Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 1, 55–73
DOI: 10.24193/subbmath.2021.1.05

Perturbed eigenvalue problems: an overview

Maria Fărcăşeanu, Andrei Grecu,
Mihai Mihăilescu and Denisa Stancu-Dumitru

Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th anniversary.

Abstract. The study of perturbed eigenvalue problems has been a very active field of investigation throughout the years. In this survey we collect several results in the field.

Mathematics Subject Classification (2010): 35D30, 35D40, 46E30, 49J40, 35A15.

Keywords: Eigenvalue problem, p-Laplace operator, nonlocal (s, p)-Laplace operator, Sobolev space, variational methods.

1. Introduction

This paper is dedicated to prof. Gheorghe Moroşanu on the occasion of his 70th birthday. The topic of our paper fits perfectly with one of prof. Moroşanu 's fields of interests, namely *the study of eigenvalue problems for elliptic operators*, on which he brought a couple of nice contributions which will be recalled in the main body of this article. It is an opportunity and an honour for us to dedicate this work to our professor and friend Gheorghe Moroşanu on the occasion of his 70th birthday.

The goal of this paper is to collect some known results on perturbed eigenvalue problems. We split the discussion in two main parts. More precisely, we will start our survey by presenting results on the classical eigenvalue problem for the p-Laplace operator in both local and nonlocal cases (including a discussion on the limiting case when $p \to \infty$), and we will continue with the case of the perturbed eigenvalue problems of the p-Laplace operator on bounded domains under different boundary conditions or on unbounded domains.

1.1. Notations

Throughout this paper Ω will stand for an open set (bounded or unbounded) of the Euclidean space $\mathbb{R}^N$. We will denote by $\partial\Omega$ the boundary of Ω while ν will stand for the unit outward normal to $\partial\Omega$ and $\frac{\partial u}{\partial \nu}$ will represent the normal derivative of u. The Euclidean norm on $\mathbb{R}^N$ will be denoted by $|\cdot|_N$.

2. Eigenvalue problems for the p-Laplace operator

2.1. The case of the (local) p-Laplace operator

For each real number $p \in (1, \infty)$ and each function $u : \Omega \to \mathbb{R}$, smooth enough, we define *the (local) p-Laplace operator* by

$$\Delta_p u := \operatorname{div}(|\nabla u|_N^{p-2} \nabla u) .$$

2.1.1. The case of bounded domains. In this section we will assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$. The *classical eigenvalue problem for the p-Laplace operator* reads as follows

$$-\Delta_p u = \lambda |u|^{p-2} u, \quad \text{in } \Omega\,, \tag{2.1}$$

where $\lambda \in \mathbb{R}$ is a real parameter. This problem was studied under different boundary conditions (see, e.g. Lê [16] for more details), such as

- Dirichlet boundary conditions

$$u = 0, \quad \text{on } \partial\Omega\,, \tag{2.2}$$

- Neumann boundary conditions

$$|\nabla u|_N^{p-2} \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega\,, \tag{2.3}$$

- Robin boundary conditions

$$|\nabla u|_N^{p-2} \frac{\partial u}{\partial \nu} + \alpha |u|^{p-2} u = 0, \quad \text{on } \partial\Omega\,, \tag{2.4}$$

where $\alpha > 0$ is a given real number, etc. In this context, a parameter λ is called an *eigenvalue* of problem (2.1) if the problem possesses a nontrivial (weak) solution u which belongs to a suitable Sobolev space denoted by $W(\Omega)$, where either $W(\Omega) = W_0^{1,p}(\Omega)$, if we are working under the Dirichlet boundary conditions, or $W(\Omega) = W^{1,p}(\Omega)$, if we are working under the Neumann or Robin boundary conditions. More precisely, if we are working under boundary conditions (2.2) or (2.3) then λ is an eigenvalue of problem (2.1) if there exists $u \in W(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|_N^{p-2} \nabla u \nabla \phi \, dx = \lambda \int_\Omega |u|^{p-2} u \phi \, dx, \quad \forall\, \phi \in W(\Omega)\,,$$

while, if we are working under boundary conditions (2.4) then λ is an eigenvalue of problem (2.1) if there exists $u \in W(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|_N^{p-2} \nabla u \nabla \phi \, dx + \alpha \int_{\partial\Omega} |u|^{p-2} u \phi \, d\sigma(x) = \lambda \int_\Omega |u|^{p-2} u \phi \, dx, \ \forall\, \phi \in W(\Omega)\,.$$

A function u as above is called an *eigenfunction* corresponding to the eigenvalue λ.

It is well-known (see, e.g. Lindqvist [18] or Lê [16]) that problem (2.1) (under any of the boundary conditions (2.2), (2.3), or (2.4)) has an increasing and unbounded sequence of nonnegative eigenvalues, say $\{\lambda_k(p;\Omega)\}_{k\geq 1}$, which can be produced using, for instance, the Ljusternik-Schnirelman theory. We recall that for each integer $k \geq 1$

the eigenvalue $\lambda_k(p;\Omega)$, under the boundary conditions (2.2) or (2.3), has the following variational characterisation, (see, e.g. [16]),

$$\lambda_k(p;\Omega) := \inf_{A\in\Sigma_k} \sup_{u\in A} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}\,, \tag{2.5}$$

while, under the boundary condition (2.4) its variational characterisation reads as

$$\lambda_k(p;\Omega) := \inf_{A\in\Sigma_k} \sup_{u\in A} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, dx + \alpha \int_{\partial\Omega} |u|^p \, d\sigma(x)}{\displaystyle\int_\Omega |u|^p \, dx}\,, \tag{2.6}$$

where

$$\begin{aligned}\Sigma_k \quad := \quad & \{A \subset W(\Omega) \mid A \text{ is symmetric and compact in the} \\ & \text{topology of } W(\Omega), \gamma(A) \geq k\}\,,\end{aligned}$$

and $\gamma(A)$ stands for the Krasnosel'skii genus of A, which is defined as the smallest integer m for which there exists a continuous odd map $f : A \to \mathbb{R}^m \setminus \{0\}$. If no such integer exists, then we set $\gamma(A) = \infty$, while $\gamma(\emptyset) = 0$. Note that in the particular cases when $p = 2$ (and $N \geq 1$), that is the case when the eigenvalue problem (2.1) is linear, or $N = 1$ (and $p \in (1,\infty)$), that is the 1-dimensional case, the sequence $\{\lambda_k(p;\Omega)\}_{k\geq 1}$ describes completely the set of eigenvalues of problem (2.1). However, when $N \geq 2$ and $p \in (1,\infty)\setminus\{2\}$ the existence of other eigenvalues in the interval $(\lambda_2(p;\Omega),\infty)$ different from those given by the sequence $\{\lambda_k(p;\Omega)\}_{k\geq 3}$ remains an open question. Actually, in the latter case it is not known if the set of all eigenvalues of the problem is discrete or not.

In order to simplify the exposition, in the rest of this paper we will use three different notations for the sequences of eigenvalues of problem (2.1) depending on the boundary conditions that will be considered. More precisely, we let $\{\lambda_k^D(p;\Omega)\}_{k\geq 1}$, $\{\lambda_k^N(p;\Omega)\}_{k\geq 1}$ and $\{\lambda_k^R(p;\Omega)\}_{k\geq 1}$ be the sequences of eigenvalues of problem (2.1) under the boundary conditions (2.2), (2.3) and (2.4), respectively.

At this point it is instructive to point out the following simple observations concerning the variational characterisations of the lowest eigenvalues of problem (2.1) under the three different boundary conditions presented above

$$\lambda_1^D(p;\Omega) := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}\,, \tag{2.7}$$

$$\lambda_1^N(p;\Omega) := \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}\,,$$

$$\lambda_1^R(p;\Omega) := \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, dx + \alpha \int_{\partial\Omega} |u|^p \, d\sigma(x)}{\displaystyle\int_\Omega |u|^p \, dx}.$$

All these minimization problems possess minimizers which are corresponding eigenfunctions for the eigenvalues $\lambda_1^D(p;\Omega)$, $\lambda_1^N(p;\Omega)$ and $\lambda_1^R(p;\Omega)$. These minimizers belong to a certain Hölder space $C^{1,\beta}(\Omega)$ (for some $\beta \in (0,1)$) and do not change sign in Ω. On the other hand, the eigenvalues $\lambda_1^D(p;\Omega)$, $\lambda_1^N(p;\Omega)$ and $\lambda_1^R(p;\Omega)$ are simple and isolated. Moreover, we recall that

$$\lambda_1^D(p;\Omega) > 0 \quad \text{and} \quad \lambda_1^R(p;\Omega) > 0, \quad \forall\, p \in (1,\infty),$$

while

$$\lambda_1^N(p;\Omega) = 0, \quad \forall\, p \in (1,\infty).$$

Since the lowest eigenvalue of problem (2.1)+(2.3) vanishes it is important to present the variational characterisation of the second eigenvalue $\lambda_2^N(p;\Omega)$ (that is the first positive eigenvalue of the problem), namely

$$\lambda_2^N(p;\Omega) := \inf_{u \in X_p(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx},$$

where $X_p(\Omega) := \{u \in W^{1,p}(\Omega) : \int_\Omega |u|^{p-2}u \, dx = 0\}$.

2.1.2. The ∞-eigenvalue problem under the Dirichlet boundary conditions. For each $p \in (1,\infty)$ we can rewrite problem (2.1)+(2.2) as

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

The asymptotic behavior as $p \to \infty$ of problems (2.8) with $\lambda = \lambda_1^D(p;\Omega)$ has been studied by Fukagai, Ito, & Narukawa [11] and Juutinen, Lindqvist, & Manfredi [15]). A first step in this direction was to show that

$$\lim_{p\to\infty} \sqrt[p]{\lambda_1^D(p;\Omega)} = \left[\max_{x\in\Omega} dist(x,\partial\Omega)\right]^{-1},$$

where $dist(\cdot,\partial\Omega)$ stands for the distance function to the boundary of Ω, (recall that $dist(x,\partial\Omega) := \inf_{y\in\partial\Omega} |x-y|_N$, for all $x \in \Omega$). Next, since the corresponding eigenfunctions of $\lambda_1^D(p;\Omega)$ are, actually, minimizers for the minimization problem (2.7) that do not change sign in Ω, we can let, for each $p \in (1,\infty)$, $u_p > 0$ to be an eigenfunction corresponding to the eigenvalue $\lambda_1^D(p;\Omega)$. Juutinen, Lindqvist & Manfredi showed in [15] that there exists a subsequence of $\{u_p\}$ which converges uniformly in Ω to a nontrivial and nonnegative viscosity solution of the limiting problem

$$\begin{cases} \min\left\{ |\nabla u|_N - \left[\max_{x\in\Omega} dist(x,\partial\Omega)\right]^{-1} u,\ -\Delta_\infty u \right\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

where Δ_∞ is the ∞-Laplace operator, which on sufficiently smooth functions $u : \Omega \to \mathbb{R}$ is given by $\Delta_\infty u := \langle D^2u\nabla u, \nabla u\rangle = \sum_{i,j=1}^N \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}$. Note that $dist(\cdot, \partial\Omega)$ is not always a viscosity solution of (2.9), but, in the particular case when Ω is a ball it turns out that $dist(\cdot, \partial\Omega)$ is the only viscosity solution of (2.9). However, for general domains Ω the convergence of the entire sequence u_p to a unique limit, as $p \to \infty$, is an open question.

2.1.3. The case of unbounded domains. In the first part of this section we will let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ be a general open set (bounded or unbounded) and $V : \Omega \to \mathbb{R}$ be a function which satisfies the hypotheses

$$\begin{cases} V \in L^1_{loc}(\Omega),\ V^+ = V_1 + V_2 \neq 0,\ V_1 \in L^{N/2}(\Omega), \\ \lim\limits_{|x|_N \to \infty} |x|_N^2 V_2(x) = 0,\ \lim\limits_{x \to y} |x-y|_N^2 V_2(x) = 0 \text{ for any } y \in \overline{\Omega}. \end{cases} \tag{2.10}$$

Note that in particular the function V may change sign in Ω.

In [25] Szulkin & Willem analyzed the eigenvalue problem

$$-\Delta u = \lambda V(x) u, \quad u \in \mathcal{D}_0^{1,2}(\Omega)\,, \tag{2.11}$$

where $\mathcal{D}_0^{1,2}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ under the L^2-norm of the gradient. Using an elementary argument based on a simple minimization procedure it was proved in [25, Theorems 2.2 & 2.3] the existence of infinitely many eigenvalues of (2.11). A similar result was obtained in the case when instead of the Laplace operator was considered the general p-Laplace operator in equation (2.11) (naturally, in this new case conditions (2.10) were slightly modified in order to be compatible with the new situation).

In the second part of this section we let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a simply connected bounded domain, containing the origin, with C^2 boundary denoted by $\partial\Omega$ and we denote by $\Omega^{\text{ext}} := \mathbb{R}^N \setminus \overline{\Omega}$ the exterior of Ω. Let $K : \Omega^{\text{ext}} \to (0, \infty)$ be a function having the property that $K \in L^\infty(\Omega^{\text{ext}}) \cap L^{N/p}(\Omega^{\text{ext}})$, for some $p \in (1, N)$. Chhetri and Drábek studied in [6] the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda K(x)|u|^{p-2}u, & \text{for} \quad x \in \Omega^{\text{ext}}, \\ u(x) = 0, & \text{for} \quad x \in \partial\Omega, \\ u(x) \to 0, & \text{as} \quad |x|_N \to \infty\,. \end{cases} \tag{2.12}$$

In particular, they showed that the lowest eigenvalue of problem (2.12) has the following variational characterization

$$\lambda_1(p; \Omega^{\text{ext}}) := \inf_{u \in C_0^\infty(\Omega^{\text{ext}}) \setminus \{0\}} \frac{\displaystyle\int_{\Omega^{\text{ext}}} |\nabla u|_N^p \mathrm{d}x}{\displaystyle\int_{\Omega^{\text{ext}}} K(x)|u|^p \mathrm{d}x}\,. \tag{2.13}$$

Moreover, $\lambda_1(p; \Omega^{\text{ext}})$ is simple, isolated and its corresponding eigenfunctions have constant sign in Ω^{ext}. In particular, the results from [6] complemented to the case of exterior domains the results obtained on the classical eigenvalue problem of the p-Laplacian on bounded domains subject to the homogeneous Dirichlet boundary conditions (that is problem (2.1)+(2.2), or, equivalently, problem (2.8)).

2.2. The case of the nonlocal p-Laplace operator

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. For each $p \in (1, \infty)$ and $s \in (0, 1)$ we define the nonlocal nonlinear operator

$$(-\Delta_p)^s u(x) := 2 \lim_{\epsilon \searrow 0} \int_{|x-y|_N \geq \epsilon} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|_N^{N+sp}} \, dy, \ \ x \in \mathbb{R}^N . \quad (2.14)$$

Since for $p = 2$ the above definition reduces to the linear fractional Laplacian, $(-\Delta)^s$, we will refer to $(-\Delta_p)^s$ as being a *fractional* (s, p)*-Laplace operator.*

The eigenvalue problem for the fractional (s, p)-Laplacian reads as follows

$$\begin{cases} (-\Delta_p)^s u(x) = \lambda |u(x)|^{p-2} u(x), & \text{for } x \in \Omega, \\ u(x) = 0, & \text{for } x \in \mathbb{R}^N \setminus \Omega . \end{cases} \quad (2.15)$$

Problem (2.15) was extensively studied in the literature in the last decade. Among the results related with this problem we just recall some facts from the paper by Lindgren & Lindqvist [17]. First, in order to explain the notion of eigenvalue for problem (2.15) let us denote by $\widetilde{W}_0^{s,p}(\Omega)$ the fractional Sobolev space where it is natural to seek weak solutions for this problem. Next, for simplicity, for each $p \in (1, \infty)$ and $s \in (0, 1)$ we will consider the notation

$$\mathcal{E}_{s,p}(u, v) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|_N^{N+sp}} \, dx \, dy \, , \quad (2.16)$$

for all $u, \ v \in \widetilde{W}_0^{s,p}(\Omega)$. A real number $\lambda \in \mathbb{R}$ will be called an *eigenvalue* of problem (2.15) if there exists a function $u \in \widetilde{W}_0^{s,p}(\Omega)$ such that

$$\mathcal{E}_{s,p}(u, v) = \lambda \int_\Omega |u(x)|^{p-2} u(x) v(x) \, dx, \quad \forall \, v \in \widetilde{W}_0^{s,p}(\Omega) . \quad (2.17)$$

Further, we define

$$\lambda_1(s, p) := \inf_{u \in \widetilde{W}_0^{s,p}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|_N^{N+sp}} \, dx \, dy}{\displaystyle\int_{\mathbb{R}^N} |u|^p \, dx} . \quad (2.18)$$

It is known that $\lambda_1(s, p)$ is attained at some $u \in \widetilde{W}_0^{s,p}(\Omega) \setminus \{0\}$ (see [17, Theorem 5]), with $\|u\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\Omega)} = 1$ and

$$\frac{\displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|_N^{N+sp}} \, dx \, dy}{\displaystyle\int_{\mathbb{R}^N} |u|^p \, dx} = \lambda_1(s, p) .$$

Moreover, it holds true that

$$\mathcal{E}_{s,p}(u, \varphi) = \lambda_1(s, p) \int_{\mathbb{R}^N} |u(x)|^{p-2} u(x) \varphi(x) dx, \quad \forall \, \varphi \in \widetilde{W}_0^{s,p}(\Omega) ,$$

which means that $\lambda_1(s, p)$ is an eigenvalue of problem (2.15).

Next, let us recall a result on an eigenvalue problem involving the fractional Laplacian studied on the whole Euclidean space $\mathbb{R}^N$. More precisely, the second author of this survey studied in [12] the eigenvalue problem

$$(-\Delta)^s u(x) = \lambda V(x)u(x), \quad \forall\, x \in \mathbb{R}^N, \tag{2.19}$$

where $s \in (0,1)$ is a given real number, λ is a real parameter and $V : \mathbb{R}^N \to \mathbb{R}$ is a function that may change sign and which satisfies the hypothesis

$$\begin{cases} V \in L^1_{\text{loc}}(\mathbb{R}^N),\ V^+ = V_1 + V_2 \neq 0,\ V_1 \in L^{\frac{N}{2s}}(\mathbb{R}^N) \text{ and} \\ \lim\limits_{x\to y} |x-y|_N^{2s} V_2(x) = 0, \text{ for all } y \in \mathbb{R}^N \text{ and } \lim\limits_{|x|_N\to\infty} |x|_N^{2s} V_2(x) = 0. \end{cases} \tag{$\widetilde{V}$}$$

It was shown in [12, Theorem 1.3] that under condition ($\widetilde{\text{V}}$) the problem (2.19) has an unbounded, increasing sequence of positive eigenvalues. In particular this result extended to the case of nonlocal operators the result by Szulkin & Willem from [25, Theorems 2.2 & 2.3].

3. Perturbed eigenvalue problems for the p-Laplace operator

In this section we will analyze some perturbations of classical eigenvalue problems. All the perturbed eigenvalue problems are, actually, *nontypical eigenvalue problems* since the differential operators involved in their constructions are inhomogeneous. However, their formulations are similar with those of the typical eigenvalue problems and for that reason we will continue to call the parameter λ involved in these equations an *eigenvalue* if the corresponding problem possesses a nontrivial weak solution.

3.1. The perturbation of the (local) p-Laplace operator

3.1.1. The case of bounded domains. In this section we will assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$. Let $p \in (1,\infty)$ be a given real number. We will call a *perturbation of the eigenvalue problem (*2.1*)* a problem of type

$$-\Delta_p u - \Delta_q u = \lambda |u|^{p-2}u, \quad \text{in } \Omega\,, \tag{3.1}$$

where $q \in (1,\infty) \setminus \{p\}$ is a given real number and $\lambda \in \mathbb{R}$ is a real parameter. Our goal will be to determine the set of all parameters λ for which problem (3.1) has nontrivial solutions, under different boundary conditions. This kind of parameters will be called eigenvalues of problem (3.1).

I. *The case of the Dirichlet boundary conditions.* We consider the case when problem (3.1) is investigated subject to the boundary conditions (2.2). More precisely, we consider the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.2}$$

For this problem a *weak solution* is a function $u \in W_0^{1,\max\{p,q\}}(\Omega)$ such that

$$\int_\Omega (|\nabla u|_N^{p-2} + |\nabla u|_N^{q-2})\nabla u \nabla\phi\, dx = \lambda \int_\Omega |u|^{p-2}u\phi\, dx, \quad \forall\, \phi \in W_0^{1,\max\{p,q\}}(\Omega)\,.$$

We will say that λ from the above relation is an *eigenvalue* of problem (3.2) if $u \in W_0^{1,\max\{p,q\}}(\Omega) \setminus \{0\}$. In that case we will refer to u as being an *eigenfunction* corresponding to the eigenvalue λ.

Independently, Tanaka [23] and Bocea and the third author of this paper [5, Theorem 1.1] proved the following result.

Theorem 3.1. *The set of eigenvalues of problem* (3.2) *is exactly given by the open interval* $(\lambda_1^D(p;\Omega), \infty)$. *Moreover, for each* $\lambda \in (\lambda_1^D(p;\Omega), \infty)$ *there exists a nontrivial and nonnegative weak solution for problem* (3.2).

Note the interesting fact that in the case of the perturbed eigenvalue problems under the Dirichlet boundary conditions, such as (3.2), the set of eigenvalues can be entirely described and it is a continuous set. In particular, this is in sharp contrast with the situation which occurs in the case of the Laplace operator when the set of eigenvalues is discrete.

We would like to point out that similar results with those obtained in Theorem 3.1 were obtained by Bhattacharya, Emamizadeh, & Farjudian in [3] and by the first author of this paper in [8, Theorem 1] but for a class of anisotropic differential operators.

Further, let us assume that for each real number $p \in (1,\infty)$ the parameter $q \in (1,\infty) \setminus \{p\}$ which is involved in the construction of problem (3.2) depends on p. In other words we assume that $q : (1,\infty) \to (1,\infty)$ is a function which depends on p, i.e. $q = q(p)$. Furthermore, we assume that $\lim_{p\to\infty} \frac{q(p)}{p} = Q \in (0,\infty) \setminus \{1\}$, and where either $q(p) < p$ if $Q \in (0,1)$ or $q(p) > p$ if $Q \in (1,\infty)$. In [5] the authors investigated the asymptotic behavior of positive solutions of the problems (3.2) as $p \to \infty$. They showed that for any $\Lambda \in [(\max_{x\in\Omega} dist(x,\partial\Omega))^{-1}, \infty)$ and each sequence $\{\lambda_p\}$, with $\lambda_p \in (\lambda_1^D(p;\Omega), \infty)$, such that $\lim_{p\to\infty}(\lambda_p)^{1/p} = \Lambda$ the sequence of positive weak solutions of (3.2) with $\lambda = \lambda_p$ possesses a subsequence which converges to a nontrivial and nonnegative viscosity solution of the limiting problem

$$\begin{cases} \min\left\{\max\{|\nabla u|_N, |\nabla u|_N^Q\} - \Lambda u, -\Delta_\infty u\right\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

On the other hand, it was shown that for all $\Lambda \in (-\infty, (\max_{x\in\Omega} dist(x,\partial\Omega))^{-1})$ there are no nonnegative and nontrivial solutions of problem (3.3). Thus, in comparison to the well-known problem (2.9), the analysis of (3.3) reveals a markedly different situation: while for the original problem a single value of Λ, namely $\left[\max_{x\in\Omega} dist(x,\partial\Omega)\right]^{-1}$, is known for which the corresponding viscosity solution is nonnegative, in the case of problem (3.3) this situation extends to the entire interval $[(\max_{x\in\Omega} dist(x,\partial\Omega))^{-1}, \infty)$ (see [5, Theorem 1.3] for details).

II. *The case of the Neumann boundary conditions.* We consider the case when problem (3.1) is investigated subject to the Neumann-type boundary conditions. More precisely, we consider the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ (|\nabla u|_N^{p-2} + |\nabla u|_N^{q-2})\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.4}$$

where p, $q \in (1,\infty)$ and $p \neq q$. For this problem a *weak solution* is a function $u \in W^{1,\max\{p,q\}}(\Omega)$ such that

$$\int_\Omega (|\nabla u|_N^{p-2} + |\nabla u|_N^{q-2})\nabla u \nabla \phi \, dx = \lambda \int_\Omega |u|^{p-2}u\phi \, dx, \quad \forall\, \phi \in W^{1,\max\{p,q\}}(\Omega)\,.$$

We will say that λ is an *eigenvalue* of problem (3.4) if $u \in W^{1,\max\{p,q\}}(\Omega) \setminus \{0\}$. In that case we will refer to u as being an *eigenfunction* corresponding to the eigenvalue λ.

Problem (3.4) was investigated in the case when $p = 2$ and $q \in (1,\infty) \setminus \{2\}$ by three of the authors of this paper in [19, Theorem 1.1] (for the case $q \in (2,\infty)$) and [9, Theorem 1] (for the case $q \in (1,2)$) while the case $p \in (2,\infty)$ and $q \in (1,\infty) \setminus \{p\}$ it was analyzed by Moroşanu and the third author of this paper in [21, Theorem 1.1]. We summarise all the results on problem (3.4) in the following theorem.

Theorem 3.2. *Assume that $p \in [2,\infty)$ and $q \in (1,\infty)\setminus\{p\}$. For each such two numbers p and q define*

$$X_{p,q}(\Omega) := \left\{ u \in W^{1,\max\{p,q\}}(\Omega) : \int_\Omega |u|^{p-2}u \, dx = 0 \right\}.$$

Then the set of eigenvalues of problem (3.4) is precisely

$$\{0\} \cup (\mu_1(p,q;\Omega), \infty)\,,$$

where

$$\mu_1(p,q;\Omega) := \inf_{u \in X_{p,q}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}\,,$$

is a positive constant.

Note that in the case when $q \in (1,p)$ and $p \geq 2$ we have $\mu_1(p,q;\Omega) = \lambda_2^N(p;\Omega)$ and thus the constant $\mu_1(p,q;\Omega)$ does not depend on q in this case. On the other hand, in the case where $q \in (p,\infty)$ the constant $\mu_1(p,q;\Omega)$ depends on q since in this case $W^{1,\max\{p,q\}}(\Omega) = W^{1,q}(\Omega)$. In that case we can deduce only the fact that $\mu_1(p,q;\Omega) \geq \lambda_2^N(p;\Omega)$.

The conclusion of Theorem 3.2 is interesting if we compare it, for example, with two classical well-known results on similar problems. First, recall the fact that when $q = p = 2$ then problem (3.4) reduces to the eigenvalue problem for the Laplace operator under the homogenous Neumann boundary conditions. In that case we recall the well-known fact that the problem possesses a discrete set of eigenvalues which can be organized as an increasing and unbounded sequence of positive real numbers. On the other hand, if we consider for instance the problem

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

with $q \in (1, \infty) \setminus \{2\}$ then the set of all parameters λ for which problem (3.5) has nontrivial weak solutions is exactly the interval $[0, \infty)$. In that case the set of eigenvalues of problem (3.5) is continuous. The case of problem (3.4) with $p = 2$ and $q \in (1, \infty) \setminus \{2\}$ brings to our attention a new situation when the set of eigenvalues of the problem possesses on the one hand, a continuous part, that is the interval $(\mu_1(2, q; \Omega), \infty)$, and, on the other hand, one more eigenvalue, i.e. $\lambda = 0$, which is isolated.

Finally, we would like to point out three similar results with those obtained in Theorem 3.2. The first result was recently obtained by Abreu & Madeira in [1] in the case when in problem (3.4) we have $p = 2$, $q \in (1, \infty) \setminus \{2\}$ but working under parametric-type boundary conditions instead of the Neumann-type boundary conditions. The other two results are due to Costea & Moroşanu [7] and Barbu & Moroşanu [2] for some Steklov-type eigenvalue problems.

III. *The case of the Robin boundary conditions.* Assume that we are working in an Euclidean space having dimension $N \geq 2$. We consider the case when problem (3.1) is investigated subject to the Robin-type boundary conditions. More precisely, for a given real number $\alpha > 0$ we consider the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ (|\nabla u|_N^{p-2} + |\nabla u|_N^{q-2})\dfrac{\partial u}{\partial \nu} + \alpha |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where p, $q \in (1, \infty)$ and $p \neq q$.
For this problem a *weak solution* is a function $u \in W^{1,\max\{p,q\}}(\Omega)$ such that

$$\int_\Omega (|\nabla u|_N^{p-2} + |\nabla u|_N^{q-2})\nabla u \nabla \phi \, dx + \alpha \int_{\partial\Omega} |u|^{p-2}u\phi \, d\sigma(x) = \lambda \int_\Omega |u|^{p-2}u\phi \, dx \,,$$

for all $\phi \in W^{1,\max\{p,q\}}(\Omega)$. We will say that λ is an *eigenvalue* of problem (3.6) if $u \in W^{1,\max\{p,q\}}(\Omega) \setminus \{0\}$. In that case we will refer to u as being an *eigenfunction* corresponding to the eigenvalue λ.

The perturbed eigenvalue problem (3.6) has been investigated by Gyulov & Moroşanu in [14]. In order to recall their result let us define two quantities which play an important role in the analysis of the problem. More precisely, we define

$$\lambda^\star := \alpha \frac{m_{N-1}(\partial\Omega)}{m_N(\Omega)} \,,$$

where $m_{N-1}(\partial\Omega)$ and $m_N(\Omega)$ denote the corresponding $N-1$ and N dimensional Lebesgue measures of the boundary $\partial\Omega$ and the set Ω, respectively, and

$$\nu_1(p,q;\Omega) := \inf_{u\in W^{1,\max\{p,q\}}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p\,dx + \alpha\int_{\partial\Omega}|u|^p\,d\sigma(x)}{\displaystyle\int_\Omega |u|^p\,dx}\,.$$

By [14, Remark 2] it is clear that $\lambda^\star > \nu_1(p,q;\Omega)$. Moreover, we point out that if $q\in(1,p)$ then $W^{1,\max\{p,q\}}(\Omega) = W^{1,p}(\Omega)$ and, consequently, in that case $\nu_1(p,q;\Omega) = \lambda_1^R(p;\Omega)$. By contrary, if $q\in(p,\infty)$ then $W^{1,\max\{p,q\}}(\Omega) = W^{1,q}(\Omega)$ and, consequently, in that case $\nu_1(p,q;\Omega)\geq\lambda_1^R(p;\Omega)$. The main result on (3.6) is a consequence of Theorems 1-3 from [14].

Theorem 3.3. *For each p, $q\in(1,\infty)$ in the interval $(-\infty,\lambda_1^R(p;\Omega)]$ there is no eigenvalue of problem (3.6). If $q\in(p,\infty)$ then each $\lambda\in(\nu_1(p,q;\Omega),\lambda^\star)$ is an eigenvalue of problem (3.6). If $q\in(1,p)$ then each $\lambda\in(\lambda_1^R(p;\Omega),\lambda^\star)$ is an eigenvalue of problem (3.6).*

The case $\lambda\geq\lambda^\star$ is open.

3.1.2. The case of unbounded domains. In the first part of this section we will let $\Omega\subseteq\mathbb{R}^N$ $(N\geq 3)$ be a general open set (bounded or unbounded) and $V:\Omega\to\mathbb{R}$ a function which satisfies the hypothesis (2.10).

Motivated by the results from [25] on the eigenvalue problem (2.11) in [22] the last two authors of this paper studied the set of parameters λ for which the following perturbed eigenvalue problem has nontrivial solutions

$$-\Delta u - \Delta_p u = \lambda V(x)u,\quad u\in\mathcal{D}_0^{1,\Phi_p}(\Omega)\,, \tag{3.7}$$

where $p\in(1,N)\setminus\{2\}$ and $\Phi_p:\mathbb{R}\to\mathbb{R}$ is given by $\Phi_p(t):=\frac{t^2}{2}+\frac{|t|^p}{p}$, the Orlicz-Sobolev type space $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ is obtained as the closure of $C_0^\infty(\Omega)$ under the Luxemburg-type norm

$$\|u\| := \inf\left\{\mu>0;\ \int_\Omega \Phi_p\left(\frac{|\nabla u(x)|_N}{\mu}\right)\,dx\leq 1\right\}\,,$$

(see [22, Section 2] for more details regarding the definition and properties of Φ_p and $\mathcal{D}_0^{1,\Phi_p}(\Omega)$). We recall that in the above framework we say that u is a *weak solution* of equation (3.7) if there exists $u\in\mathcal{D}_0^{1,\Phi_p}(\Omega)\setminus\{0\}$ such that

$$\int_\Omega \nabla u\nabla w\,dx + \int_\Omega |\nabla u|_N^{p-2}\nabla u\nabla w\,dx = \lambda\int_\Omega V(x)uw\,dx,\quad \forall\, w\in\mathcal{D}_0^{1,\Phi_p}(\Omega).$$

The main result on problem (3.7) is formulated in the following theorem.

Theorem 3.4. *Assume condition (2.10) is fulfilled. Then the set of parameters λ for which problem (3.7) possesses nontrivial solutions is exactly the open interval*

$(\lambda_1, +\infty)$, *where* λ_1 *is given by*

$$\lambda_1 := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u(x)|_N^2 \, dx}{\displaystyle\int_\Omega V(x) u^2(x)\, dx} \,. \tag{3.8}$$

Note that by [25, Theorem 2.2] it is obvious that λ_1 defined in (3.8) is achieved in $\mathcal{D}_0^{1,2}(\Omega)$ which is larger than $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ (see [22, Section 2] for details).

In the second part of this section we let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a simply connected bounded domain, containing the origin, with C^2 boundary denoted by $\partial\Omega$ and we denote by $\Omega^{\text{ext}} := \mathbb{R}^N \setminus \overline{\Omega}$ the exterior of Ω. Let $K : \Omega^{\text{ext}} \to (0, \infty)$ be a function having the property that $K \in L^\infty(\Omega^{\text{ext}}) \cap L^{N/p}(\Omega^{\text{ext}})$ for some $p \in (1, N)$. Let $\lambda_1(p; \Omega^{\text{ext}})$ be the first eigenvalue of problem (2.12) given by relation (2.13). In [13] the second author of this paper investigated a perturbation of problem (2.12) obtained when we perturb the p-Laplacian by a q-Laplacian with $q \neq p$. More precisely, he studied the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda K(x)|u|^{p-2}u, & \text{for} \quad x \in \Omega^{\text{ext}}, \\ u(x) = 0, & \text{for} \quad x \in \partial\Omega, \\ u(x) \to 0, & \text{as} \quad |x|_N \to \infty, \end{cases} \tag{3.9}$$

where $p, q \in (1, N)$ with $p \neq q$. Note that the natural function space framework for problem (3.9) is given by the Orlicz-Sobolev space $W_0^{1,\Psi_{p,q}}(\Omega^{\text{ext}})$ constructed with the aid of the N-function $\Psi_{p,q} : [0, \infty) \to \mathbb{R}$, given by $\Psi_{p,q}(t) := \frac{t^p}{p} + \frac{t^q}{q}$. In that framework, we say that $u \in W_0^{1,\Psi_{p,q}}(\Omega^{\text{ext}})$ is a weak solution of problem (3.9), if the following relation holds

$$\int_{\Omega^{\text{ext}}} (|\nabla u|_N^{p-2} + |\nabla u|_N^{q-2}) \nabla u \nabla \varphi \mathrm{d}x = \lambda \int_{\Omega^{\text{ext}}} K(x)|u|^{p-2}u\varphi \mathrm{d}x,$$

for all $\varphi \in W_0^{1,\Psi_{p,q}}(\Omega^{\text{ext}})$.

The main result on problem (3.9) is given by the following theorem.

Theorem 3.5. *The set of all parameters* λ *for which problem* (3.9) *possesses nontrivial weak solutions is the open interval* $(\lambda_1(p; \Omega^{\text{ext}}), \infty)$.

3.2. The perturbation of the nonlocal (s, p)-Laplace operator

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. In [10] three of the authors of this paper studied a perturbation of the eigenvalue problem (2.15), namely

$$\begin{cases} (-\Delta_p)^s u(x) + (-\Delta_q)^t u(x) = \lambda |u(x)|^{r-2} u(x), & \text{for } x \in \Omega, \\ u(x) = 0, & \text{for } x \in \mathbb{R}^N \setminus \Omega \,, \end{cases} \tag{3.10}$$

where s, t, p and q are real numbers satisfying the assumption

$$0 < t < s < 1, \quad 1 < p < q < \infty, \quad s - \frac{N}{p} = t - \frac{N}{q} \,, \tag{3.11}$$

$r \in \{p, q\}$ and $\lambda \in \mathbb{R}$ is a parameter. The goal was to determine all the parameters λ for which problem (3.10) possesses nontrivial weak solutions. By a *weak solution of problem* (3.10) we understand a function $u \in \widetilde{W}_0^{s,p}(\Omega)$ such that

$$\mathcal{E}_{s,p}(u,v) + \mathcal{E}_{t,q}(u,v) = \lambda \int_\Omega |u(x)|^{r-2}u(x)v(x)\,dx, \quad \forall\, v \in \widetilde{W}_0^{s,p}(\Omega)\,, \tag{3.12}$$

where the quantities $\mathcal{E}_{s,p}(u,v)$ and $\mathcal{E}_{t,q}(u,v)$ are given by relation (2.16).

Define

$$\overline{\lambda}_1 := \begin{cases} \lambda_1(s,p), & \text{if} \quad r = p, \\ \lambda_1(t,q), & \text{if} \quad r = q\,, \end{cases} \tag{3.13}$$

where $\lambda_1(s,p)$ and $\lambda_1(t,q)$ are given by relation (2.18). The main result on problem (3.10) is given by the following theorem (see [10, Theorem 1.1]).

Theorem 3.6. *Assume condition* (3.11) *is fulfilled. Then the set of all real parameters* λ *for which problem* (3.10) *has at least a nontrivial weak solution is the interval* $(\overline{\lambda}_1, \infty)$, *with* $\overline{\lambda}_1$ *defined by relation* (3.13)*. Moreover, the weak solution could be chosen to be non-negative.*

Next, we recall a result obtained by the second author of this paper in [12] on a perturbation of problem (2.19), namely

$$(-\Delta)^s\, u(x) + (-\Delta_p)^t\, u(x) = \lambda V(x)u(x), \quad \forall\, x \in \mathbb{R}^N, \tag{3.14}$$

under the assumption

$$0 < t < s < 1 \text{ and } s - \frac{N}{2} = t - \frac{N}{p}, \tag{3.15}$$

where λ is a real parameter and $V : \mathbb{R}^N \to [0, \infty)$ is a function satisfying the hypothesis $(\widetilde{\mathrm{V}})$. Note that in the case of problem (3.14) we have $V = V^+$. We will say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (3.14), if there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\mathcal{E}_{s,2}(u,\varphi) + \mathcal{E}_{t,p}(u,\varphi) = \lambda \int_{\mathbb{R}^N} V(x)u(x)\varphi(x)\,dx\,, \tag{3.16}$$

for all $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$, where the quantities $\mathcal{E}_{s,2}(u,v)$ and $\mathcal{E}_{t,q}(u,v)$ are given by relation (2.16). Furthermore, u from the above relation will be called an eigenfunction corresponding to the eigenvalue λ.

Define

$$\tilde{\lambda}_1 := \inf_{u \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|_N^{N+2s}}\,dxdy}{\displaystyle\int_{\mathbb{R}^N} V(x)u^2\,dx}. \tag{3.17}$$

The main result regarding problem (3.14) is given by the following theorem (see [12, Theorem 1.5]).

Theorem 3.7. *Assume that* $V : \mathbb{R}^N \to [0, \infty)$ *is a function which satisfies condition* $(\widetilde{V})$*. Under assumption* (3.15)*, the set of eigenvalues of problem* (3.14) *is the open interval* $(\tilde{\lambda}_1, \infty)$*. Moreover, the corresponding eigenfunctions can be chosen to be non-negative.*

Remark. A simple analysis of the proof of Theorem 1.3 from [12] shows that in the case when function V satisfies $V(x) \geq 0$, for all $x \in \mathbb{R}^N$, then $\tilde{\lambda}_1$ defined in relation (3.17) is the smallest eigenvalue of problem (2.19).

3.3. A perturbed eigenvalue problem involving rapidly growing operators in divergence form

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. In this section our goal is to recall some results on the perturbation of the classical eigenvalue problem of the Laplace operator subject to the homogenous Dirichlet boundary conditions (that is problem (2.8) with $p = 2$) with a so-called rapidly growing operator in divergence form (that is $\operatorname{div}(e^{|\nabla u|_N^2 - 1}\nabla u)$). More precisely, we are concerned with the problem

$$\begin{cases} -\operatorname{div}(e^{|\nabla u|_N^2 - 1}\nabla u) - \Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

This problem was investigated by Bocea and the third author of this paper in [4]. Using a similar terminology as in the case of the classical eigenvalue problems a real number λ is called an eigenvalue of problem (3.18) if the problem possesses a nontrivial weak solution. Note the fact that the nature of the problem asks for a function space framework involving an Orlicz-Sobolev space, say $X_0 := W_0^{1,\Psi}(\Omega)$ which is constructed with the aid of the N-function $\Psi : [0,\infty) \to \mathbb{R}$, given by $\Psi(t) := e^{t^2} - 1$.

Next, note that the Euler-Lagrange functional associated to the problem (3.18) is $\Lambda : X_0 \to \mathbb{R}$ defined by

$$\Lambda(u) := \frac{1}{2}\int_\Omega \Phi(|\nabla u(x)|_N)\, dx + \frac{e}{2}\int_\Omega |\nabla u(x)|_N^2\, dx - \lambda\frac{e}{2}\int_\Omega |u(x)|^2\, dx.$$

If Λ was smooth on X_0, then one could define an eigenvalue for (3.18) as a real number λ for which there exists a function $u \in X_0 \setminus \{0\}$ such that

$$\int_\Omega e^{|\nabla u|_N^2}\nabla u \nabla v\, dx + e\int_\Omega \nabla u \nabla v\, dx - \lambda e\int_\Omega uv\, dx = 0, \quad \forall\, v \in X_0\,.$$

Unfortunately, in our framework, the functional Λ is not smooth on X_0. However, the functional $g : X_0 \to \mathbb{R}$ defined by

$$g(u) := \frac{e}{2}\int_\Omega |u(x)|^2 dx \tag{3.19}$$

is of class $C^1(X_0, \mathbb{R})$, and we have $\langle g'(u), v\rangle = e\displaystyle\int_\Omega uv\, dx$ for all $u, v \in X_0$. On the other hand, the functional $f : X_0 \to \mathbb{R}$ given by

$$f(u) := \frac{1}{2}\int_\Omega \Phi(|\nabla u(x)|_N) dx + \frac{e}{2}\int_\Omega |\nabla u(x)|_N^2 dx \tag{3.20}$$

is convex, weakly* lower semicontinuous, and coercive but $f \notin C^1(X_0, \mathbb{R})$. To overcome this drawback, we will work with the following reformulation (*à la* Szulkin [24]) of the problem (3.18) as a variational inequality:

$$\begin{cases} f(v) - f(u) - \lambda\langle g^{'}(u), v-u\rangle \geq 0, & \forall\, v \in X_0, \\ u \in X_0. \end{cases} \tag{3.21}$$

A real number λ such that (3.21) has nontrivial solutions $u \in X_0$ is called an *eigenvalue* for the problem (3.21). In this context the main result on problem (3.18) is given by the following theorem (see [4, Theorem 1])

Theorem 3.8. *The set of eigenvalues for problem (*3.18*) is the open interval*

$$\left(\left(1+\frac{1}{e}\right)\lambda_1^D(2;\Omega), \infty\right),$$

*where $\lambda_1^D(2;\Omega)$ stands for the first eigenvalue of the Laplace operator under the homogenous Dirichlet boundary conditions (see relation (*2.7*) with $p=2$).*

3.4. The spectrum of the relativistic mean curvature operator

In this section our goal is to characterize the spectrum of the *relativistic mean curvature operator*, i.e.

$$\mathcal{M}u := -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|_N^2}}\right),$$

acting on maps u defined in an open, bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, subject to the homogeneous Dirichlet boundary conditions. More precisely, our goal is to analyze the problem

$$\begin{cases} \mathcal{M}u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.22}$$

The starting point in the study of problem (3.22) is to explain the function space framework that will be considered in the sequel. Thus, we note that the structure of the relativistic mean curvature operator asks for a condition of type $|\nabla u(x)|_N \leq 1$ for a.e. $x \in \Omega$. That fact and the homogeneous Dirichlet boundary conditions involved in problem (3.22) imply that a good candidate for the functional space framework would be a subset of

$$W_0^{1,\infty}(\Omega) := \{u \in W^{1,\infty}(\Omega) :\ u = 0, \text{ on } \partial\Omega\},$$

namely

$$K_0 := \{u \in W_0^{1,\infty}(\Omega) :\ |\nabla u(x)|_N \leq 1, \text{ a.e. } x \in \Omega\}.$$

We remark that K_0 is a convex and closed subset of $W^{1,\infty}(\Omega)$ which is the dual of a separable Banach space. This leads to the idea of constructing the Euler-Lagrange functional associated to the relativistic mean curvature operator as $I : W^{1,\infty}(\Omega) \to [0,\infty]$ defined by

$$I(u) := \begin{cases} \displaystyle\int_\Omega F(|\nabla u|_N)\, dx & \text{if } u \in K_0, \\ +\infty & \text{if } u \in W^{1,\infty}(\Omega) \setminus K_0, \end{cases}$$

where $F : [-1,1] \to \mathbb{R}$ is given by $F(t) := 1 - \sqrt{1-t^2}$ for all $t \in [-1,1]$. Then, the Euler-Lagrange functional associated to the problem (3.22) is $J_\lambda : W^{1,\infty}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) := I(u) - \frac{\lambda}{2}\int_\Omega u^2\,dx, \quad \forall\, u \in W^{1,\infty}(\Omega)\,.$$

We observe that J_λ is the sum of a convex, lower semi-continuous function and a C^1-functional, and, consequently, it has the structure required by *Szulkin's critical point theory* (see [24]). More precisely, the functional J_λ is the sum of the functional $h_\lambda : W^{1,\infty}(\Omega) \to \mathbb{R}$ defined by

$$h_\lambda(u) := -\frac{\lambda}{2}\int_\Omega u^2\,dx\,,$$

which belongs to $C^1(W^{1,\infty}(\Omega), \mathbb{R})$ and has the derivative given by

$$\langle h'_\lambda(u), v\rangle = -\lambda\int_\Omega uv\,dx, \quad \forall\, u,\ v \in W^{1,\infty}(\Omega)\,,$$

with the functional I which is convex and weakly* lower semicontinuous. Then, we will work with a reformulation of problem (3.22) as a *variational inequality*, namely

$$\begin{cases} I(v) - I(u_\lambda) + \langle h'_\lambda(u_\lambda), v - u_\lambda\rangle \geq 0 \quad \text{for all } v \in W^{1,\infty}(\Omega), \\ u_\lambda \in W^{1,\infty}(\Omega)\,. \end{cases} \tag{3.23}$$

or, equivalently,

$$\begin{cases} I(v) - I(u_\lambda) + \langle h'_\lambda(u_\lambda), v - u_\lambda\rangle \geq 0 \quad \text{for all } v \in K_0, \\ u_\lambda \in K_0\,. \end{cases} \tag{3.24}$$

In this context a real number $\lambda \in \mathbb{R}$ is called an *eigenvalue* for problem (3.22) if problem (3.24) has a nontrivial solution $u_\lambda \in K_0$. u_λ will be called an *eigenfunction* corresponding to the eigenvalue λ. According to the terminology from [24], we refer to u_λ as being a *critical point* of functional J_λ.

The main result on problem (3.22) is given by the following theorem (see [20, Theorem 1.1]).

Theorem 3.9. *The set of eigenvalues for problem (*3.22*) is the open interval* $(\lambda_1^D(2;\Omega), \infty)$ *where* $\lambda_1^D(2;\Omega)$ *stands for the principal frequency of the Laplace operator in* Ω *subject to the homogeneous Dirichlet boundary conditions (see relation (*2.7*) with* $p = 2$*). Moreover, for each eigenvalue* λ *we can choose a corresponding eigenfunction* $u_\lambda \in K_0$ *which is nonnegative on* Ω *and minimizes* J_λ*.*

Note that problem (3.22) can be regarded as a perturbation of the classical eigenvalue problem of the Laplace operator subject to the homogenous Dirichlet boundary conditions (that is problem (2.8) with $p = 2$). Indeed, first note that the function $F : [-1,1] \to \mathbb{R}$, given by $F(t) := 1 - \sqrt{1-t^2}$, for all $t \in [-1,1]$, admits the following extension into power series

$$F(t) = \frac{1}{2}t^2 + \sum_{n\geq 2} a_n t^{2n}, \quad \forall\, t \in [-1,1]\,,$$

where for each integer $n \geq 2$ we let $a_n := \frac{(2n-3)!!}{2^n n!}$. Thus, the above simple remark suggests to us that the differential operator,

$$u \mapsto -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|_N^2}}\right),$$

on the left hand side of the PDE in (3.22) can be regarded as being equivalent with the differential operator

$$u \mapsto -\Delta u - \sum_{n\geq 2} a_n \Delta_{2n} u\,,$$

where $\Delta_{2n} u$ stands for the $2n$-Laplacian of u (i.e. $\Delta_{2n} u = \mathrm{div}(|\nabla u|_N^{2n-2}\nabla u)$), for each positive integer n. Thus, problem (3.22) can be reformulated as

$$\begin{cases} -\Delta u - \sum\limits_{n\geq 2} a_n \Delta_{2n} u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Acknowledgements. AG, MF and DS-D have been partially supported by CNCS-UEFISCDI Grant No. PN-III-P1-1.1-TE-2019-0456.

References

[1] Abreu, J., Madeira, G.F., *Generalized eigenvalues of the $(p,2)$-Laplacian under a parametric boundary condition*, Proc. Edinb. Math. Soc., **63**(2020), 287-303.

[2] Barbu, L., Moroşanu, G., *Eigenvalues of the negative (p,q)-Laplacian under a Steklov-like boundary condition*, Complex Var. Elliptic Equ., **64**(2019), 685-700.

[3] Bhattacharya, T., Emamizadeh, B., Farjudian, A., *Existence of continuous eigenvalues for a class of parametric problems involving the $(p,2)$-Laplacian operator*, Acta Appl. Math., **165**(2020), 65-79.

[4] Bocea, M., Mihăilescu, M., *Eigenvalue problems in Orlicz-Sobolev spaces for rapidly growing operators in divergence form*, J. Differential Eqnuations, **256**(2014), 640-657.

[5] Bocea, M., Mihăilescu, M., *Existence of nonnegative viscosity solutions for a class of problems involving the ∞-Laplacian*, Nonlinear Differential Equations and Applications (NoDEA), **23**(2016), Art. 11.

[6] Chhetri, M., Drábek, P., *Principal eigenvalue of p-Laplacian operator in exterior domain*, Results Math., **66**(2014), 461-468.

[7] Costea, N., Moroşanu, G., *Steklov-type eigenvalues of $\Delta_p + \Delta_q$*, Pure Appl. Funct. Anal., **3**(2018), 75-89.

[8] Fărcăşeanu, M., *An eigenvalue problem involving an anisotropic differential operator*, Complex Variables and Elliptic Equations, **62**(2017), 297-306.

[9] Fărcăşeanu, M., Mihăilescu, M., Stancu-Dumitru, D., *On the set of eigenvalues of some PDEs with homogeneous Neumann boundary condition*, Nonlinear Analysis, **116**(2015), 19-25.

[10] Fărcăşeanu, M., Mihăilescu, M., Stancu-Dumitru, D., *Perturbed fractional eigenvalue problems*, Discrete and Continuous Dynamical Systems - Series A, **37**(2017), 6243-6255.

[11] Fukagai, N., Ito, M., Narukawa, K., *Limit as $p \to \infty$ of p-Laplace eigenvalue problems and L^∞-inequality of Poincaré type*, Differential Integral Equations, **12**(1999), 183-206.

[12] Grecu, A., *Fractional eigenvalue problems on* $\mathbb{R}^N$, Electronic Journal of Qualitative Theory of Differential Equations, **26**(2020), 1-17.

[13] Grecu, A., *A perturbed eigenvalue problem in exterior domain*, submitted.

[14] Gyulov, T., Moroşanu, G., *Eigenvalues of* $-(\Delta_p + \Delta_q)$ *under a Robin-like boundary condition*, Ann. Acad. Rom. Sci. Ser. Math. Appl., **8**(2016), 114-132.

[15] Juutinen, P., Lindqvist, P., Manfredi, J.J., *The* ∞*-eigenvalue problem*, Arch. Rational Mech. Anal., **148**(1999), 89-105.

[16] Lê, A., *Eigenvalue problems for the p-Laplacian*, Nonlinear Analysis, **64**(2006), 1057-1099.

[17] Lindgren, E., Lindqvist, P., *Fractional eigenvalues*, Calc. Var., **49**(2014), 795-826.

[18] Lindqvist, P., *A nonlinear eigenvalue problem*, In: Ciatti, Paolo (ed.) et al., "Topics in Mathematical Analysis", 175-203, Hackensack, NJ: World Scientific (ISBN 978-981-281-105-9/hbk). Series on Analysis Applications and Computation 3, 2008.

[19] Mihăilescu, M., *An eigenvalue problem possessing a continuous family of eigenvalues plus an isolated eigenvalue*, Communications on Pure and Applied Analysis, **10**(2011), 701-708.

[20] Mihăilescu, M., *The Spectrum of the Mean Curvature Operator*, accepted for publication in: Proceedings of the Royal Society of Edinburgh Section A: Mathematics. (DOI: 10.1017/prm.2020.25).

[21] Mihăilescu, M., Moroşanu, G., *Eigenvalues of* $-\Delta_p - \Delta_q$ *under Neumann boundary condition*, Canadian Mathematical Bulletin, **59**(2016), 606-616.

[22] Mihăilescu, M., Stancu-Dumitru, D., *A perturbed eigenvalue problem on general domains*, Annals of Functional Analysis, **7**(2016), 529-542.

[23] Tanaka, M., *Generalized eigenvalue problems for* (p, q)*-Laplacian with indefinite weight*, J. Math. Anal. Appl., **419**(2014), 1181-1192.

[24] Szulkin, A., *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **3**(1986), 77-109.

[25] Szulkin, A., Willem, M., *Eigenvalue problems with indefinite weights*, Studia Mathematica, **135**(1999), 191-201.

Maria Fărcăşeanu
The University of Sydney, School of Mathematics and Statistics,
NSW 2006, Australia
and
University Politehnica of Bucharest,
Research group of the project PN-III-P1-1.1-TE-2019-0456,
060042 Bucharest, Romania
e-mail: `maria.farcaseanu@sydney.edu.au`

Andrei Grecu
University of Craiova, Department of Mathematics,
200585 Craiova, Romania
and
University Politehnica of Bucharest,
Research group of the project PN-III-P1-1.1-TE-2019-0456,
060042 Bucharest, Romania
e-mail: andreigrecu.cv@gmail.com

Mihai Mihăilescu
University of Craiova, Department of Mathematics,
200585 Craiova, Romania
e-mail: mmihailes@yahoo.com

Denisa Stancu-Dumitru
University Politehnica of Bucharest,
Research group of the project PN-III-P1-1.1-TE-2019-0456,
060042 Bucharest, Romania
e-mail: denisa.stancu@yahoo.com