Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 4, 817–826
DOI: 10.24193/subbmath.2023.4.10

Oscillatory behavior of a fifth-order differential equation with unbounded neutral coefficients

John R. Graef, Hakan Avcı, Osman Özdemir and Ercan Tunç

Abstract. The authors study the oscillatory behavior of solutions to a class of fifth-order differential equations with unbounded neutral coefficients. The results are obtained by a comparison with first-order delay differential equations whose oscillatory characters are known. Two examples illustrating the results are provided, one of which is applied to Euler type equations.

Mathematics Subject Classification (2010): 34C10, 34K11, 34K40.

Keywords: Oscillation, fifth-order, neutral differential equation.

1. Introduction

In this paper, we are concerned with the oscillatory behavior of all solutions of the fifth-order neutral differential equation

$$z^{(5)}(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, and the following conditions are assumed to hold throughout:

(C1) p, $q : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $p(t) \geq 1$, $p(t) \not\equiv 1$ for all large t, $q(t) \geq 0$, and $q(t)$ is not identically zero for all large t;
(C2) τ, $\sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions such that $\tau(t) \leq t$, $\sigma(t) \leq t$, τ is strictly increasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;
(C3) $h(t) := \tau^{-1}(\sigma(t)) \leq t$ and $\lim_{t\to\infty} h(t) = \infty$, where τ^{-1} is the inverse function of τ.

Received 25 September 2020; Accepted 19 January 2021.

By a *solution* of equation (1.1), we mean a function $x \in C([t_x, \infty), \mathbb{R})$ for some $t_x \geq t_0$ such that $z \in C^5([t_x, \infty), \mathbb{R})$ and x satisfies (1.1) on $[t_x, \infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_x, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \quad \text{for any} \quad T_1 \geq t_x,$$

and moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, i.e., for any $t_1 \in [t_x, \infty)$ there exists $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1.1) is termed oscillatory if all its solutions are oscillatory.

Recently there has been a great deal of work on the oscillation of solutions of neutral differential equations. A neutral differential equation is a differential equation in which the highest order derivative of the unknown function is evaluated both at the present state t and at one or more past or future states. Besides its theoretical interest, the study of neutral equations has some importance in applications; for example, see Hale's monograph [15] for some applications in science and technology.

Among numerous papers dealing with the oscillation of the solutions of third and higher odd-order neutral differential equations, we refer the reader to the papers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 13, 16, 17, 21, 22, 23, 25, 26, 27, 28, 29, 30] and the references cited therein as examples of recent results on this topic. However, except for the papers [3, 4, 14, 30] in which third order equations are studied, the results obtained in these other papers are for the case where p is bounded, i.e., the cases $0 \leq p(t) \leq p_0 < 1$, $-1 < p_0 \leq p(t) \leq 0$, or $0 \leq p(t) \leq p_0 < \infty$. To the best of our knowledge, there appears to be no results for fifth and/or higher odd-order differential equations with unbounded neutral coefficients. The aim of the present paper is to initiate the study of the oscillatory behavior of (1.1) and to provide new results that can be applied not only to the case where $p(t) \to \infty$ as $t \to \infty$ but also to the case where $p(t)$ is a bounded function. Since the equation considered here is linear, it is possible to extend our results to more general differential equations (see Remark 2.8 below). It is our belief that the present paper will contribute significantly to the study of oscillatory behavior of solutions of fifth and higher odd-order differential equations with unbounded neutral coefficients.

In the sequel, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of (1.1), since if $x(t)$ is a solution of (1.1), then $-x(t)$ is also a solution.

2. Main results

We begin with the following auxiliary lemmas that are essential in the proofs of our main results.

Lemma 2.1 ([1, Lemma 2.2.3]). *Let* $f \in C^n([t_0, \infty), (0, \infty))$ *such that* $f^{(n)}(t) f^{(n-1)}(t) \leq 0$ *for* $t \geq t_x$ *for some* $t_x \geq t_0$, *and assume that* $\lim_{t\to\infty} f(t) \neq 0$. *Then for every*

$\lambda \in (0,1)$, *there exists a* $t_\lambda \in [t_x, \infty)$ *such that, for all* $t \in [t_\lambda, \infty)$,

$$f(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| f^{(n-1)}(t) \right| .$$

Lemma 2.2. *(Kiguradze and Chanturia* [19]*). Let the function* f *satisfy* $f^{(i)}(t) > 0$, $i = 0, 1, 2, \ldots, m$ *and* $f^{(m+1)}(t) \leq 0$ *eventually. Then, for every* $l \in (0,1)$,

$$\frac{f(t)}{f'(t)} \geq \frac{lt}{m}$$

eventually.

To prove our results we will make use of the additional hypothesis:

(C4) There exist real numbers l_1, $l_2 \in (0,1)$ such that

$$\psi_1(t) := \frac{1}{p(\tau^{-1}(t))} \left[1 - \left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)} \right)^{4/l_1} \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))} \right] \geq 0, \tag{2.1}$$

$$\psi_2(t) := \frac{1}{p(\tau^{-1}(t))} \left[1 - \left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)} \right)^{2/l_2} \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))} \right] \geq 0, \tag{2.2}$$

and

$$\psi_3(t) := \frac{1}{p(\tau^{-1}(t))} \left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))} \right) \geq 0, \tag{2.3}$$

for all sufficiently large t.

The following lemma is a consequence of a well known result of Kiguradze [18].

Lemma 2.3. *Let conditions (C1)–(C3) be satisfied and assume that* x *is an eventually positive solution of equation* (1.1)*. Then, there exists* $t_1 \in [t_0, \infty)$ *such that the corresponding function* z *satisfies one of the following three cases:*

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) > 0$, $z''''(t) > 0$, *and* $z^{(5)}(t) \leq 0$,
(II) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) < 0$, $z''''(t) > 0$, *and* $z^{(5)}(t) \leq 0$,
(III) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, $z'''(t) < 0$, $z''''(t) > 0$, *and* $z^{(5)}(t) \leq 0$,

for $t \geq t_1$.

Theorem 2.4. *Let conditions (C1)–(C4) hold and assume that there exists a function* $\eta \in C([t_0, \infty), \mathbb{R})$ *such that* $h(t) \leq \eta(t) \leq t$ *for* $t \geq t_0$*. If there exist constants* $\lambda_1, \lambda_2 \in (0,1)$ *such that the first-order delay differential equations*

$$w'(t) + \frac{\lambda_1}{24} q(t) \psi_1(\sigma(t)) h^4(t) w(h(t)) = 0, \tag{2.4}$$

$$y'(t) + \frac{\lambda_2}{24} q(t) \psi_2(\sigma(t)) h^4(t) y(h(t)) = 0, \tag{2.5}$$

and

$$\varphi'(t) + \frac{1}{24} q(t) \psi_3(\sigma(t)) (\eta(t) - h(t))^4 \varphi(\eta(t)) = 0 \tag{2.6}$$

are oscillatory, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, from Lemma 2.3, $z(t)$ satisfies one of cases (I)-(III) for $t \geq t_1$.

First, we consider case (I). From the definition of z, we have

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right] \\ &\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))}z(\tau^{-1}(\tau^{-1}(t))). \end{aligned} \tag{2.7}$$

Now $\tau(t) \leq t$ and τ is strictly increasing, so τ^{-1} is increasing and $t \leq \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t)). \tag{2.8}$$

In view of (I) and Lemma 2.2 with $m = 4$, there exists $t_2 \in [t_1, \infty)$ such that, for every $l_1 \in (0, 1)$,

$$\frac{z(t)}{z'(t)} \geq l_1 \frac{t}{4} \quad \text{for } t \geq t_2,$$

which yields

$$\left(\frac{z(t)}{t^{4/l_1}}\right)' = \frac{z'(t) - \frac{4}{l_1 t}z(t)}{t^{4/l_1}} \leq 0,$$

i.e, $z(t)/t^{4/l_1}$ is nonincreasing for $t \geq t_2$. Using the monotonicity of $z(t)/t^{4/l_1}$, it follows from (2.8) that

$$z\left(\tau^{-1}(\tau^{-1}(t))\right) \leq \frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^{4/l_1} z(\tau^{-1}(t))}{\left(\tau^{-1}(t)\right)^{4/l_1}}. \tag{2.9}$$

Using (2.9) in (2.7) yields

$$x(t) \geq \psi_1(t) z(\tau^{-1}(t)) \quad \text{for } t \geq t_2. \tag{2.10}$$

Since $\lim_{t\to\infty} \sigma(t) = \infty$, we can choose $t_3 \geq t_2$ such that $\sigma(t) \geq t_2$ for all $t \geq t_3$. Thus, from (2.10) we have

$$x(\sigma(t)) \geq \psi_1(\sigma(t)) z(\tau^{-1}(\sigma(t))) \quad \text{for } t \geq t_3. \tag{2.11}$$

Using (2.11) in (1.1) gives

$$z^{(5)}(t) + q(t)\psi_1(\sigma(t)) z(\tau^{-1}(\sigma(t))) \leq 0. \tag{2.12}$$

Now $z(t) > 0$ and $z'(t) > 0$ on $[t_3, \infty) \subseteq [t_2, \infty)$, so

$$\lim_{t\to\infty} z(t) \neq 0,$$

and hence by Lemma 2.1 with $n = 5$ and case (I), for every λ, $0 < \lambda < 1$, there exists $t_\lambda \geq t_3$ such that

$$z(t) \geq \frac{\lambda}{24} t^4 z''''(t) \quad \text{for } t \geq t_\lambda, \tag{2.13}$$

from which we see that

$$z(\tau^{-1}(\sigma(t))) \geq \frac{\lambda}{24} (\tau^{-1}(\sigma(t)))^4 z''''(\tau^{-1}(\sigma(t))) \quad \text{for } t \geq t_5, \tag{2.14}$$

where $\tau^{-1}(\sigma(t)) \geq t_\lambda$ for $t \geq t_5$ for some $t_5 \geq t_\lambda$. Using (2.14) in (2.12) yields

$$z^{(5)}(t) + \frac{\lambda}{24} q(t)\psi_1(\sigma(t))(\tau^{-1}(\sigma(t)))^4 z''''(\tau^{-1}(\sigma(t))) \leq 0, \tag{2.15}$$

for every λ with $0 < \lambda < 1$. Letting $w(t) = z''''(t)$, we see that w is a positive solution of the first-order delay differential inequality

$$w'(t) + \frac{\lambda}{24} q(t)\psi_1(\sigma(t))h^4(t)w(h(t)) \leq 0 \quad \text{for } t \geq t_5. \tag{2.16}$$

It follows from [24, Theorem 1] that the delay differential equation (2.4) corresponding to (2.16) also has a positive solution for all $\lambda_1 \in (0,1)$, but this contradicts our assumption on Eq. (2.4).

Next, we consider case (II). Since $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, and $z'''(t) < 0$, by Lemma 2.2 with $m = 2$, there exists $t_2 \in [t_1, \infty)$ such that, for every $l_2 \in (0,1)$,

$$\frac{z(t)}{z'(t)} \geq l_2 \frac{t}{2} \quad \text{for } t \geq t_2,$$

which yields

$$\left(\frac{z(t)}{t^{2/l_2}}\right)' = \frac{z'(t) - \frac{2}{l_2 t} z(t)}{t^{2/l_2}} \leq 0,$$

i.e, $z(t)/t^{2/l_2}$ is nonincreasing for $t \geq t_2$. Using the fact that $z(t)/t^{2/l_2}$ is nonincreasing, it follows from (2.8) that

$$z\left(\tau^{-1}(\tau^{-1}(t))\right) \leq \frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^{2/l_2} z(\tau^{-1}(t))}{\left(\tau^{-1}(t)\right)^{2/l_2}}. \tag{2.17}$$

Using (2.17) in (2.7) yields

$$x(t) \geq \psi_2(t) z(\tau^{-1}(t)). \tag{2.18}$$

Using (2.18) in (1.1) gives

$$z^{(5)}(t) + q(t)\psi_2(\sigma(t)) z(\tau^{-1}(\sigma(t))) \leq 0 \tag{2.19}$$

for $t \geq t_3$ for some $t_3 \geq t_2$. Now $z(t) > 0$ and $z'(t) > 0$ on $[t_3, \infty) \subseteq [t_2, \infty)$, so

$$\lim_{t\to\infty} z(t) \neq 0,$$

and hence by Lemma 2.1 with $n = 5$ and case (II), for every λ, $0 < \lambda < 1$, there exists $t_\lambda \geq t_3$ such that

$$z(t) \geq \frac{\lambda}{24} t^4 z''''(t) \quad \text{for } t \geq t_\lambda, \tag{2.20}$$

so

$$z(\tau^{-1}(\sigma(t))) \geq \frac{\lambda}{24} (\tau^{-1}(\sigma(t)))^4 z''''(\tau^{-1}(\sigma(t))) \quad \text{for } t \geq t_5, \tag{2.21}$$

where $\tau^{-1}(\sigma(t)) \geq t_\lambda$ for $t \geq t_5$ for some $t_5 \geq t_\lambda$. Using (2.21) in (2.19) gives

$$z^{(5)}(t) + \frac{\lambda}{24} q(t)\psi_2(\sigma(t))(\tau^{-1}(\sigma(t)))^4 z''''(\tau^{-1}(\sigma(t))) \leq 0, \tag{2.22}$$

for every λ with $0 < \lambda < 1$. Letting $y(t) = z''''(t)$, we see that y is a positive solution of the first-order delay differential inequality

$$y'(t) + \frac{\lambda}{24} q(t)\psi_2(\sigma(t))h^4(t)y(h(t)) \le 0 \quad \text{for } t \ge t_5. \tag{2.23}$$

As in case (I), we conclude that there exists a positive solution $y(t)$ of (2.5) for all $\lambda_2 \in (0,1)$, which contradicts the fact that equation (2.5) is oscillatory.

Finally, we consider case (III). Since $z'(t) < 0$, it follows from (2.8) that

$$z(\tau^{-1}(t)) \ge z(\tau^{-1}(\tau^{-1}(t))),$$

and so inequality (2.7) takes the form

$$x(t) \ge \psi_3(t)z(\tau^{-1}(t)). \tag{2.24}$$

Using (2.24) in (1.1) gives

$$z^{(5)}(t) + q(t)\psi_3(\sigma(t))z(h(t)) \le 0 \tag{2.25}$$

for $t \ge t_2$ for some $t_2 \ge t_1$. Since $(-1)^k z^{(k)}(t) > 0$ for $k = 0,1,2,3,4$ and $z^{(5)}(t) \le 0$, for $t_2 \le u \le v$, we can easily see that

$$z(u) \ge \frac{(v-u)^4}{24} z''''(v). \tag{2.26}$$

Letting $u = h(t)$ and $v(t) = \eta(t)$ in (2.26), we obtain

$$z(h(t)) \ge \frac{(\eta(t)-h(t))^4}{24} z''''(\eta(t)),$$

and using this in (2.25), we arrive at

$$z^{(5)}(t) + \frac{1}{24} q(t)\psi_3(\sigma(t))(\eta(t)-h(t))^4 z''''(\eta(t)) \le 0.$$

With $\varphi(t) = z''''(t)$, we see that φ is a positive solution of the first-order delay differential inequality

$$\varphi'(t) + \frac{1}{24} q(t)\psi_3(\sigma(t))(\eta(t)-h(t))^4 \varphi(\eta(t)) \le 0. \tag{2.27}$$

As before, we conclude that equation (2.6) has a positive solution, which is a contradiction. This completes the proof of the theorem. □

It is well known from [20] (see also [1, Lemma 2.2.9] that if

$$\liminf_{t\to\infty} \int_{g(t)}^{t} a(s)ds > \frac{1}{e}, \tag{2.28}$$

then the first-order delay differential equation

$$x'(t) + a(t)x(g(t)) = 0 \tag{2.29}$$

is oscillatory, where $a, g \in C([t_0,\infty),\mathbb{R})$ with $a(t) \ge 0$, $g(t) < t$, and $\lim_{t\to\infty} g(t) = \infty$. Thus, from Theorem 2.4, we have the following oscillation result for equation (1.1).

Corollary 2.5. *Let conditions (C1)–(C4) hold and assume that there exists a function $\eta \in C([t_0,\infty),\mathbb{R})$ such that $h(t) < \eta(t) < t$ for $t \geq t_0$. If*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} q(s)\psi_1(\sigma(s))h^4(s)ds > \frac{24}{e}, \tag{2.30}$$

$$\liminf_{t\to\infty} \int_{h(t)}^{t} q(s)\psi_2(\sigma(s))h^4(s)ds > \frac{24}{e}, \tag{2.31}$$

and

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} q(s)\psi_3(\sigma(s))(\eta(s)-h(s))^4ds > \frac{24}{e}, \tag{2.32}$$

then equation (1.1) *is oscillatory.*

Proof. From (2.30), one can choose a positive constant λ_1 with $0 < \lambda_1 < 1$ such that

$$\liminf_{t\to\infty} \lambda_1 \int_{h(t)}^{t} q(s)\psi_1(\sigma(s))h^4(s)ds > \frac{24}{e}. \tag{2.33}$$

Now, in view of (2.28)–(2.29), inequality (2.33) ensures that equation (2.4) is oscillatory. Again, in view of (2.28)–(2.29), inequalities (2.31) and (2.32) guarantee that equations (2.5) and (2.6) are oscillatory, respectively. So, by Theorem 2.4, the conclusion of Corollary 2.5 holds. □

We conclude this paper with the following examples and remarks to illustrate the above results. Our first example is concerned with an equation with bounded neutral coefficients in the case where p is a constant function; the second example is for an equation with unbounded neutral coefficients where $p(t) \to \infty$ as $t \to \infty$.

Example 2.6. Consider the fifth-order differential equation of Euler type

$$[x(t) + 128x(t/2)]^{(5)} + \frac{q_0}{t^5}x(t/6) = 0, \quad t \geq 1. \tag{2.34}$$

Here $p(t) = 128$, $q(t) = q_0/t^5$, $\tau(t) = t/2$, and $\sigma(t) = t/6$. Then, it is easy to see that conditions (C_1)–(C_3) hold, and

$$\tau^{-1}(t) = 2t,\ \tau^{-1}(\tau^{-1}(t)) = 4t, \quad \text{and} \quad h(t) = t/3.$$

Choosing $l_1 = l_2 = 2/3$, we see that

$$\psi_1(t) = 1/2^8, \quad \psi_2(t) = 15/2^{11} \quad \text{and} \quad \psi_3(t) = 127/2^{14},$$

i.e., condition (C4) holds. With $\eta(t) = t/2$, we have $h(t) < \eta(t) < t$ for $t \geq 1$. Then, by Corollary 2.5, Eq. (2.34) is oscillatory for

$$q_0 > \max\left\{\frac{2^{11}3^5}{e\ln 3}, \frac{2^{14}3^4}{5e\ln 3}, \frac{2^{21}3^5}{127e\ln 2}\right\} = \frac{2^{21}3^5}{127e\ln 2} \approx 2.1297 \times 10^6.$$

Example 2.7. Consider the equation

$$[x(t) + tx(t/2)]^{(5)} + \frac{q_0}{t^4}x(t/4) = 0, \quad t \geq 128. \tag{2.35}$$

Here $p(t) = t$, $q(t) = q_0/t^4$, $\tau(t) = t/2$, and $\sigma(t) = t/4$. Then, it is easy to see that conditions $(C1)$–$(C3)$ hold, and

$$\tau^{-1}(t) = 2t, \ \tau^{-1}(\tau^{-1}(t)) = 4t, \quad \text{and} \quad h(t) = t/2.$$

Choosing $l_1 = l_2 = 1/2$, we see that

$$\psi_1(t) \geq 1/4t, \quad \psi_2(t) \geq 31/64t \quad \text{and} \quad \psi_3(t) \geq 511/2^{10}t,$$

so (C4) holds. With $\eta(t) = 2t/3$, it is easy to see that all conditions of Corollary 2.5 hold, and so Eq. (2.35) is oscillatory if

$$q_0 > \max\left\{\frac{3 \cdot 2^7}{e \ln 2}, \frac{3 \cdot 2^{11}}{31e \ln 2}, \frac{3^5 \cdot 2^{15}}{511e \ln \frac{3}{2}}\right\} = \frac{2^{15} \cdot 3^5}{511e \ln \frac{3}{2}} \approx 14138.$$

Remark 2.8. The results of this paper can be extended to the fifth-order differential equation with unbounded neutral coefficients

$$\left(r(t)\left(z''''(t)\right)^{\gamma}\right)' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

under each of the conditions

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt = \infty$$

or

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt < \infty,$$

where $r \in C\left([t_0, \infty), (0, \infty)\right)$, γ and β are the ratios of odd positive integers, and the other functions in the equation are defined as in this paper.

Remark 2.9. Since it is known that $p(t) \equiv -1$ is a bifurcation point for the behavior of solutions of neutral differential equations (see [12, 13]), it would be of interest to study the oscillatory behavior of all solutions of (1.1) for $p(t) \leq -1$ with $p(t) \not\equiv -1$ for large t.

References

[1] Agarwal, R.P., Grace, S.R., O'Regan, D., *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer, Dordrecht, 2010.

[2] Baculíková, B., Džurina, J., *Oscillation of third-order neutral differential equations*, Math. Comput. Model., **52**(2010), 215-226.

[3] Chatzarakis, G.E., Džurina, J., Jadlovská, I., *Oscillatory properties of third-order neutral delay differential equations with noncanonical operators*, Mathematics, **7**(2019), no. 12, 1-12.

[4] Chatzarakis, G.E., Grace, S.R., Jadlovská, I., Li, T., Tunç, E., *Oscillation criteria for third-order Emden-Fowler differential equations with unbounded neutral coefficients*, Complexity, **2019**(2019), Article ID 5691758, 1-7.

[5] Chen, Da-X., Liu, J.-C., *Asymptotic behavior and oscilation of solutions of third-order nonlinear neutral delay dynamic equations on time scales*, Canad. Appl. Math. Quart., **16**(2008), 19-43.

[6] Das, P., *Oscillation criteria for odd order neutral equations*, J. Math. Anal. Appl., **188**(1994), 245-257.

[7] Došlá, Z., Liška, P., *Comparison theorems for third-order neutral differential equations*, Electron. J. Differ. Equ., **2016**(2016), no. 38, 1-13.

[8] Džurina, J., Grace, S.R., Jadlovská, I., *On nonexistence of Kneser solutions of third-order neutral delay differential equations*, Appl. Math. Lett., **88**(2019), 193-200.

[9] Grace, S.R., Graef, J.R., El-Beltagy, M.A., *On the oscillation of third order neutral delay dynamic equations on time scales*, Comput. Math. Appl., **63**(2012), 775-782.

[10] Grace, S.R., Graef, J.R., Tunç, E., *Oscillatory behavior of third order nonlinear differential equations with a nonlinear nonpositive neutral term*, J. Taibah Univ. Sci., **13**(2019), 704-710.

[11] Graef, J.R., Savithri, R., Thandapani, E., *Oscillatory properties of third order neutral delay differential equations*, Proceedings of the Fourth International Conference on Dynamical Systems and Differential Equations, May 24-27, 2002, Wilmington, NC, USA, 342-350.

[12] Graef, J.R., Spikes, P.W., *Some asymptotic properties of solutions of a neutral delay equation with an oscillatory coefficient*, Canad. Math. Bull., **36**(1993), 263-272.

[13] Graef, J.R., Spikes, P.W., Grammatikopoulos, M.K., *Asymptotic behavior of nonoscillatory solutions of neutral delay differential equations of arbitrary order*, Nonlinear Anal., **21**(1993), 23-42.

[14] Graef, J.R., Tunç, E., Grace, S.R., *Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation*, Opuscula Math., **37**(2017), 839-852.

[15] Hale, J.K., *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[16] Jiang, Y., Jiang, C., Li, T., *Oscillatory behavior of third-order nonlinear neutral delay differential equations*, Adv. Differ. Equ., **2016**(2016), Article ID 171, 1-12.

[17] Karpuz, B., Öcalan, Ö., Öztürk, S., *Comparison theorems on the oscillation and asymptotic behaviour of higher-order neutral differential equations*, Glasgow Math. J., **52**(2010), 107-114.

[18] Kiguradze, I.T., *On the oscillatory character of solutions of the equation* $d^m u/dt^m + a(t)|u|^n \operatorname{sign} u = 0$, Mat. Sb. (N.S.), **65**(1964), 172-187.

[19] Kiguradze, I.T., Chanturia, T.A., *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer, Dordrecht, 1993, Translated from the 1985 Russian original.

[20] Koplatadze, R.G., Chanturiya, T. A., *Oscillating and monotone solutions of first-order differential equations with deviating argument* (in Russian), Differ. Uravn., **18**(1982), 1463-1465.

[21] Li, T., Rogovchenko, Yu. V., *Asymptotic behavior of higher-order quasilinear neutral differential equations*, Abstr. Appl. Anal., **2014**(2014), Article ID 395368, 1-11.

[22] Mihalíková, B., Kostiková, E., *Boundedness and oscillation of third order neutral differential equations*, Tatra Mt. Math. Publ., **43**(2009), 137-144.

[23] Moaaz, O., Awrejcewicz, J., Muhib, A., *Establishing new criteria for oscillation of odd-order nonlinear differential equations*, Mathematics, **8**(2020), no. 5, 1-15.

[24] Philos, Ch. G., *On the existence of nonoscillatory solutions tending to zero at* ∞ *for differential equations with positive delays*, Arch. Math. (Basel), **36**(1981), 168-178.

[25] Saker, S.H., Graef, J.R., *Oscillation of third-order nonlinear neutral functional dynamic equations on time scales*, Dynam. Syst. Appl., **21**(2012), 583-606.

[26] Sun, Y., Hassan, T.S., *Comparison criteria for odd order forced nonlinear functional neutral dynamic equations*, Appl. Math. Comput., **251**(2015), 387-395.

[27] Sun, Y., Zhao, Y., *Oscillatory behavior of third-order neutral delay differential equations with distributed deviating arguments*, J. Inequal. Appl., **2019**(2019), Article ID 207, 1-16.

[28] Thandapani, E., Li, T., *On the oscillation of third-order quasi-linear neutral functional differential equations*, Arch. Math. (Brno), **47**(2011), 181-199.

[29] Thandapani, E., Padmavathy, S., Pinelas, S., *Oscillation criteria for odd-order nonlinear differential equations with advanced and delayed arguments*, Electron. J. Differ. Equ., **2014**(2014), no. 174, 1-13.

[30] Tunç, E., *Oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments*, Electron. J. Differ. Equ., **2017**(2017), no. 16, 1-12.

John R. Graef
University of Tennessee at Chattanooga,
Department of Mathematics,
Chattanooga, TN 37403, USA
e-mail: John-Graef@utc.edu

Hakan Avcı
Samsun University,
Department of Basic Sciences,
Faculty of Engineering,
Samsun, Turkey
e-mail: hakan.avci@samsun.edu.tr

Osman Özdemir
Tokat Gaziosmanpaşa University,
Department of Mathematics,
Faculty of Arts and Sciences,
60240, Tokat, Turkey
e-mail: osman.ozdemir@gop.edu.tr

Ercan Tunç
Tokat Gaziosmanpaşa University,
Department of Mathematics,
Faculty of Arts and Sciences,
60240, Tokat, Turkey
e-mail: ercantunc72@yahoo.com