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On some evolution inclusions in non separable Banach spaces

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Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th anniversary.

Abstract. We study a Cauchy problem of a class of nonconvex second-order integro-differential inclusions and a boundary value problem associated to a semilinear evolution inclusion defined by nonlocal conditions in non-separable Banach spaces. The existence of mild solutions is established under Filippov type assumptions.

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1. Introduction

In this note we study two classes of evolution differential inclusions. First we consider the problem

$$x''(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, x'(0) = y_0, \tag{1.1}$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set-valued map lipschitzian with respect to the second variable, X is a Banach space, $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{G(t,s)\}_{t,s\in[0,T]}$, $\Delta = \{(t,s) \in [0,T] \times [0,T]; t \geq s\}$, $K(.,.) : \Delta \to \mathbb{R}$ is continuous and $x_0, y_0 \in X$. The general framework of evolution operators $\{A(t)\}_{t\geq 0}$ that define problem (1.1) has been developed by Kozak ([19]) and improved by Henriquez ([17]).

Existence results and some qualitative properties of the mild solutions of problem (1.1) may be found in [14] in the case when X is a separable Banach space.

De Blasi and Pianigiani ([15]) obtained the existence of mild solutions for semilinear differential inclusions on an arbitrary, not necessarily separable, Banach space X. Even if Filippov's ideas ([16]) are still present, the approach in [15] is fundamental different: it consists in the construction of the measurable selections of the multifunction. This construction does not use classical selection theorems such as Kuratowski and Ryll-Nardzewski's ([20]) or Bressan and Colombo's ([7]).

The aim of this note is to obtain an existence result for problem (1.1) similar to the one in [15]. We will prove the existence of solutions for problem (1.1) in an arbitrary space X under Filippov-type assumptions on F.

In several recent papers ([2, 3, 5, 12, 13, 17, 18]) existence results and qualitative properties of mild solutions have been obtained for the following problem

$$x''(t) \in A(t)x(t) + F(t, x(t)), \quad x(0) = x_0, x'(0) = y_0, \tag{1.2}$$

with $A(.)$ and $F(.,.)$ as above.

On one hand, the result in the present paper extends to the integro-differential framework (1.1) the result in [12] obtained for problem (1.2) and, on the other hand, this paper extends to second-order integro-differential inclusions a similar result in [10] obtained for a class of first-order integro-differential inclusions.

The second class of evolution inclusions that we are considering is

$$x' \in Ax + F(t, x) \quad a.e. \ ([0, T]), \tag{1.3}$$

$$x(0) + \sum_{i=1}^{m} a_i x(t_i) = x_0, \tag{1.4}$$

where X is a real separable Banach space, $a_i \in \mathbb{R}$, $a_i \neq 0$, $i = \overline{1, m}$, $x_0 \in X$, $0 < t_1 < t_2 < ... < t_m < T$, $F : [0, T] \times X \to \mathcal{P}(X)$ is a set-valued map and A is the infinitesimal generator of a linear semigroup $\{\mathcal{G}(t);\ t \geq 0\}$.

The nonlocal condition (1.4) was used by Byszewski ([8, 9]). If $a_i \neq 0$, $i = \overline{1, m}$ the results can be applied in kinematics to determine the evolution $t \to x(t)$ of the location of a physical object for which the positions $x(0), x(t_1), ..., x(t_m)$ are unknown but it is known the condition (1.4). Consequently, to describe some physical phenomena the nonlocal condition may be more useful than the standard initial condition $x(0) = x_0$. Obviously, when $a_i = 0$, $i = \overline{1, m}$, one has the classical initial condition.

Existence of mild solutions of problem (1.3)-(1.4) has been obtained in [4, 6] for convex as well as nonconvex set-valued maps. All these results are based on some suitable theorems of fixed point theory. In our recent paper [11] it is shown that Filippov's ideas ([1, 16]) can be suitably adapted in order to prove the existence of solutions to problem (1.3)-(1.4) provided the Banach space X is separable.

The result that we established in non separable Banach spaces for problem (1.3)-(1.4) may be interpreted as extension of the result in [15] from Cauchy problems to boundary value problems defined by nonlocal conditions and as an extension of the result in [11] to non separable Banach spaces.

The paper is organized as follows: in Section 2 we present the notations, definitions and preliminary results to be used in the sequel and in Section 3 we prove the main results.

2. Preliminaries

Consider X, an arbitrary real Banach space with norm $|.|$ and with the corresponding metric $d(.,.)$. Let $\mathcal{P}(X)$ be the space of all bounded nonempty subsets of X endowed with the Hausdorff pseudometric

$$\mathrm{d}_H(A,B) = \max\{\mathrm{d}^*(A,B), \mathrm{d}^*(B,A)\}, \quad \mathrm{d}^*(A,B) = \sup_{a\in A} \mathrm{d}(a,B),$$

where $\mathrm{d}(x,A) = \inf_{a\in A} |x-a|$, $A \subset X, x \in X$.

Let $\mathcal{L}$ be the σ-algebra of the (Lebesgue) measurable subsets of R and, for $A \in \mathcal{L}$, let $\mu(A)$ be the Lebesgue measure of A.

Let X be a Banach space and Y be a metric space. An open (resp., closed) ball in Y with center y and radius r is denoted by $B_Y(y,r)$ (resp., $\overline{B}_Y(y,r)$. In what follows, $B = B_X(0,1)$.

A multifunction $F : Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be d_H-continuous at $y_0 \in Y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in B_Y(y_0,r)$ there is $\mathrm{d}_H(F(y), F(y_0)) \le \varepsilon$. F is called d_H-continuous if it is so at each point $y_0 \in Y$.

Let $A \in \mathcal{L}$, with $\mu(A) < \infty$. A multifunction $F : Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be *Lusin measurable* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset A$, with $\mu(A\backslash K_\varepsilon) < \varepsilon$ such that F restricted to K_ε is d_H-continuous.

It is clear that if $F, G : A \to \mathcal{P}(X)$ and $f : A \to X$ are Lusin measurable, then so are F restricted to B ($B \subset A$ measurable), $F+G$ and $t \to \mathrm{d}(f(t), F(t))$. Moreover, the uniform limit of a sequence of Lusin measurable multifunctions is Lusin measurable, too.

Let I stand for the interval $[0,T]$, $T > 0$, $C(I,X)$ is the Banach space of all continuous functions from I to X with the norm $||x||_C = \sup_{t\in I} |x(t)|$ and $L^1(I,X)$ is the Banach space of (Bochner) integrable functions $u(.) : I \to X$ endowed with the norm $||u||_1 = \int_0^T |u(t)|dt$. Denote by $B(X)$ the Banach space of bounded linear operators from X into X with the norm $||N|| = \sup\{|N(y)|;\ |y| = 1\}$.

In what follows $\{A(t)\}_{t\ge 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{G(t,s)\}_{t,s\in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t.

Definition 2.1. ([17, 19]) A family of bounded linear operators $G(t,s) : X \to X$, $(t,s) \in \Delta := \{(t,s) \in I \times I; s \le t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t) \tag{2.1}$$

if

i) For any $x \in X$, the map $(t,s) \to G(t,s)x$ is continuously differentiable and

a) $G(t,t) = 0$, $t \in I$.

b) If $t \in I, x \in X$ then $\frac{\partial}{\partial t}G(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}G(t,s)x|_{t=s} = -x$.

ii) If $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}G(t,s)x \in D(A(t))$, the map $(t,s) \to G(t,s)x$ is of class C^2 and

a) $\frac{\partial^2}{\partial t^2}G(t,s)x \equiv A(t)G(t,s)x$.

b) $\frac{\partial^2}{\partial s^2}G(t,s)x \equiv G(t,s)A(t)x$.

c) $\frac{\partial^2}{\partial s \partial t}G(t,s)x|_{t=s} = 0$.

iii) If $(t,s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2 \partial s}G(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}G(t,s)x$ and

a) $\frac{\partial^3}{\partial t^2 \partial s}G(t,s)x \equiv A(t)\frac{\partial}{\partial s}G(t,s)x$ and the map $(t,s) \to A(t)\frac{\partial}{\partial s}G(t,s)x$ is continuous.

b) $\frac{\partial^3}{\partial s^2 \partial t}G(t,s)x \equiv \frac{\partial}{\partial t}G(t,s)A(s)x$.

As an example for equation (2.1) one may consider the problem (e.g., [19])

$$\frac{\partial^2 z}{\partial t^2}(t,\tau) = \frac{\partial^2 z}{\partial \tau^2}(t,\tau) + a(t)\frac{\partial z}{\partial t}(t,\tau), \quad t \in [0,T], \tau \in [0,2\pi],$$

$$z(t,0) = z(t,\pi) = 0, \quad \frac{\partial z}{\partial \tau}(t,0) = \frac{\partial z}{\partial \tau}(t,2\pi),\ t \in [0,T],$$

where $a(.) : I \to \mathbb{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbb{R},\mathbb{C})$ of 2π-periodic 2-integrable functions from $\mathbb{R}$ to $\mathbb{C}$, $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$ with domain $H^2(\mathbb{R},\mathbb{C})$ the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbb{R},\mathbb{C})$. It is well known that$A_1$ is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbb{Z}$ with associated eigenvectors

$$z_n(\tau) = \frac{1}{\sqrt{2\pi}}e^{in\tau},\ n \in \mathbb{N}.$$

The set $z_n,\ n \in \mathbb{N}$ is an orthonormal basis of X. In particular,

$$A_1 z = \sum_{n \in \mathbb{Z}} -n^2 < z, z_n > z_n,\ z \in D(A_1).$$

The cosine function is given by

$$C(t)z = \sum_{n \in \mathbf{Z}} \cos(nt) < z, z_n > z_n$$

with the associated sine function

$$S(t)z = t < z, z_0 > z_0 + \sum_{n \in \mathbf{Z}^*} \frac{\sin(nt)}{n} < z, z_n > z_n.$$

For $t \in I$ define the operator $A_2(t)z = a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbb{R},\mathbb{C})$. Set $A(t) = A_1 + A_2(t)$. It has been proved in [19] that this family generates an evolution operator as in Definition 2.1.

Definition 2.2. A continuous mapping $x(.) \in C(I,X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that

$$f(t) \in F(t,x(t)) \quad a.e.\,(I), \tag{2.2}$$

$$x(t) = -\frac{\partial}{\partial s}G(t,0)x_0 + G(t,0)y_0 + \int_0^t G(t,s)\int_0^s K(s,\tau)f(\tau)d\tau,\ t \in I. \tag{2.3}$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) if $f(.)$ verifies (2.2) and $x(.)$ is defined by (2.3).

We note that condition (2.3) can be rewritten as

$$x(t) = -\frac{\partial}{\partial s}G(t,0)x_0 + G(t,0)y_0 + \int_0^t U(t,s)f(s)ds \quad \forall t \in I, \tag{2.4}$$

where $U(t,s) = \int_s^t G(t,\tau)K(\tau,s)d\tau$.

Hypothesis H1. i) There exists an evolution operator $\{G(t,s)\}_{t,s\in I}$ associated to the family $\{A(t)\}_{t\geq 0}$.

ii) There exist $M, M_0 \geq 0$ such that $|G(t,s)|_{B(X)} \leq M$, $|\frac{\partial}{\partial s}G(t,s)| \leq M_0$, for all $(t,s) \in \Delta$.

iii) $K(.,.) : \Delta \to \mathbb{R}$ is continuous.

Hypothesis H2. i) A is the infinitesimal generator of a strongly continuous and compact semigroup $\{\mathcal{G}(t);\ t \geq 0\}$ in X.

ii) There exists an operator $C : X \to X$ defined by

$$C = [I + \sum_{i=1}^m a_i \mathcal{G}(t_i)]^{-1}.$$

Let $m_0 \geq 0$ be such that $|\mathcal{G}(t)| \leq m_0$ $\forall t \in I$.

According to [4] if we assume that $\sum_{i=1}^m |a_i| < \frac{1}{m_0}$ then there exists C as in Hypothesis H2 ii).

Definition 2.3. A continuous mapping $x(.) \in C(I,X)$ is called a mild solution of problem (1.3)-(1.4) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that

$$f(t) \in F(t,x(t)) \quad a.e.\,(I) \tag{2.5}$$

$$x(t) = \mathcal{G}(t)Cx_0 - \sum_{i=1}^m a_i \mathcal{G}(t)C\int_0^{t_i} \mathcal{G}(t_i-u)f(u)du + \int_0^t \mathcal{G}(t-u)f(u)du, t \in I. \tag{2.6}$$

Remark 2.4. If we denote

$$H(t,s) = \sum_{i=1}^m a_i \mathcal{G}(t)C\mathcal{G}(t_i - s)\chi_{[0,t_i]}(s) + \mathcal{G}(t-s)\chi_{[0,t]}(s),$$

where $\chi_S(\cdot)$ is the characteristic function of the set S, then the solution $x(\cdot)$ in Definition 2.3 may be written as

$$x(t) = \mathcal{G}(t)Cx_0 - \int_0^T H(t,s)f(s)ds. \tag{2.7}$$

Obviously,

$$|H(t,s)| \leq \sum_{i=1}^m |a_i| m_0^2 ||C|| + m_0 =: m \quad \forall\, t,s \in I.$$

In what follows X is a real Banach space and we assume the following hypotheses.

Hypothesis H3. i) *$F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed bounded values and for any $x \in X$ $F(.,x)$ is Lusin measurable on I.*

ii) *There exists $l(.) \in L^1(I,(0,\infty))$ such that, $\forall t \in I$*

$$\mathrm{d}_H(F(t,x_1),F(t,x_2)) \le l(t)|x_1 - x_2|, \quad \forall\, x_1,x_2 \in X.$$

iii) *There exists $q(.) \in L^1(I,(0,\infty))$ such that $\forall t \in I$ we have*

$$F(t,0) \subset q(t)B.$$

Denote $L = \int_0^T l(s)ds$.

The technical results summarized in the following lemma are essential in the proof of our results. For the proof, we refer the reader to [15].

Lemma 2.5. *i) Let $F_i : I \to \mathcal{P}(X)$, i=1,2 be two Lusin measurable multifunctions and let $\varepsilon_i > 0$, i=1,2 be such that*

$$H_1(t) := (F_1(t) + \varepsilon_1 B) \cap (F_2(t) + \varepsilon_2 B) \neq \emptyset, \quad \forall t \in I.$$

Then the multifunction $H_1 : I \to \mathcal{P}(X)$ has a Lusin measurable selection $h : I \to X$.

ii) Assume that Hypothesis H3 is satisfied. Then for any continuous $x(.) : I \to X$, $u(.) : I \to X$ measurable and any $\varepsilon > 0$ one has

a) the multifunction $t \to F(t,x(t))$ is Lusin measurable on I.

b) the multifunction $H_2 : I \to \mathcal{P}(X)$ defined by

$$H_2(t) := (F(t,x(t)) + \varepsilon B) \cap B_X(u(t), d(u(t),F(t,x(t))) + \varepsilon)$$

has a Lusin measurable selection $g : I \to X$.

3. The results

Set $n(t) = \int_0^t l(u)\mathrm{d}u$, $t \in I$, denote $K_0 := \sup_{(t,s)\in\Delta} |K(t,s)|$ and note that

$$|U(t,s)| \le MK_0(t-s) \le MK_0T.$$

Theorem 3.1. *We assume that Hypotheses H1 and H3 are satisfied. Then, for every $x_0, y_0 \in X$, Cauchy problem (1.1) has a mild solution $x(.) \in C(I,X)$.*

Proof. Let us first note that if $z(.) : I \to X$ is continuous, then every Lusin measurable selection $u : I \to X$ of the multifunction $t \to F(t,z(t)) + B$ is Bochner integrable on I. More precisely, for any $t \in I$, there holds

$$\begin{aligned}|u(t)| &\le d_H(F(t,z(t)) + B, 0) \le d_H(F(t,z(t)),F(t,0)) + d_H(F(t,0),0) + 1\\ &\le l(t)|z(t)| + q(t) + 1.\end{aligned}$$

Let $0 < \varepsilon < 1$, $\quad \varepsilon_n = \frac{\varepsilon}{2^{n+2}}$.

Consider $f_0(.) : I \to X$, an arbitrary Lusin measurable, Bochner integrable function, and define

$$x_0(t) = -\frac{\partial}{\partial s}G(t,0)x_0 + G(t,0)y_0 + \int_0^t U(t,s)f_0(s)ds, \quad t \in I.$$

Since $x_0(.)$ is continuous, by Lemma 2.5 ii) there exists a Lusin measurable function $f_1(.) : I \to X$ which, for $t \in I$, satisfies

$$f_1(t) \in (F(t, x_0(t)) + \varepsilon_1 B) \cap B(f_0(t), d(f_0(t), F(t, x_0(t))) + \varepsilon_1)$$

Obviously, $f_1(.)$ is Bochner integrable on I. Define $x_1(.) : I \to X$ by

$$x_1(t) = -\frac{\partial}{\partial s} G(t,0)x_0 + G(t,0)y_0 + \int_0^t U(t,s)f_1(s)ds, \quad t \in I.$$

By induction, we construct a sequence $x_n : I \to X$, $n \geq 2$ given by

$$x_n(t) = -\frac{\partial}{\partial s} G(t,0)x_0 + G(t,0)y_0 + \int_0^t U(t,s)f_n(s)ds, \quad t \in I, \tag{3.1}$$

where $f_n(.) : I \to X$ is a Lusin measurable function which, for $t \in I$, satisfies:

$$f_n(t) \in (F(t, x_{n-1}(t)) + \varepsilon_n B) \cap B(f_{n-1}(t), d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n). \tag{3.2}$$

At the same time, as we saw at the beginning of the proof, $f_n(.)$ is also Bochner integrable.
From (3.2), for $n \geq 2$ and $t \in I$, we obtain

$$\begin{aligned} |f_n(t) - f_{n-1}(t)| &\leq d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n \\ &\leq d(f_{n-1}(t), F(t, x_{n-2}(t))) + d_H(F(t, x_{n-2}(t)), F(t, x_{n-1}(t))) + \varepsilon_n \\ &\leq \varepsilon_{n-1} + l(t)|x_{n-1}(t) - x_{n-2}(t)| + \varepsilon_n. \end{aligned}$$

Since $\varepsilon_{n-1} + \varepsilon_n < \varepsilon_{n-2}$, for $n \geq 2$, we deduce that

$$|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)|x_{n-1}(t) - x_{n-2}(t)|. \tag{3.3}$$

Denote $p_0(t) := d(f_0(t), F(t, x_0(t))), t \in I$. We next prove by recurrence, that for $n \geq 2$ and $t \in I$

$$\begin{aligned} |x_n(t) - x_{n-1}(t)| \leq \sum_{k=0}^{n-2} \int_0^t \varepsilon_{n-2-k} &\frac{(MK_0T)^{k+1}(n(t) - n(u))^k}{k!} du \\ &+ \varepsilon_0 \int_0^t \frac{(MK_0T)^n(n(t) - n(u))^{n-1}}{(n-1)!} du \\ &+ \int_0^t \frac{(MK_0T)^n(n(t) - n(u))^{n-1}}{(n-1)!} p_0(u) du. \end{aligned} \tag{3.4}$$

We start with $n = 2$. In view of (3.1), (3.2) and (3.3), for $t \in I$, there is

$$|x_2(t) - x_1(t)| \leq \int_0^t |U(t,s)| \cdot |f_2(s) - f_1(s)| ds$$

$$\leq \int_0^t MK_0T[\varepsilon_0 + l(s)|x_1(s) - x_0(s)|] ds$$

$$\leq \varepsilon_0 MK_0Tt + \int_0^t \left[MK_0Tl(s) \int_0^s |U(s,r)| \cdot |f_1(r) - f_0(r)| dr \right] ds$$

$$\leq \varepsilon_0 MK_0Tt + \int_0^t \left[(MK_0T)^2 l(s) \int_0^s (p_0(u) + \varepsilon_1) du \right] ds$$

$$\leq \varepsilon_0 MK_0Tt + \int_0^t \left[(MK_0T)^2(p_0(u)+\varepsilon_1)\int_u^t l(s)ds\right] du$$
$$= \varepsilon_0 MK_0Tt + \int_0^t (MK_0T)^2(n(t)-n(s))[p_0(s)+\varepsilon_0]ds,$$

i.e, (3.4) is verified for $n=2$.
Using again (3.3) and (3.4), we conclude

$$|x_{n+1}(t)-x_n(t)| \leq \int_0^t |U(t,s)|.|f_{n+1}(s)-f_n(s)|ds$$
$$\leq \int_0^t MK_0T[\varepsilon_{n-1}+l(s)|x_n(s)-x_{n-1}(s)|]ds$$
$$\leq \varepsilon_{n-1}MK_0Tt + \int_0^t l(s)\left[\sum_{k=0}^{n-2}\int_0^s \varepsilon_{n-2-k}\frac{(MK_0T)^{k+2}(n(s)-n(u))^k}{k!}du\right.$$
$$\left.+\int_0^s \frac{(MK_0T)^{n+1}(n(s)-n(u))^{n-1}}{(n-1)!}(p_0(u)+\varepsilon_0)du\right] ds$$
$$= \varepsilon_{n-1}MK_0Tt + \sum_{k=0}^{n-2}\varepsilon_{n-2-k}\int_0^t\left[\int_0^s \frac{(MK_0T)^{k+2}(n(s)-n(u))^k}{k!}l(s)du\right] ds$$
$$+\int_0^t l(s)\left(\int_0^s \frac{(MK_0T)^{n+1}(n(s)-n(u))^{n-1}}{(n-1)!}l(s)[p_0(u)+\varepsilon_0]du\right) ds$$
$$= \varepsilon_{n-1}MK_0Tt + \sum_{k=0}^{n-2}\varepsilon_{n-2-k}\int_0^t\left(\int_u^t \frac{(MK_0T)^{k+2}(n(s)-n(u))^k}{k!}l(s)ds\right) du$$
$$+\int_0^t \left(\int_u^t \frac{(MK_0T)^{n+1}(n(s)-n(u))^{n-1}}{(n-1)!}l(s)ds\right)[p_0(u)+\varepsilon_0]du$$
$$= \varepsilon_{n-1}MK_0Tt + \sum_{k=0}^{n-2}\varepsilon_{n-2-k}\int_0^t \frac{(MK_0T)^{k+2}(n(s)-n(u))^{k+1}}{(k+1)!}du$$
$$+\int_0^t \frac{(MK_0T)^{n+1}(n(s)-n(u))^{n}}{n!}[p_0(u)+\varepsilon_0]du$$
$$= \sum_{k=0}^{n-1}\varepsilon_{n-1-k}\cdot\int_0^t \frac{(MK_0T)^{k+1}(n(s)-n(u))^{k}}{k!}du$$
$$+\int_0^t \frac{(MK_0T)^{n+1}(n(s)-n(u))^{n}}{n!}[p_0(u)+\varepsilon_0]du$$

and statement (3.8) it is true for $n+1$.

From (3.8) it follows that for $n\geq 2$ and $t\in I$

$$|x_n(t)-x_{n-1}(t)| \leq a_n, \tag{3.5}$$

where

$$a_n = \sum_{k=0}^{n-2}\varepsilon_{n-2-k}\frac{(MK_0T)^{k+1}n(T)^k}{k!} + \frac{(MK_0T)^n n(T)^{n-1}}{(n-1)!}\left[\int_0^1 p_0(u)du+\varepsilon_0\right],$$

Obviously, the series whose n-th term is a_n converges. So, from (3.5) we infer that $x_n(.)$ converges to a continuous function, $x(.) : I \to X$, uniformly on I.

On the other hand, in view of (3.3) there is

$$|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)a_{n-1}, \quad t \in I, n \geq 3$$

which implies that the sequence $f_n(.)$ converges to a Lusin measurable function $f(\cdot) : I \to X$.

Since $x_n(.)$ is bounded and

$$|f_n(t)| \leq l(t)|x_{n-1}(t)| + q(t) + 1,$$

we infer that $f(.)$ is also Bochner integrable.

Passing with $n \to \infty$ in (3.1) and using the Lebesgue dominated convergence theorem, we obtain

$$x(t) = -\frac{\partial}{\partial s}G(t,0)x_0 + G(t,0)y_0 + \int_0^t U(t,s)f(s)ds, \quad t \in I.$$

On the other hand, from (3.2) we get

$$f_n(t) \in F(t, x_n(t)) + \varepsilon_n B, \quad t \in I, n \geq 1$$

and letting $n \to \infty$ we obtain

$$f(t) \in F(t, x(t)), \quad t \in I,$$

which completes the proof. □

Theorem 3.2. *Assume that Hypotheses H2 and H3 are satisfied and $mL < 1$.*

Then, for every $x_0 \in X$ problem (1.3)-(1.4) has a solution $x(.) : I \to X$.

Proof. The proof follows the same pattern as in the proof of Theorem 3.1. This time

$$x_n(t) = \mathcal{G}(t)Cx_0 - \int_0^T H(t,s)f_n(s)\mathrm{d}s, \quad \forall t \in I,$$

with $f_n(\cdot)$ as before and

$$|x_n(t) - x_{n-1}(t)| \leq \sum_{j=0}^{n-2} \varepsilon_{n-2-j}m^{j+1}L^jT + m^nL^{n-1}\int_0^T (p_0(s) + \varepsilon_0)ds$$

for $n \geq 2$ and $t \in I$. The estimate in (3.5) becames

$$|x_n(t) - x_{n-1}(t)| \leq a_n,$$

where

$$a_n = \sum_{j=0}^{n-2} \varepsilon_{n-2-j}m^{j+1}L^jT + m^nL^{n-1}\int_0^T (p_0(s) + \varepsilon_0)ds$$

Taking into account the fact that $mL < 1$, we deduce that the series whose n-th term is a_n is convergent. □

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