2.9}$$

and the class of admissible controls $u(\cdot) \in L_2([t_0,\infty); \mathbb{R}^m)$. The problem of the optimal regulator requires the finding of the conditions which guarantee the existence of a control $\tilde{u}(\cdot)$ which minimizes the cost function (2.9) along of the trajectories of the controlled system (2.8) determined by the all admissible controls $u(\cdot)$. One shows that if $R(t) > 0$, $M(t) - L(t)R^{-1}(t)L^T(t) \geq 0$, for all $t \geq t_0$, then the optimal control may be computed using a special solution named "stabilizing solution" of the matrix differential equation

$$\begin{aligned}-\dot{X}(t) =&A^T(t)X(t) + X(t)A(t) - (X(t)B(t) + L(t))R^{-1}(t)\\ &\cdot (X(t)B(t) + L(t))^T + M(t).\end{aligned} \tag{2.10}$$

The problem of the existence of the optimal control which solves the optimization problem described by (2.8)-(2.9) reduces to the problem of the existence of the stabilizing solution of the matrix differential equation (2.10). The differential equation (2.10) may be regarded as a special case of (2.1) when $N = 1$, $\Pi_k(t)[\mathbf{X}] = 0$, $1 \le k \le 3$, $\mathbf{X} \in \mathcal{S}_n^1$. Since, for $n = 1$ the differential equation (2.10) reduces to the well known scalar differential equation studied by the mathematician Jacopo Francesco Riccati in the first part of the 18^{th} century, the equation (2.10) was called "matrix Riccati differential equation".

B. The stochastic linear quadratic optimal regulator.

The case when the controlled system is affected by multiplicative white noise perturbations.
In this case, the optimal control problem consists of the finding of a control $\tilde{u}(\cdot)$ which minimizes the cost function

$$J(x_0, u) = \mathbb{E}\left[\int_0^\infty (x^T(t)M(t)x(t) + 2x^T(t)L(t)u(t) + u^T(t)R(t)u(t))dt\right] \tag{2.11}$$

along of the trajectories of the controlled system

$$\begin{aligned}&dx(t) = (A(t)x(t) + B(t)u(t))dt + (C(t)x(t) + D(t)u(t))dw(t)\\ &x(0) = x_0 \in \mathbb{R}^n\end{aligned} \tag{2.12}$$

determined by the admissible controls $u(t)$ from a class of admissible stochastic processes $\mathcal{U}(x_0)$. Here and in the sequel $\mathbb{E}[\cdot]$ stands for the mathematical expectation. In (2.12) $\{w(t)\}_{t\ge 0}$ is an 1-dimensional standard Wiener process defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ (see [11], [16]). In 1968, to solve this problem W. M. Wonham (see [20]) introduced the following matrix differential equation

$$\begin{aligned}-\dot{X}(t) =&A^T(t)X(t) + X(t)A(t) + C^T(t)X(t)C(t)\\ &- (X(t)B(t) + C^T(t)X(t)D(t) + L(t))\\ &\cdot (D^T(t)X(t)D(t) + R(t))^{-1}(B^T(t)X(t) + D^T(t)X(t)C(t)\\ &+ L^T(t)) + M(t).\end{aligned} \tag{2.13}$$

Since in the special case $C(t) = 0$, $D(t) = 0$, the differential equation (2.13) reduces to the matrix Riccati differential equation (2.10), it was called "matrix Riccati differential equation of stochastic control".

The problem of the stochastic linear quadratic optimal regulator in the case of the simultaneous presence of multiplicative white noise perturbations and of the perturbation modeled by a Markov process.

In this case, the controlled system is of the form:

$$dx(t) = (A(t,\eta_t)x(t) + B(t,\eta_t)u(t))dt + (C(t,\eta_t)x(t) + D(t,\eta_t)u(t))dw(t) \tag{2.14}$$

and the performance criterion:

$$J(x_0,u) = \mathbb{E}[\int_0^\infty (x^T(t)M(t,\eta_t)x(t) + 2x^T(t)L(t,\eta_t)u(t) + u^T(t)R(t,\eta_t)u(t))dt]. \tag{2.15}$$

Here, $\{w(t)\}_{t\geq 0}$ is 1-dimensional standard Wiener process as before, and $\{\eta_t\}_{t\geq 0}$ is a standard Markov process defined by the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$, taking values in the finite set $\mathcal{N} = \{1, 2, \ldots, N\}$ and having the transition semigroup $P(t) = e^{Qt}$. The elements q_{ij} of the generator matrix Q have the properties:

$$q_{ij} \geq 0, \text{ if } i \neq j, \qquad \sum_{l=1}^{N} q_{il} = 0, \text{ for all } i,j \in \mathcal{N}.$$

For more details we refer to [2], [3], [7], [11]. It is assumed that the processes $\{w(t)\}_{t\geq 0}$ and $\{\eta_t\}_{t\geq 0}$ are independent stochastic processes.

In the computation of the control which minimizes the cost functional (2.15) along of the trajectories of the controlled system (2.14) one uses the stabilizing solution of the following system of matrix Riccati type differential equations:

$$\begin{aligned}
-\dot{X}(t,i) =& A^T(t,i)X(t,i) + X(t,i)A(t,i) + C^T(t,i)X(t,i)C(t,i) \\
&- (X(t,i)B(t,i) + C^T(t,i)X(t,i)D(t,i) + L(t,i)) \\
&\cdot (D^T(t,i)X(t,i)D(t,i) + R(t,i))^{-1}(B^T(t,i)X(t,i) \\
&+ D^T(t,i)X(t,i)C(t,i) + L^T(t,i)) + \sum_{j=1}^{N} q_{ij}X(t,j) + M(t,i).
\end{aligned} \tag{2.16}$$

$1 \leq i \leq N$. One sees that (2.16) is a special case of (2.1) with $A(t,i)$ replaced by $A(t,i) + \frac{1}{2}q_{ii}I_n$,

$$\Pi_1(t)[\mathbf{X}](i) = C^T(t,i)X(i)C(t,i) + \sum_{\substack{j=1\\ j\neq i}}^{N} q_{ij}X(j),$$

$$\Pi_2(t)[\mathbf{X}](i) = C^T(t,i)X(i)D(t,i)$$

$$\Pi_3(t)[\mathbf{X}](i) = D^T(t,i)X(i)D(t,i), \ \ 1 \leq i \leq N,$$

$$\mathbf{X} = (X(1), X(2), \ldots, X(N)) \in \mathcal{S}_n^N.$$

Remark 2.2. From the previous examples, one sees that the differential equation (2.1) contains as special cases different types of Riccati differential equations arising in both deterministic and stochastic framework. That is why, in the sequel we shall call the differential equation (2.1) "generalized Riccati differential equation" (GRDE).

2.3. The stabilizing solution of a GRDE

Let $\mathcal{F}_{ad}$ be the set of all continuous and θ periodic functions $\mathbf{F} : \mathbb{R} \to \mathcal{M}_{mn}^{N}$. In the developments from this work these functions will be named "admissible feedback gains".

If $\mathbf{\Pi}(\cdot)$ is the operator valued function described in (2.5) and $\mathbf{F}(\cdot)$ is an arbitrary admissible feedback gain, we associate a new operator valued function $\mathbf{\Pi_F}(\cdot)$ defined as follows:

$$\mathbf{\Pi_F}(t)[\mathbf{X}] = (\Pi_{\mathbf{F}}(t)[\mathbf{X}](1), \Pi_{\mathbf{F}}(t)[\mathbf{X}](2), \dots, \Pi_{\mathbf{F}}(t)[\mathbf{X}](N)) \qquad (2.17a)$$

$$\Pi_{\mathbf{F}}(t)[\mathbf{X}](i) = (I_n \quad F^T(t,i))\Pi(t)[\mathbf{X}](i)(I_n \quad F^T(t,i))^T, \;\; 1 \le i \le N. \qquad (2.17b)$$

Remark 2.3. **(a).** From (2.5) and (2.17) we infer that $\mathbf{X} \to \mathbf{\Pi_F}(t)[\mathbf{X}] \in \mathcal{B}(\mathcal{S}_n^N)$. Moreover, if the assumption (**H**) **(c)** is fulfilled, then $\mathbf{\Pi_F}(t)[\mathbf{X}] \succeq 0$ whenever $\mathbf{X} \succeq 0$.

(b). In the sequel, $\mathbf{\Pi_F^*}(t)[\cdot]$ denotes the adjoint operator of $\mathbf{\Pi_F}(t)[\cdot]$ with respect to the inner product (1.1). Thus, in the special case when $\mathbf{\Pi}(t)[\cdot]$ is associated to the Riccati differential equation (2.13) we infer that

$$\mathbf{\Pi_F^*}(t)[\mathbf{X}] = (C(t) + D(t)F(t))\mathbf{X}(C(t) + D(t)F(t))^T, \;\; \text{for all } \; \mathbf{X} \in \mathcal{S}_n^1, \qquad (2.18)$$

and in the case when the operator $\mathbf{\Pi}(t)[\cdot]$ is associated to the Riccati differential equation (2.16), one obtains that

$$\begin{aligned}\mathbf{\Pi_F^*}(t)[\mathbf{X}](i) =& (C(t,i) + D(t,i)F(t,i))X(i)(C(t,i) + D(t,i)F(t,i))^T \\ &+ \sum_{\substack{j=1 \\ j \neq i}}^{N} q_{ji} X(j), \;\; \text{for all } \; \mathbf{X} \in \mathcal{S}_n^N. \end{aligned} \qquad (2.19)$$

Employing the triple $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ described in (2.4a) and (2.5), for each $\mathbf{F} \in \mathcal{F}_{ad}$ we may define the operator valued function $\mathcal{L}_{\mathbf{F}} : \mathbb{R} \to \mathcal{B}(\mathcal{S}_n^N)$ by

$$\mathcal{L}_{\mathbf{F}}(t)[\mathbf{X}] = (\mathbf{A}(t) + \mathbf{B}(t)\mathbf{F}(t))\mathbf{X} + \mathbf{X}(\mathbf{A}(t) + \mathbf{B}(t)\mathbf{F}(t))^T + \mathbf{\Pi_F^*}(t)[\mathbf{X}], \qquad (2.20)$$

for all $\mathbf{X} \in \mathcal{S}_n^N$.

Remark 2.4. From (2.18)-(2.20) one sees that in the case when the GRDE (2.1) takes one of the special forms (2.13) or (2.16), the linear operator $\mathcal{L}_{\mathbf{F}}(t)[\cdot]$ defined in (2.20) recover the well known Lyapunov type operators involved in the definition of the property of the stochastic stabilizability of the systems (2.12) and (2.14), respectively.

Now we are in position to define the notion of "stabilizing solution" of the GRDE (2.1).

Definition 2.5. A solution $\tilde{\mathbf{X}}(\cdot) : \mathbb{R} \to \mathcal{S}_n^N$ of the GRDE (2.1),

$$\tilde{\mathbf{X}}(\cdot) = (\tilde{X}(\cdot, 1), \dots, \tilde{X}(\cdot, N))$$

is named **stabilizing solution** if the linear differential equation on the linear space $\mathcal{S}_n^N$

$$\dot{\mathbf{Y}}(t) = \mathcal{L}_{\tilde{\mathbf{F}}}(t)[\mathbf{Y}(t)] \qquad (2.21)$$

is exponentially stable, where $\mathcal{L}_{\tilde{\mathbf{F}}}[\cdot]$ is defined as in (2.20) for $\mathbf{F}(\cdot)$ replaced by

$$\tilde{\mathbf{F}}(\cdot) = (\tilde{F}(\cdot, 1), \dots, \tilde{F}(\cdot, N))$$

described by

$$\begin{aligned}\tilde{F}(t,i) = &- (\Pi_3(t)[\tilde{\mathbf{X}}(t)](i) + R(t,i))^{-1}(\tilde{X}(t,i)B(t,i) + \Pi_2(t)[\tilde{\mathbf{X}}(t)](i) \\ &+ L(t,i))^T. \end{aligned} \tag{2.22}$$

Invoking Remark 2.4, we may infer that in the case when the GRDE (2.1) takes one of the special forms (2.10), (2.13) or (2.16), the concept of stabilizing solution introduced in the Definition 2.5, recover the traditional definition of the stabilizing solution of a Riccati differential equation from deterministic / stochastic control. Applying Corollary 5.4.2 and Theorem 5.4.3 from [9] we obtain:

Corollary 2.6. *Under the assumption* ($\mathbf{H}$) *the GRDE (*2.1*) has at most one bounded and stabilizing solution. Moreover, this solution, if it exists is a periodic function with period* θ.

Our aim is to provide a set of necessary and sufficient conditions for the existence of the stabilizing solution of a GRDE of type (2.1).

3. Some auxiliary issues

3.1. Monodromy operators. Characteristic multipliers

In this subsection we recall several definitions and results regarding the linear differential equations with periodic coefficients specialized to the case of linear differential equations on $\mathcal{S}_n^N$ of the form

$$\dot{\mathbf{Y}}(t) = \mathcal{L}_{\mathbf{F}}(t)[\mathbf{Y}(t)] \tag{3.1}$$

when $\mathcal{L}_{\mathbf{F}}(t)[\cdot]$ is associated via (2.20) to the triple $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ and to an arbitrary $\mathbf{F}(\cdot) \in \mathcal{F}_{ad}$.

For $t, t_0 \in \mathbb{R}$ we denote $\mathbf{T}_{\mathbf{F}}(t,t_0)$ the linear evolution operator on $\mathcal{S}_n^N$ defined by the linear differential equation (3.1) by

$$\mathbf{T}_{\mathbf{F}}(t,t_0)\mathbf{X}_0 = \mathbf{Y}(t;t_0,\mathbf{X}_0)$$

where $\mathbf{Y}(\cdot;t_0,\mathbf{X}_0)$ is the solution of the linear differential equation (3.1) with the initial condition $\mathbf{Y}(t_0;t_0,\mathbf{X}_0) = \mathbf{X}_0$. The main properties of the operator $\mathbf{T}_{\mathbf{F}}(t,t_0)$ involved in the developments of this paper, are summarized in the following lemma.

Lemma 3.1. **(a).** $\mathbf{T}_{\mathbf{F}}(t,t_0) = \mathcal{I}_{\mathcal{S}_n^N}$, *for all* $t_0 \in \mathbb{R}$, $\mathcal{I}_{\mathcal{S}_n^N}$ *is the identity operator on* $\mathcal{S}_n^N$;
(b). $\mathbf{T}_{\mathbf{F}}(t,t_1)\mathbf{T}_{\mathbf{F}}(t_1,t_0) = \mathbf{T}_{\mathbf{F}}(t,t_0)$, *for all* $t, t_1, t_0 \in \mathbb{R}$;
(c). $\mathbf{T}_{\mathbf{F}}^{-1}(t,t_0) = \mathbf{T}_{\mathbf{F}}(t_0,t)$, *for all* $t, t_0 \in \mathbb{R}$;
(d). *if the assumption* ($\mathbf{H}$) *is fulfilled, then* $\mathbf{T}_{\mathbf{F}}(t+j\theta, t_0+j\theta) = \mathbf{T}_{\mathbf{F}}(t,t_0)$, *for all* $t, t_0 \in \mathbb{R}$, $j \in \mathbb{Z}$;
(e). *for each* $t \geq t_0 \in \mathbb{R}$, $\mathbf{T}_{\mathbf{F}}(t,t_0) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ *is a positive operator, i.e.* $\mathbf{T}_{\mathbf{F}}(t,t_0)\mathbf{X} \succeq 0$, *for all* $t \geq t_0$ *if* $\mathbf{X} \succeq 0$.

Proof. The proof of these properties is omitted because they are known in a more general framework for linear differential equations in both finite and infinite dimensional case, see e.g. Chapter 3 from [4] for infinite dimensional case and Chapter 1.3. from

[12] for finite dimensional case. Assertion **(e)** may be obtained applying Corollary 2.2.6 from [9]. □

In the rest of the paper we assume that (**H**) is fulfilled. For each $t_0 \in \mathbb{R}$, we set

$$\mathbb{T}_{\mathbf{F}}(t_0) \triangleq \mathbf{T}_{\mathbf{F}}(t_0+\theta, t_0). \tag{3.2}$$

According to the terminology used in connection with the linear differential equations with periodic coefficients, $\mathbb{T}_{\mathbf{F}}(t_0)$ will be named the monodromy operator associated to the linear operator $\mathcal{L}_{\mathbf{F}}(\cdot)$ or, equivalently the monodromy operator associated to the linear differential equation (3.1). From Lemma 3.1 **(d), (e)** and (3.2) one gets that $\mathbb{T}_{\mathbf{F}}(\cdot)$ is an operator valued function periodic of period θ and for each $t_0 \in \mathbb{R}$, $\mathbb{T}_{\mathbf{F}}(t_0)$ is a positive operator on the ordered space $(\mathcal{S}_n^N, \mathcal{S}_n^{N+})$. The elements of the spectrum $\sigma(\mathbb{T}_{\mathbf{F}}(t_0))$ are named characteristic multipliers of the linear differential equation (3.1). If $\lambda \in \sigma(\mathbb{T}_{\mathbf{F}}(t_0))$ then there exists $0 \neq \mathbf{X} \in \mathcal{S}_n^N$ such that

$$\mathbb{T}_{\mathbf{F}}(t_0)\mathbf{X} = \lambda\mathbf{X}. \tag{3.3}$$

By direct calculations based on Lemma 3.1 we obtain that (3.3) is equivalent to

$$\mathbb{T}_{\mathbf{F}}(t_1)\mathbf{T}_{\mathbf{F}}(t_1, t_0)\mathbf{X} = \lambda\mathbf{T}_{\mathbf{F}}(t_1, t_0)\mathbf{X}, \text{ for all } t_1 \in \mathbb{R}. \tag{3.4}$$

From the equivalence of (3.3) and (3.4) we may conclude that the spectrum of the monodromy operator does not depend upon t_0. In our development an important role will be played by the characteristic multipliers from the subset

$$\sigma^+(\mathbb{T}_{\mathbf{F}}(t_0)) = \{\mu \in \sigma(\mathbb{T}_{\mathbf{F}}(t_0)) | \exists 0 \neq \mathbf{Y} \in \mathcal{S}_n^{N+} \text{ such that } \mathbb{T}_{\mathbf{F}}(t_0)\mathbf{Y} = \mu\mathbf{Y}\}.$$

If $\mu \in \sigma^+(\mathbb{T}_{\mathbf{F}}(t_0))$ we denote

$$\mathcal{V}_{\mathbf{F}}(\mu, t_0) = \{\mathbf{Y} \in \mathcal{S}_n^{N+} | \ \mathbf{Y} \neq 0, \ \mathbb{T}_{\mathbf{F}}(t_0)\mathbf{Y} = \mu\mathbf{Y}\}.$$

One sees that $\sigma^+(\mathbb{T}_{\mathbf{F}}(t_0)) \subset [0, \infty)$. In this work, the elements of $\sigma^+(\mathbb{T}_{\mathbf{F}}(t_0))$ will be named distinctive characteristic multipliers. These represent the time-varying counter part of the concept of distinctive eigenvalues introduced in [23] to characterize a subset of the spectrum of a Lyapunov type operators arising in stochastic control.

Let $\rho_{\mathbf{F}}$ be the spectral radius of the monodromy operator. Applying, for example, Theorem 2.6 from [5] in the case of linear operator $\mathbb{T}_{\mathbf{F}}^*(t_0)$ defined on the ordered space $(\mathcal{S}_n^N, \mathcal{S}_n^{N+})$ we obtain that there exists $0 \neq \mathbf{Y} \in \mathcal{S}_n^{N+}$ such that $\mathbb{T}_{\mathbf{F}}(t_0)\mathbf{Y} = \rho_{\mathbf{F}}\mathbf{Y}$, so $\rho_{\mathbf{F}} \in \sigma^+(\mathbb{T}_{\mathbf{F}}(t_0))$, for all $t_0 \in \mathbb{R}$.

3.2. Unobservable characteristic multipliers

Let $\mathbf{C}(\cdot) : \mathbb{R} \to \mathcal{M}_{pn}^N$ be a continuous function which is periodic of period θ.

Definition 3.2. (a). We say that the distinctive characteristic multiplier μ is an unobservable characteristic multiplier at t_0 with respect to the pair $(\mathbf{C}(\cdot), \mathcal{L}_{\mathbf{F}}(\cdot))$ if there exists $\mathbf{Y} \in \mathcal{V}_{\mathbf{F}}(\mu, t_0)$ such that

$$\mathbf{C}(t)\mathbf{T}_{\mathbf{F}}(t, t_0)\mathbf{Y} = 0, \ \forall\, t \in [t_0, t_0+\theta]. \tag{3.5}$$

(b). A distinctive characteristic multiplier μ is named unobservable characteristic multiplier for the pair $(\mathbf{C}(\cdot), \mathcal{L}_{\mathbf{F}}(\cdot))$ if it is an unobservable characteristic multiplier at any $t_0 \in \mathbb{R}$.

Remark 3.3. **(a).** Even if the property of unobservability could be defined for any characteristic multiplier we preferred to restrict the definition to the distinctive characteristic multipliers, because, in this framework this property will be involved in the next sections.

(b). From the periodicity property of the functions involved in Definition 3.2, one obtaines that (3.5) holds for any $t \geq t_0$ if it is true for $t \in [t_0, t_0 + \theta]$.

3.3. An useful representation formula of the operator $\mathcal{R}$

Using Lemma 5.1.1 from [9] applied to the operator (2.2), we obtain:

Corollary 3.4. *Let $\mathbf{F}(\cdot) = (F(\cdot, 1), F(\cdot, 2), \ldots, F(\cdot, N))$ be an admissible feedback gain. Under the assumption $(\mathbf{H})$ we have the following representation of the operator $\mathcal{R}_i(\cdot, \cdot)$:*

$$\begin{aligned}\mathcal{R}_i(t, \mathbf{X}) =& \mathcal{L}^*_{\mathbf{F}}(t)[\mathbf{X}](i) - (F(t,i) - F^{\mathbf{X}}(t,i))^T (\Pi_3(t)[\mathbf{X}](i) + R(t,i) \\ & \times (F(t,i) - F^{\mathbf{X}}(t,i)) + M(t,i) - L(t,i) R^{-1}(t,i) L^T(t,i) \\ & + (F(t,i) + R^{-1}(t,i) L^T(t,i))^T R(t,i) (F(t,i) + R^{-1}(t,i) L^T(t,i))\end{aligned} \tag{3.6}$$

for all $1 \leq i \leq N$, $(t, \mathbf{X}) \in Dom\ \mathcal{R}$, where $\mathcal{L}^*_{\mathbf{F}}[\mathbf{X}] = (\mathcal{L}^*_{\mathbf{F}}(t)[\mathbf{X}](1), \ldots, \mathcal{L}^*_{\mathbf{F}}(t)[\mathbf{X}](N))$ is the adjoint of the operator $\mathcal{L}_{\mathbf{F}}(t)[\cdot]$ associated via (2.20) to the triple $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ and to the admissible feedback gain $\mathbf{F}(\cdot)$, while,

$$\begin{aligned}F^{\mathbf{X}}(t,i) \triangleq& - (\Pi_3(t)[\mathbf{X}](i) + R(t,i))^{-1} (X(i) B(t,i) + \Pi_2(t)[\mathbf{X}](i) \\ & + L(t,i))^T.\end{aligned} \tag{3.7}$$

Employing the convention of notation established in Subsection 1.1, we may rewrite (3.6) and (3.7) in a compact form:

$$\begin{aligned}\mathcal{R}(t, \mathbf{X}) =& \mathcal{L}^*_{\mathbf{F}}(t)[\mathbf{X}] - (\mathbf{F}(t) - \mathbf{F}^{\mathbf{X}}(t))^T (\mathbf{\Pi}_3(t)[\mathbf{X}] + \mathbf{R}(t)) \\ & \cdot (\mathbf{F}(t) - \mathbf{F}^{\mathbf{X}}(t)) + \mathbf{M}(t) - \mathbf{L}(t) \mathbf{R}^{-1}(t) \mathbf{L}^T(t) \\ & + (\mathbf{F}(t) + \mathbf{R}^{-1}(t) \mathbf{L}^T(t))^T \mathbf{R}(t) (\mathbf{F}(t) + \mathbf{R}^{-1}(t) \mathbf{L}^T(t))\end{aligned} \tag{3.8}$$

$$\mathbf{F}^{\mathbf{X}}(t) = -(\mathbf{\Pi}_3(t)[\mathbf{X}] + \mathbf{R}(t))^{-1} (\mathbf{X}\mathbf{B}(t) + \mathbf{\Pi}_2(t)[\mathbf{X}] + \mathbf{L}(t))^T, \tag{3.9}$$

for all $(t, \mathbf{X}) \in Dom\ \mathcal{R}$.

4. The main result

4.1. The statement of the main result

First we recall the definition of the concept of stabilizability of a triple $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ where $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$ are defined in (2.4a) and $\mathbf{\Pi}(\cdot)$ is described by (2.5). The next definition is an adaptation of the Definition 5.3.2 from [9] to the triples involved here.

Definition 4.1. We say that $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ is stabilizable if there exists an admissible feedback gain $\mathbf{F}(\cdot)$ with the property that the linear differential equation on $\mathcal{S}_n^N$:

$$\dot{\mathbf{Y}} = \mathcal{L}_{\mathbf{F}}(t)[\mathbf{Y}](t) \tag{4.1}$$

is exponentially stable, $\mathcal{L}_{\mathbf{F}}(t)[\cdot]$ being the linear operator associated via (2.20) to $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ and to the admissible feedback gain $\mathbf{F}(\cdot)$. The admissible feedback gains for which the linear differential equation (4.1) is exponentially stable will be named stabilizing admissible feedback gains.

Remark 4.2. When the triple $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ is associated to the Riccati equations (2.10), (2.13) or (2.16), respectively, the concept of stabilizability introduced by Definition 4.1 recover the notions of stabilizability known in the deterministic and/or stochastic control.

The main result of this work is stated in the following theorem:

Theorem 4.3. *Under the assumption* (**H**) *the following are equivalent:*

(i). *the GRDE (*2.1*) has a bounded and stabilizing solution*

$$\tilde{\mathbf{X}}(\cdot) = (\tilde{X}(\cdot, 1), \dots, \tilde{X}(\cdot, N))$$

with $\tilde{X}(t, i) \geq 0$, *for all* $t \in \mathbb{R}$, $1 \leq i \leq N$;

(ii). **(a).** *the triple* $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ *is stabilizable;*

(b). $\mu = 1$ *is not an unobservable distinctive characteristic multiplier for the pair* $(\tilde{\mathbf{C}}(\cdot), \mathcal{L}_{-\mathbf{R}^{-1}\mathbf{L}^T}(\cdot))$ *where* $\tilde{\mathbf{C}}(\cdot) = (\tilde{C}(\cdot, 1), \dots, \tilde{C}(\cdot, N))$,

$$\tilde{C}(t, i) = (M(t, i) - L(t, i)R^{-1}(t, i)L^T(t, i))^{1/2} \tag{4.2}$$

and $\mathcal{L}_{-\mathbf{R}^{-1}\mathbf{L}^T}(\cdot)$ *is the linear operator of type (*2.20*) associated to the triple* $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ *and the feedback gains* $F(t, i) = -R^{-1}(t, i)L^T(t, i)$.

4.2. Several intermediate results

For the proof of Theorem 4.3 we need several results which can be interesting by themselves. In this subsection we present their proofs.

Lemma 4.4. *Let* $\mathbf{F}_k(\cdot) = (F_k(\cdot, 1), \dots, F_k(\cdot, N))$, $k = 1, 2$ *be two admissible feedback gains. Let* $\mathcal{L}_{\mathbf{F}_k}(t) : \mathcal{S}_n^N \to \mathcal{S}_n^N$, $k = 1, 2$ *be the linear operators associated to the triple* $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ *and the feedback gain* $\mathbf{F}_k(\cdot)$ *as in (*2.20*). We denote* $\mathbf{T}_{\mathbf{F}_k}(t, t_0)$ *the linear evolution operator on* $\mathcal{S}_n^N$ *defined by the linear differential equation*

$$\dot{\mathbf{X}}(t) = \mathcal{L}_{\mathbf{F}_k}(t)[\mathbf{X}(\mathbf{t})], \quad k = 1, 2. \tag{4.3}$$

If there exists $\mathbf{Y} \in \mathcal{S}_n^{N+} \setminus \{0\}$ *with the property that*

$$\mathbf{F}_1(t)\mathbf{T}_{\mathbf{F}_1}(t, t_0)\mathbf{Y} = \mathbf{F}_2(t)\mathbf{T}_{\mathbf{F}_1}(t, t_0)\mathbf{Y}, \quad t \in [t_0, t_0 + \theta], \quad t_0 \in \mathbb{R} \tag{4.4}$$

then we have

$$\mathbf{T}_{\mathbf{F}_1}(t, t_0)\mathbf{Y} = \mathbf{T}_{\mathbf{F}_2}(t, t_0)\mathbf{Y}, \quad \textit{for all } \ t \in [t_0, t_0 + \theta]. \tag{4.5}$$

Proof. Let $\mathbf{X}_k(t) = \mathbf{X}_k(t; t_0, \mathbf{Y}) \triangleq \mathbf{T}_{\mathbf{F}_k}(t, t_0)\mathbf{Y}$, $k = 1, 2$. The linear differential equation (4.3) satisfied by $\mathbf{X}_k(\cdot)$ may be rewritten as:

$$\begin{aligned}\dot{\mathbf{X}}_k(t) =& \mathbf{A}(t)\mathbf{X}_k(t) + \mathbf{X}_k(t)\mathbf{A}^T(t) + \mathbf{B}(t)\mathbf{F}_k(t)\mathbf{X}_k(t) \\ &+ (\mathbf{B}(t)\mathbf{F}_k(t)\mathbf{X}_k(t))^T + \mathbf{\Pi}^*_{\mathbf{F}_k}(t)[\mathbf{X}_k(t)], \quad k = 1, 2.\end{aligned} \tag{4.6}$$

Substituting (4.4) in (4.6) written for $k=1$, we obtain that $\mathbf{X}_1(\cdot)$ satisfies the differential equation:

$$\begin{aligned}\dot{\mathbf{X}}_1(t) =&(\mathbf{A}(t)+\mathbf{B}(t)\mathbf{F}_2(t))\mathbf{X}_1(t)+\mathbf{X}_1(t)(\mathbf{A}(t)+\mathbf{B}(t)\mathbf{F}_2(t))^T\\ &+\mathbf{\Pi}^*_{\mathbf{F}_1}(t)[\mathbf{X}_1(t)],\ \ t\in[t_0,t_0+\theta].\end{aligned} \tag{4.7}$$

We show now that if (4.4) holds, then

$$\mathbf{\Pi}^*_{\mathbf{F}_1}(t)[\mathbf{X}_1(t)]=\mathbf{\Pi}^*_{\mathbf{F}_2}(t)[\mathbf{X}_1(t)],\ \ t\in[t_0,t_0+\theta]. \tag{4.8}$$

To this end we set $\mathbf{X}_1(t)=(X_1(t,1),\ldots,X_1(t,N))$. This allows us to write the componentwise version of (4.4) as

$$F_1(t,i)X_1(t,i)=F_2(t,i)X_1(t,i),\ \ t\in[t_0,t_0+\theta],\ \ 1\le i\le N. \tag{4.9}$$

Let $\mathbf{Z}=(Z(1),\ldots,Z(N))\in\mathcal{S}_n^N\setminus\{0\}$ be arbitrary. Using the definition of the adjoint operator with respect to the inner product (1.1) we may write via (2.17) that

$$\begin{aligned}\langle\mathbf{Z},\mathbf{\Pi}^*_{\mathbf{F}_1}(t)[\mathbf{X}_1(t)]\rangle=\langle\mathbf{\Pi}^*_{\mathbf{F}_1}(t)[\mathbf{Z}],\mathbf{X}_1(t)\rangle&=\sum_{i=1}^N Tr[\mathbf{\Pi}^*_{\mathbf{F}_1}(t)[\mathbf{Z}](i)X_1(t,i)]\\ &=\sum_{i=1}^N Tr[\mathbf{\Pi}(t)[\mathbf{Z}](i)\cdot\Psi(X_1(t,i),F_1(t,i))]\end{aligned} \tag{4.10}$$

where

$$\Psi(X_1(t,i),F_1(t,i))\triangleq(I_n\ \ F_1^T(t,i))^TX_1(t,i)(I_n\ \ F_1^T(t,i))\in\mathcal{S}_{n+m}. \tag{4.11}$$

Substituting (4.9) in (4.11) we obtain

$$\Psi(X_1(t,i),F_1(t,i))=\Psi(X_1(t,i),F_2(t,i)). \tag{4.12}$$

Plugging (4.12) in (4.10), we obtain after some calculations based on the properties of the trace operator that

$$\langle\mathbf{Z},\mathbf{\Pi}^*_{\mathbf{F}_1}(t)[\mathbf{X}_1(t)]\rangle=\langle\mathbf{Z},\mathbf{\Pi}^*_{\mathbf{F}_2}(t)[\mathbf{X}_1(t)]\rangle.$$

This equality confirms the validity of (4.8) because $\mathbf{Z}$ is arbitrary in $\mathcal{S}_n^N$. Now, (4.8) allows us to rewrite (4.7) as

$$\begin{aligned}\dot{\mathbf{X}}_1(t) =&(\mathbf{A}(t)+\mathbf{B}(t)\mathbf{F}_2(t))\mathbf{X}_1(t)+\mathbf{X}_1(t)(\mathbf{A}(t)+\mathbf{B}(t)\mathbf{F}_2(t))^T\\ &+\mathbf{\Pi}^*_{\mathbf{F}_2}(t)[\mathbf{X}_1(t)],\ \ t\in[t_0,t_0+\theta].\end{aligned} \tag{4.13}$$

Writing (4.6) for $k=2$ and comparing with (4.13) we remark that $\mathbf{X}_2\triangleq\mathbf{T}_{\mathbf{F}_2}(t,t_0)\mathbf{Y}$ is also a solution of the linear differential equation (4.13). On the other hand $\mathbf{X}_1(t_0)=\mathbf{X}_2(t_0)=\mathbf{Y}$. From the uniqueness of the solution of an initial value problem we deduce that

$$\mathbf{X}_2(t;t_0,\mathbf{Y})=\mathbf{X}_1(t;t_0,\mathbf{Y}),\ \ t\in[t_0,t_0+\theta]$$

which shows that (4.5) is true. Thus the proof is complete. □

Proposition 4.5. *Let* $\tilde{\mathbf{X}}(\cdot) = (\tilde{X}(\cdot,1),\dots,\tilde{X}(\cdot,N))$ *be a* $\theta-$*periodic solution of the GRDE (*2.1*) such that* $\tilde{X}(t,i) \geq 0$*, for all* $t \in \mathbb{R}$, $1 \leq i \leq N$*. Let*

$$\tilde{\mathbf{F}}(t) = (\tilde{F}(t,1),\dots,\tilde{F}(t,N))$$

*be the corresponding feedback gain (*2.22*) associated to the solution* $\tilde{\mathbf{X}}(\cdot)$*. Under the assumption* $(\mathbf{H})$*, if* $\mu \in \sigma^{+}(\mathbb{T}_{\tilde{\mathbf{F}}}(t_0))$ *is such that* $\mu \geq 1$ *then* μ *is an unobservable at* t_0 *distinctive characteristic multiplier for the pair* $(\tilde{\mathbf{C}}(\cdot), \mathcal{L}_{-\mathbf{R}^{-1}\mathbf{L}^T}(\cdot))$ *where* $\tilde{\mathbf{C}}(\cdot)$ *is defined as in (*4.2*).*

Proof. Comparing (2.22) and (3.7) we see that $\tilde{\mathbf{F}}(t) = \mathbf{F}^{\mathbf{X}}(t)$, for all $t \in \mathbb{R}_+$. Using (3.8) with $\mathbf{F}(t) = \tilde{\mathbf{F}}(t)$ we obtain that the equation (2.1) satisfied by $\tilde{\mathbf{X}}(\cdot)$ becomes

$$-\frac{d}{dt}\tilde{\mathbf{X}}(t) = \mathcal{L}^{*}_{\tilde{\mathbf{F}}}(t)[\tilde{\mathbf{X}}(t)] + \mathbf{W}(t) \tag{4.14}$$

where we denote

$$\begin{aligned}\mathbf{W}(t) =&\mathbf{M}(t) - \mathbf{L}(t)\mathbf{R}^{-1}(t)\mathbf{L}^T(t) + (\tilde{\mathbf{F}}(t) + \mathbf{R}^{-1}(t)\mathbf{L}^T(t))^T \\ &\cdot \mathbf{R}(t)(\tilde{\mathbf{F}}(t) + \mathbf{R}^{-1}(t)\mathbf{L}(t)).\end{aligned} \tag{4.15}$$

From (4.14) we deduce

$$\tilde{\mathbf{X}}(t_0) = \mathbb{T}^{*}_{\tilde{\mathbf{F}}}(t_0)\tilde{\mathbf{X}}(t_0+\theta) + \int_{t_0}^{t_0+\theta} \mathbf{T}^{*}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{W}(t)dt \tag{4.16}$$

with $\mathbb{T}^{*}_{\tilde{\mathbf{F}}}(t_0)$ and $\mathbf{T}^{*}_{\tilde{\mathbf{F}}}(t,t_0)$ are the adjoint operators of $\mathbb{T}_{\tilde{\mathbf{F}}}(t_0)$ and $\mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)$, respectively, with respect to the inner product (1.1). From (2.7) we infer that $\mathbf{W}(t) \succeq 0$, for all $t \in [t_0, t_0+\theta]$. This leads to

$$\Psi \triangleq \int_{t_0}^{t_0+\theta} \mathbf{T}^{*}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{W}(t)dt \succeq 0 \tag{4.17}$$

because $\mathbf{T}^{*}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{W}(t) \succeq 0$, for all $t \in [t_0, t_0+\theta]$.
On the other hand from $\mu \in \sigma^{+}(\mathbb{T}_{\tilde{\mathbf{F}}}(t_0))$ we deduce that there exists $\mathbf{Y} \in \mathcal{S}_n^{N+} \setminus \{0\}$ such that

$$\mathbb{T}_{\tilde{\mathbf{F}}}(t_0)\mathbf{Y} = \mu\mathbf{Y}. \tag{4.18}$$

From (1.1), (4.16), (4.17) and (4.18) together with $\tilde{\mathbf{X}}(t_0) = \tilde{\mathbf{X}}(t_0+\theta)$, we get

$$\begin{aligned}0 \leq&\langle\Psi,\mathbf{Y}\rangle = \langle\tilde{\mathbf{X}}(t_0),\mathbf{Y}\rangle - \langle\mathbb{T}^{*}_{\tilde{\mathbf{F}}}(t_0)\tilde{\mathbf{X}}(t_0+\theta),\mathbf{Y}\rangle \\ &= \langle\tilde{\mathbf{X}}(t_0),\mathbf{Y}\rangle - \langle\tilde{\mathbf{X}}(t_0),\mathbb{T}_{\tilde{\mathbf{F}}}(t_0)\mathbf{Y}\rangle = (1-\mu)\langle\tilde{\mathbf{X}}(t_0),\mathbf{Y}\rangle \leq 0.\end{aligned}$$

Hence, $\langle\Psi,\mathbf{Y}\rangle = 0$, which yields to

$$\int_{t_0}^{t_0+\theta} \langle\mathbf{W}(t),\mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y}\rangle dt = 0. \tag{4.19}$$

From (4.15) and (4.19) we deduce that

$$\int_{t_0}^{t_0+\theta} \langle \mathbf{M}(t) - \mathbf{L}(t)\mathbf{R}^{-1}(t)\mathbf{L}^T(t), \mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y}\rangle dt = 0, \tag{4.20a}$$

$$\int_{t_0}^{t_0+\theta} \langle \mathbf{W}_1(t), \mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y}\rangle dt = 0. \tag{4.20b}$$

where $\mathbf{W}_1(t) = (\tilde{\mathbf{F}}(t) + \mathbf{R}^{-1}(t)\mathbf{L}^T(t))^T\mathbf{R}(t)(\tilde{\mathbf{F}}(t) + \mathbf{R}^{-1}(t)\mathbf{L}^T(t))$.
From (2.7a) and (4.20a) we get

$$\langle \mathbf{M}(t) - \mathbf{L}(t)\mathbf{R}^{-1}(t)\mathbf{L}^T(t), \mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y}\rangle = 0.$$

Further, from (1.1), (4.2) one gets

$$\sum_{i=1}^{N} Tr\left[\tilde{C}(t,i)(\mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y})(i)\tilde{C}^T(t,i)\right] = 0, \text{ for all } t \in [t_0, t_0+\theta].$$

Since $\tilde{C}(t,i)(\mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y})(i)\tilde{C}^T(t,i) \geq 0$ we may conclude that

$$\tilde{\mathbf{C}}(t)\mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y} = 0, \text{ for all } t \in [t_0, t_0+\theta]. \tag{4.21}$$

Proceeding analogously we obtain from (4.20b) that

$$\tilde{\mathbf{F}}(t)\mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y} = -\mathbf{R}^{-1}(t)\mathbf{L}^T(t)\mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y}, \text{ for all } t \in [t_0, t_0+\theta]. \tag{4.22}$$

Applying Lemma 4.4 with $\mathbf{F}_1(t) = \tilde{\mathbf{F}}(t)$ and $\mathbf{F}_2(t) = -\mathbf{R}^{-1}(t)\mathbf{L}^T(t)$ in the case of the equality (4.22) we may infer that

$$\mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y} = \mathbf{T}_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{Y}, \text{ for all } t \in [t_0, t_0+\theta]. \tag{4.23}$$

From (4.18) together with (4.23) written for $t = t_0+\theta$, we obtain

$$\mathbb{T}_{-\mathbf{R}^{-1}\mathbf{L}^T}(t_0)\mathbf{Y} = \mu\mathbf{Y}. \tag{4.24}$$

Finally, from (4.21) and (4.23) we obtain that

$$\tilde{\mathbf{C}}(t)\mathbf{T}_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{Y} = 0, \text{ for all } t \in [t_0, t_0+\theta]. \tag{4.25}$$

From (4.24) and (4.25) we may conclude that μ is a distinctive characteristic multiplier unobservable at instance time t_0 for $(\tilde{\mathbf{C}}(\cdot), \mathcal{L}_{-\mathbf{R}^{-1}\mathbf{L}^T}(\cdot))$ which ends the proof. □

Proposition 4.6. *Under the assumption* (**H**), *if* $\mu = 1$ *is a distinctive characteristic multiplier unobservable at instance time* t_0 *for the pair* $(\tilde{\mathbf{C}}(\cdot), \mathcal{L}_{-\mathbf{R}^{-1}\mathbf{L}^T}(\cdot))$ *then for any positive semidefinite and* $\theta-$*periodic solution* $\tilde{\mathbf{X}}(\cdot)$ *of the GRDE* (2.1) *we have that* $1 \in \sigma^+(\mathbb{T}_{\tilde{\mathbf{F}}}(t_0))$ *where* $\tilde{\mathbf{F}}(\cdot)$ *is the feedback gain associated via* (3.9) *to the solution* $\tilde{\mathbf{X}}(\cdot)$ *and* $\tilde{\mathbf{C}}(\cdot)$ *is defined in* (4.2).

Proof. If $\mu = 1$ is a distinctive characteristic multiplier unobservable at t_0 for $(\tilde{\mathbf{C}}(\cdot), \mathcal{L}_{-\mathbf{R}^{-1}\mathbf{L}^T}(\cdot))$, there exists $\mathbf{Y} \in \mathcal{S}_n^{N+} \setminus \{0\}$ with the properties

$$\mathbb{T}_{-\mathbf{R}^{-1}\mathbf{L}^T}(t_0)\mathbf{Y} = \mathbf{Y} \tag{4.26}$$

and

$$\tilde{\mathbf{C}}(t)\mathbf{T}_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{Y}=0,\ \text{ for all }\ t\in[t_0,t_0+\theta]. \tag{4.27}$$

Let $\tilde{\mathbf{X}}(\cdot)$ be an arbitrary positive semidefinite and $\theta-$periodic solution of GRDE (2.1). Applying Corollary 3.4, taking $\mathbf{F}(t)=-\mathbf{R}^{-1}(t)\mathbf{L}^T(t)$ we may rewrite the equation (2.1) satisfied by $\tilde{\mathbf{X}}(\cdot)$ as:

$$\begin{aligned}-\frac{d}{dt}\tilde{\mathbf{X}}(t)&=\mathcal{L}^*_{-\mathbf{R}^{-1}\mathbf{L}^T}(t_0)[\tilde{\mathbf{X}}(t)]+\mathbf{M}(t)-\mathbf{L}(t)\mathbf{R}^{-1}(t)\mathbf{L}^T(t)\\&\quad-(\mathbf{R}^{-1}(t)\mathbf{L}^T(t)+\tilde{\mathbf{F}}(t))^T(\mathbf{\Pi}_3(t)[\tilde{\mathbf{X}}]+\mathbf{R}(t))(\mathbf{R}^{-1}(t)\mathbf{L}^T(t)+\tilde{\mathbf{F}}(t)).\end{aligned}$$

This allows us to write the following representation of the solution $\tilde{\mathbf{X}}(\cdot)$:

$$\begin{aligned}\tilde{\mathbf{X}}(t_0)&=\mathbb{T}^*_{-\mathbf{R}^{-1}\mathbf{L}^T}(t_0)\tilde{\mathbf{X}}(t_0+\theta)+\int_{t_0}^{t_0+\theta}\mathbf{T}^*_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\tilde{\mathbf{C}}^2(t)dt\\&\quad-\int_{t_0}^{t_0+\theta}\mathbf{T}^*_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{\Upsilon}(t)dt \qquad (4.28a)\\ \mathbf{\Upsilon}(t)&=(\mathbf{R}^{-1}(t)\mathbf{L}^T(t)+\tilde{\mathbf{F}}(t))^T(\mathbf{\Pi}_3(t)[\tilde{\mathbf{X}}(t)]\\&\quad+\mathbf{R}(t))(\mathbf{R}^{-1}(t)\mathbf{L}^T(t)+\tilde{\mathbf{F}}(t)). \qquad (4.28b)\end{aligned}$$

Employing (4.26), (4.27) together with periodicity property of $\tilde{\mathbf{X}}(\cdot)$, we get

$$\langle\mathbb{T}^*_{-\mathbf{R}^{-1}\mathbf{L}^T}(t_0)\tilde{\mathbf{X}}(t_0+\theta)-\tilde{\mathbf{X}}(t_0),\mathbf{Y}\rangle+\left\langle\int_{t_0}^{t_0+\theta}\mathbf{T}^*_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\tilde{\mathbf{C}}^2(t)dt,\mathbf{Y}\right\rangle=0. \tag{4.29}$$

Further, (4.28a) and (4.29) yield

$$\int_{t_0}^{t_0+\theta}\langle\mathbf{T}^*_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{\Upsilon}(t),\mathbf{Y}\rangle dt=0. \tag{4.30}$$

From (1.1), (4.28b) we infer that (4.30) is true, if and only if

$$\langle\mathbf{T}^*_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{\Upsilon}(t),\mathbf{Y}\rangle=0,\ \text{ for all }\ t\in[t_0,t_0+\theta].$$

This is equivalent to

$$\langle\mathbf{\Upsilon}(t),\mathbf{T}_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{Y}\rangle=0,\ \text{ for all }\ t\in[t_0,t_0+\theta]. \tag{4.31}$$

Combining (1.1), (2.7a), (4.28b) we may conclude that (4.31) is true if and only if

$$-\mathbf{R}^{-1}(t)\mathbf{L}^T(t)\mathbf{T}_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{Y}=\tilde{\mathbf{F}}(t)\mathbf{T}_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{Y}, \tag{4.32}$$

for all $t\in[t_0,t_0+\theta]$.

Applying Lemma 4.4 taking $\mathbf{F}_1(t)=-\mathbf{R}^{-1}(t)\mathbf{L}^T(t)$ and $\mathbf{F}_2(t)=\tilde{\mathbf{F}}(t)$, we conclude from (4.32) that

$$\mathbf{T}_{-\mathbf{R}^{-1}\mathbf{L}^T}(t,t_0)\mathbf{Y}=\mathbf{T}_{\tilde{\mathbf{F}}}(t,t_0)\mathbf{Y}. \tag{4.33}$$

Finally, (4.26) together with (4.33) written for $t=t_0+\theta$, yield $\mathbb{T}_{\tilde{\mathbf{F}}}(t_0)\mathbf{Y}=\mathbf{Y}$. This shows that $\mu=1$ lies in $\sigma^+(\mathbb{T}_{\tilde{\mathbf{F}}}(t_0))$ which complete the proof. □

4.3. The proof of the main result

First we prove the implication **(i)** $\Longrightarrow$ **(ii)**. The stabilizability of the triple $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ is a necessary condition for the existence of the stabilizing solution of the GRDE (2.1) because the feedback gain associated via (2.22) to the stabilizing solution stabilizes this triple in the sense of the Definition 4.1. Let us assume by contrary that **(ii) (b)** from the statement of the Theorem 4.3 is not true. This means that there exists $t_0 \in \mathbb{R}$ such that $\mu = 1$ is a distinctive characteristic multiplier unobservable at t_0 for $(\tilde{\mathbf{C}}(\cdot), \mathcal{L}_{-\mathbf{R}^{-1}\mathbf{L}^T}(\cdot))$. From Proposition 4.6 we deduce that $\mu = 1 \in \sigma^+(\mathbb{T}_{\tilde{\mathbf{F}}}(t_0))$ where $\tilde{\mathbf{F}}(\cdot)$ is the stabilizing feedback gain associated via (2.22) to the solution $\tilde{\mathbf{X}}(\cdot)$ of the GRDE (2.1). Hence $\rho_{\tilde{\mathbf{F}}} \geq 1$, which contradicts the fact that $\tilde{\mathbf{X}}(\cdot)$ is the stabilizing solution. So, we have shown that the implication **(i)** $\Longrightarrow$ **(ii)** holds.

Now we prove **(ii)** $\Longrightarrow$ **(i)**. Applying Theorem 5.3.5 from [9] we deduce that under the assumption **(H)** if the triple $(\mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{\Pi}(\cdot))$ is stabilizable then the GRDE (2.1) has a solution $\tilde{\mathbf{X}}(\cdot)$ which is maximal in the class of positive semidefinite solutions of (2.1). Moreover, $\tilde{\mathbf{X}}(\cdot)$ is a periodic function of period θ. Let $\tilde{\mathbf{F}}(t)$ be the feedback gain associated to the solution $\tilde{\mathbf{X}}(\cdot)$ via (3.9) written for $\mathbf{X}$ replaced by $\tilde{\mathbf{X}}(t)$. We denote $\mathbb{T}_{\tilde{\mathbf{F}}}(t_0)$, $t_0 \in \mathbb{R}$, the monodromy operator defined by the linear differential equation with periodic coefficients of type (3.1) when $\mathbf{F}(t)$ is replaced by $\tilde{\mathbf{F}}(t)$. In the proof of Theorem 5.3.5 from [9] $\mathbb{T}_{\tilde{\mathbf{F}}}(t_0)$ is obtained as the limit of a sequence of linear operators having the spectral radius strictly less than 1. Hence, the spectral radius $\rho_{\tilde{\mathbf{F}}}$ of the monodromy operator $\mathbb{T}_{\tilde{\mathbf{F}}}(t_0)$ satisfies $\rho_{\tilde{\mathbf{F}}} \leq 1$. To show that the maximal solution $\tilde{\mathbf{X}}(\cdot)$ is just the stabilizing solution we have to show that $\rho_{\tilde{\mathbf{F}}} < 1$. Let us assume by contrary that $\rho_{\tilde{\mathbf{F}}} = 1$. In this case, $\mu = 1 \in \sigma^+(\mathbb{T}_{\tilde{\mathbf{F}}}(t_0))$, for all $t_0 \in \mathbb{R}$. Then from Proposition 4.5 it follows that $\mu = 1$ is a distinctive characteristic multiplier unobservable for the pair $(\tilde{\mathbf{C}}(\cdot), \mathcal{L}_{-\mathbf{R}^{-1}\mathbf{L}^T}(\cdot))$ which contradicts **(ii) (b)**. Thus, $\rho_{\tilde{\mathbf{F}}} < 1$ which complete the proof.

5. Conclusions

This paper studies the stabilizing solution for a class of continuous-time backward nonlinear equations. The concept of unobservable characteristic multipliers for a pair adequately chosen is introduced. Based on this spectrum technique, we obtain a necessary and sufficient condition for the existance of the stabilizing solution of the GRDE. The proposed techniques are formulated as the solvability of a linear equation.

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