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Growth properties of solutions of linear difference equations with coefficients having φ-order

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Abstract. In this paper, we investigate the relations between the growth of entire coefficients and that of solutions of complex homogeneous and non-homogeneous linear difference equations with entire coefficients of φ-order by using a slow growth scale, the φ-order, where φ is a non-decreasing unbounded function. We extend some precedent results due to Zheng and Tu (2011) [15] and others.

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1. Introduction and preliminaries

We assume that the readers are familiar with the fundamental results and standard notations of the Nevanlinna's value distribution theory of entire and meromorphic functions. In addition, let us recall some notations such as $m(r,f)$ and $N(r,f)$ (see [8, 10]). Let $n(r,f)$ be the number of poles of a function f (counting multiplicities) in $|z| \leq r$. Then we define the integrated counting function $N(r,f)$ by

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r,$$

and we define the proximity function $m(r,f)$ by

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\phi}\right)\right| d\phi,$$

where $\log^+ x = \max\{0, \log x\}$. We should think of $m(r,f)$ as a measure of how close f is to infinity on $|z| = r$. Nevertheless, within that context, we recall that $T(r,f)$ stands

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for the Nevanlinna characteristic function of the meromorphic function f defined on each positive real value r by

$$T(r,f)=m(r,f)+N(r,f).$$

And $M(r,f)$ stands for the so called maximum modulus function defined for each non-negative real value r by

$$M(r,f)=\max_{|z|=r}|f(z)|.$$

The applications of Nevanlinna's value distribution theory has been developed since 1960's. Recently, the properties of meromorphic solutions of complex linear difference equations have become a subject of great interest from the viewpoint of Nevanlinna's theory and its difference analogues. Since then, many authors investigated the linear difference equations for example, [3, 11, 12]. Moreover, we use notations $\sigma(f)$ for the order of a meromorphic function $f(z)$ and defined as

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

We denote the linear measure for a set $E\subset[0,\infty)$, by $m(E)=\int_E dt$ and logarithmic measure for a set $E\subset(1,\infty)$, by $m_l(E)=\int_E\frac{dt}{t}$. The upper density of a set $E\subset[0,\infty)$ is defined as

$$\overline{dens}\ E=\limsup_{r\to\infty}\frac{m(E\cap[0,r])}{r},$$

and the upper logarithmic density of a set $E\subset(1,\infty)$ is defined as

$$\overline{\log dens}\ (E)=\limsup_{r\to\infty}\frac{m_l(E\cap[1,r])}{\log r}.$$

Proposition 1.1. [1]*For all $H\subset[1,\infty)$ the following statements hold:*
(i) If $m_l(H)=\infty$, then $m(H)=\infty$;
(ii) If $\overline{dens}H>0$, then $m(H)=\infty$;
(iii) If $\overline{\log dens}H>0$, then $m_l(H)=\infty$.

In 2008, Chiang and Feng [3] investigated the proximity function and point wise estimates of $\frac{f(z+\eta)}{f(z)}$, which are discrete versions of the classical logarithmic derivative estimates of $f(z)$. They also applied their results to obtain growth estimates of meromorphic solutions to higher order homogeneous and non-homogeneous linear difference equations

$$A_n(z)f(z+n)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=0 \tag{1.1}$$

and

$$A_n(z)f(z+n)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=F(z), \tag{1.2}$$

where the coefficients $A_0(z),...,A_n(z)$ and $F(z)\,(\not\equiv 0)$ are entire functions and they obtained the following result.

Theorem 1.2. [3] *Let $A_0(z), ..., A_n(z)$ be entire functions such that there exists an integer l $(0 \leq l \leq n)$ such that*

$$\max_{0 \leq j \leq n} \{\sigma(A_j); j \neq l\} < \sigma(A_l),$$

then every meromorphic solution of equation (1.1) *satisfies* $\sigma(f) \geq \sigma(A_l) + 1$.

Above results occur when there exists only one dominant coefficient. In the case that there are more than one dominant coefficients, Laine and Yang [11] obtained the following result.

Theorem 1.3. [11] *Let $A_0(z), ..., A_n(z)$ be entire functions of finite order such that among those having the maximal order $\sigma = \max_{0 \leq j \leq n} \sigma(A_j)$, exactly one has its type strictly greater than the others. Then for any meromorphic solution f $(\not\equiv 0)$ of equation* (1.1)*, we have* $\sigma(f) \geq \sigma + 1$.

Recently, In 2011, Zheng and Tu [15], studied the growth of meromorphic solutions of homogeneous or non-homogeneous linear difference equations and improved the previous results due to Chiang and Feng [3] and Laine and Yang [11]. In the case there are more than one coefficients of equation (1.1) which have the maximal orders Zheng and Tu [15] obtained the following results.

Theorem 1.4. [15] *Let H be a set of complex numbers satisfying* $\overline{\log dens} \{|z| : z \in H\} > 0$ *and let $A_j(z)$ $(j = 0, 1, ..., n)$ be entire functions satisfying* $\max\{\sigma(A_j), j = 0, 1, ..., n\} \leq \alpha_1$. *If there exists an integer l $(0 \leq l \leq n)$ and a positive constant α_2 $(\alpha_2 < \alpha_1)$ such that for any given $\varepsilon\,(0 < \varepsilon < \alpha_2 - \alpha_1)$, we have*

$$|A_l(z)| \geq \exp\left\{r^{\alpha_1 - \varepsilon}\right\}$$

and

$$|A_j(z)| \leq \exp\{r^{\alpha_2}\}, \quad (j \neq l),$$

as $|z| = r \to +\infty$ for $z \in H$, then every meromorphic solution f $(\not\equiv 0)$ of equation (1.1) *satisfies* $\sigma(f) \geq \sigma(A_l) + 1$.

Recently, Chyzhykov et al. [4] introduced the definition of φ-order of $f(z)$ in a unit disc, where $\varphi : [0, 1) \to (0, \infty)$ is a non-decreasing unbounded function and $f(z)$ is a meromorphic function in the unit disc and Shen et al. [14], introduced $[p, q]-\varphi$ order of entire and meromorphic functions in the complex plane $\mathbb{C}$ where $\varphi : [0, \infty) \to (0, \infty)$ is a non-decreasing unbounded function. Since then many researchers investigated the growth oscillation of solutions of linear differential equations and linear difference equations {cf. [2, 5, 6, 13]}. Revisiting their ideas of φ-order we would like to prove some results using the concepts of slow growth scale, the φ-order in the complex plane. To investigate the growth of meromorphic solutions of equations (1.1) and (1.2) more precisely, we recall the following definitions.

Definition 1.5. ([14, 4]) Let $\varphi : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function, the φ-order of a meromorphic function f is defined as

$$\sigma(f, \varphi) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log \varphi(r)}.$$

If f is an entire function, then

$$\sigma(f,\varphi)=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log\varphi(r)}.$$

Definition 1.6. ([4]) If f be a meromorphic function satisfying $0<\sigma(f,\varphi)=\sigma<\infty$. Then φ-type of f is defined as

$$\tau(f,\varphi)=\limsup_{r\to\infty}\frac{T(r,f)}{\varphi(r)^{\sigma}}.$$

If f is an entire function, then

$$\tau(f,\varphi)=\limsup_{r\to\infty}\frac{\log M(r,f)}{\varphi(r)^{\sigma}}.$$

Remark 1.7. If $\varphi(r)=r$ in the Definitions 1.5 and 1.6, then we obtain the standard definition of the order and type of a function f respectively.

Remark 1.8. Throughout this paper, we assume that $\varphi:[0,\infty)\to(0,\infty)$ is a non-decreasing unbounded function and always satisfies the following two conditions without special instruction:

(i) $\lim\limits_{r\to+\infty}\frac{\log\log r}{\log\varphi(r)}=0$.

(ii) $\lim\limits_{r\to+\infty}\frac{\log\varphi(\alpha r)}{\log\varphi(r)}=1$ for some $\alpha>1$.

Thus, a natural problem arises that: how to express the growth of solutions of homogeneous and non-homogeneous linear difference equations (1.1) and (1.2) when the coefficients $A_j(z)$ $(j=0,1,...,n)$ and $F(z)\,(\not\equiv 0)$ be entire functions of φ-order in a slow growth scale φ-order. The main purpose of this paper is to make use of the concept of φ-order due to Chyzhykov et al. [4] to extend previous results for solutions to equations (1.1) and (1.2) in the complex plane $\mathbb{C}$.

2. Main results

The main purpose of this paper is to used the concept of φ-order in the complex plane $\mathbb{C}$ to investigate the growth of solutions of homogeneous and non-homogeneous linear difference equations (1.1) and (1.2). In this direction we obtain the following results.

The Theorem 2.1 investigate the order of meromorphic solutions of homogeneous linear difference equation (1.1) in the case when there are more than one coefficients which have the maximal orders.

Theorem 2.1. *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z|:z\in H\}>0$ and let $A_j(z)$ $(j=0,1,...,n)$ be entire functions satisfying*

$$\max\{\sigma(A_j,\varphi),j=0,1,...,n\}\le\sigma.$$

If there exists an integer l $(0 \le l \le n)$ such that for some constants α and β with $0 \le \beta < \alpha$ and ε $(0 < \varepsilon < \sigma)$ sufficiently small, we have

$$T(r, A_l) \ge \exp\left\{\alpha (\varphi(r))^{\sigma-\varepsilon}\right\} \tag{2.1}$$

and

$$T(r, A_j) \le \exp\left\{\beta (\varphi(r))^{\sigma-\varepsilon}\right\}, \quad (j \ne l), \tag{2.2}$$

as $z \to \infty$ for $z \in H$, then every meromorphic solution f $(\not\equiv 0)$ of equation (1.1) *satisfies $\sigma(f, \varphi) \ge \sigma(A_l, \varphi) + 1$.*

Remark 2.2. By the assumptions of Theorem 2.1, we obtain that $\sigma(A_l, \varphi) = \sigma$. Indeed, we have $\sigma(A_l, \varphi) \le \sigma$, suppose that $\sigma(A_l, \varphi) = \eta < \sigma$. Then by Definition 1.5 of φ-order and (2.1), we have for any given $\varepsilon\left(0 < \varepsilon < \frac{\sigma-\eta}{2}\right)$

$$\exp\left\{\alpha (\varphi(r))^{\sigma-\varepsilon}\right\} \le T(r, A_l) \le \exp\left\{(\varphi(r))^{\eta+\varepsilon}\right\}, \tag{2.3}$$

as $|z| = r \to \infty$ for $z \in H$. So by $\varepsilon\left(0 < \varepsilon < \frac{\sigma-\eta}{2}\right)$ we get a contradiction from (2.3) as $r \to \infty$. Hence $\sigma(A_l, \varphi) = \sigma$.

The following example illustrate the sharpness of Theorem 2.1.

Example 2.3. The function $f(z) = e^{z^2-3z}$ satisfies the equation

$$e^{-z} f(z+2) + e^{z} f(z+1) - 2e^{3z-2} f(z) = 0.$$

Here $A_2(z) = e^{-z}$, $A_1(z) = e^{z}$, $A_0(z) = -2e^{3z-2}$, we take $\varphi(z) = z$, then we obtain that $\sigma(A_2, \varphi) = \sigma(A_1, \varphi) = \sigma(A_0, \varphi) = 1$. Now set $H = \{z : \arg z = \pi\}$ and $l = 2$, then it is clear that $\overline{dens}\{|z| = r : z \in H\} = 1 > 0$. Moreover, $A_2(z)$, $A_1(z)$ and $A_0(z)$ satisfy the assumptions (2.1) and (2.2) of Theorem 2.1. Therefore, we get $\sigma(f, \varphi) = 2 = \sigma(A_2, \varphi) + 1$.

Secondly, we consider the growth of entire solutions of non-homogeneous linear difference equation (1.2). Note that the above result may not be applicable to the equation (1.2) to which equation (1.1) is the corresponding homogeneous equation (see the following Example 2.5). But we can obtain similar results with some additional conditions.

Theorem 2.4. *Let $A_j(z)$ $(j = 0, 1, ..., n)$ and $F(z)\,(\not\equiv 0)$ be entire functions such that there exists an integer l $(0 \le l \le n)$ satisfying*

$$b = \max\{\sigma(A_j, \varphi), \sigma(F, \varphi), j \ne l,\} < \sigma(A_l, \varphi) < \frac{1}{2}, \tag{2.4}$$

then every nontrivial entire solution f $(\not\equiv 0)$ of equation (1.2) *satisfies $\sigma(f, \varphi) \ge \sigma(A_l, \varphi) + 1$.*

Example 2.5. Take $\varphi(z) = z$ and the function $f(z) = e^{z}$ satisfies the equation

$$f(z+2) - ef(z+1) + f(z) = e^{z}$$

and

$$f(z+2) - ef(z+1) + e^{-z} f(z) = 1.$$

Though there is only one dominant coefficient such that the assumptions in Theorems 2.1 hold, we cannot get similar results in the non-homogeneous equation case.

Theorem 2.6. *Let $A_j(z)$ $(i = 0, 1, ..., n)$ and $F(z) (\not\equiv 0)$ entire functions such that there exists an integer $(0 \leq l \leq n)$ satisfying*

$$b = \max\{\sigma(A_j, \varphi), \sigma(F, \varphi), j \neq l,\} < \sigma(A_l, \varphi) < \infty.$$

Also suppose that $A_l(z) = \sum_{n=1}^{\infty} C_{\lambda_n} z^{\lambda_n}$ satisfies that the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap condition $\frac{\lambda_n}{n} \to \infty$ as $n \to \infty$, then every nontrivial entire solution f $(\not\equiv 0)$ of equation (1.2) satisfies $\sigma(f, \varphi) \geq \sigma(A_l, \varphi) + 1$.

3. Preliminary lemmas

To prove the above theorems, we need some lemmas as follows.

Lemma 3.1. [3] *Let f be a meromorphic function, η be a non-zero complex number and let $\gamma > 1$ and $\varepsilon > 0$ be given real constants. Then there exist a subset $E_1 \subset (1, +\infty)$ of finite logarithmic measure and a constant A depending only on γ and η, such that for all $|z| = r \notin E_1 \cup [0, 1]$, we have*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right),$$

where $n(t) = n(t, \infty, f) + n\left(t, \infty, \frac{1}{f}\right)$.

Lemma 3.2. [7] *Let f be a transcendental meromorphic function and let j be a non-negative integer, let a be a value in the extended complex plane and let $\alpha > 1$ be a real constant. Then there exists a constant $R > 0$ such that for all $r > R$, we have*

$$n\left(r, a, f^{(j)}\right) \leq \frac{2j+6}{\log\alpha} T(\alpha r, f).$$

Lemma 3.3. *Let f be a meromorphic function and η be a non-zero complex number and let $\varepsilon > 0$ be given real constants. Then there exists a subset $E_2 \subset (1, +\infty)$ of finite logarithmic measure, such that if f has finite φ-order σ, then for all $|z| = r \notin E_2 \cup [0, 1]$, we have*

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\}.$$

Proof. By Lemma 3.1, there exist a subset there exist a subset $E_2 \subset (1, +\infty)$ of finite logarithmic measure and a constant A depending only on γ and η, such that for all $|z| = r \notin E_2 \cup [0, 1]$, we have

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right), \tag{3.1}$$

where $n(t) = n(t, \infty, f) + n\left(t, \infty, \frac{1}{f}\right)$.

Using (3.1) and Lemma 3.2, we obtain that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r} + \frac{12}{\log\alpha}\frac{T(\alpha\gamma r, f)}{r}\log^{\gamma} r \log^{+}\left(\frac{12}{\log\alpha}T(\alpha\gamma r, f)\right)\right)$$

$$\leq B\left(\frac{T(\beta r, f)}{r} + \frac{\log^{\beta} r}{r} T(\beta r, f)\log T(\beta r, f)\right), \tag{3.2}$$

for all $|z| = r \notin [0,1] \cup E_2$ with $m_l(E_2) < +\infty$, where $B > 0$ is some constant and $\beta = \alpha\gamma > 1$.

Again, since f has finite φ-order $\sigma(f, \varphi) = \sigma < +\infty$, so given $\varepsilon\,(0 < \varepsilon < 2)$, for sufficiently large r, we have

$$T(r, f) < (\varphi(r))^{\sigma + \frac{\varepsilon}{2}}. \tag{3.3}$$

Then by substituting (3.3) into (3.2), we get that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq B\left(\frac{(\varphi(\beta r))^{\sigma+\frac{\varepsilon}{2}}}{r} + \frac{\log^{\beta} r}{r}(\varphi(\beta r))^{\sigma+\frac{\varepsilon}{2}}\log(\varphi(\beta r))^{\sigma+\frac{\varepsilon}{2}}\right)$$

$$\leq \frac{(\varphi(r))^{\sigma+\varepsilon}}{r}. \tag{3.4}$$

From (3.4), we obtain that

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\}.$$

This proves the lemma. □

Lemma 3.4. *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let f be a meromorphic function of finite φ-order σ and let $\varepsilon > 0$ be given. Then there exists a subset $E_3 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_3$, we have*

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\}.$$

Proof. We can write

$$\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| = \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right|, \quad (\eta_1 \neq \eta_2).$$

Then by using Lemma 3.3, there exists a subset $E_3 \subset (1, +\infty)$ such that for any $\varepsilon > 0$ and all $|z+\eta_2| = R \notin E_3 \cup [0,1]$, with $m_l(E_3) < \infty$, we get

$$\begin{aligned}
\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\} &\leq \exp\left\{-\frac{(\varphi(|z|+|\eta_2|))^{\sigma+\frac{\varepsilon}{2}}}{|z+\eta_2|}\right\} \\
&= \exp\left\{-\frac{(\varphi(R))^{\sigma+\frac{\varepsilon}{2}}}{R}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \\
&= \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\varphi(R))^{\sigma+\frac{\varepsilon}{2}}}{R}\right\} \\
&\leq \exp\left\{\frac{(\varphi(|z|+|\eta_2|))^{\sigma+\varepsilon}}{|z+\eta_2|}\right\} \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\},
\end{aligned}$$

where $|z| = r \notin [0,1] \cup E_3$.

This proves the lemma. □

Lemma 3.5. [5]*Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$, and let f be a meromorphic function of finite φ-order. Let σ be the φ-order of $f(z)$. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left((\varphi(r))^{\sigma-1+\varepsilon}\right).$$

Lemma 3.6. [9] *Let $f(z) = \sum_{n=1}^{\infty} C_{\lambda_n} z^{\lambda_n}$ be an entire function and the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap condition $\frac{\lambda_n}{n} \to \infty$ as $n \to \infty$. Then for any given $\varepsilon > 0$*

$$\log L(r,f) > (1-\varepsilon)\log M(r,f),$$

holds outside a set E_4 of finite logarithmic measure, where $M(r,f) = \sup_{|z|=r} |f(z)|$ and $L(r,f) = \inf_{|z|=r} |f(z)|$.

Lemma 3.7. *Let $f(z)$ be an entire function of finite φ-order satisfying $0 < \sigma(f,\varphi) < \infty$, where $\varphi(r)$ only satisfies $\lim_{r\to+\infty} \frac{\log\varphi(\alpha r)}{\log\varphi(r)} = 1$ for some $\alpha > 1$. Then for any given $\beta < \sigma(f,\varphi)$, there exists a set $E_5 \subset (1,\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_5$ we have*

$$M(r,f) > \exp\left\{(\varphi(r))^{\beta}\right\}.$$

Proof. By the Definition 1.5 of the φ-order, there exists an increasing sequence $\{r_n\}$ $(r_n \to \infty)$ satisfying $\left(1+\frac{1}{n}\right) r_n < r_{n+1}$ and

$$\sigma(f,\varphi) = \lim_{r_n\to\infty} \frac{\log\log M(r_n,f)}{\log\varphi(r_n)}.$$

Then, there exists a positive integer n_0 such that for all $n \geq n_0$ and for any given $\varepsilon > 0$, we have

$$M(r_n,f) > \exp\left\{(\varphi(r_n))^{\sigma(f,\varphi)-\varepsilon}\right\}. \tag{3.5}$$

Now we have

$$\lim_{n\to\infty} \frac{\log\varphi\left(\left(1+\frac{1}{n}\right) r\right)}{\log\varphi(r)} = 1.$$

Since $\beta < \sigma(f,\varphi)$, then we can choose sufficiently small $\varepsilon > 0$ to satisfy $0 < \varepsilon < \sigma(f,\varphi) - \beta$, so there exists a positive integer n_1 such that for all $n > n_1$, we have

$$\frac{\log\varphi\left(\left(1+\frac{1}{n}\right) r\right)}{\log\varphi(r)} > \frac{\beta}{\sigma(f,\varphi)-\varepsilon},$$

which implies that

$$(\sigma(f,\varphi)-\varepsilon)\log\varphi\left(\left(1+\frac{1}{n}\right) r\right) > \beta\log\varphi(r)$$

$$\Rightarrow \left(\varphi\left(\left(1+\frac{1}{n}\right) r\right)\right)^{(\sigma(f,\varphi)-\varepsilon)} > \varphi(r)^{\beta}. \tag{3.6}$$

Taking $n \geq n_2 = \max\{n_0, n_1\}$ and $E_5 = \bigcup\limits_{n=n_2}^{\infty} I_n$, where $I_n = \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$. Then by (3.5) and (3.6), we get for $r \in \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$ that

$$M(r,f) \geq M(r_n, f) > \exp\left\{(\varphi(r_n))^{\sigma(f,\varphi)-\varepsilon}\right\}$$

$$\geq \exp\left\{\left(\varphi\left(\left(1+\frac{1}{n}\right) r\right)\right)^{\sigma(f,\varphi)-\varepsilon}\right\} > \exp\left\{\varphi(r)^{\beta}\right\}.$$

Now we obtain that

$$m_l(E_5) = \sum_{n=n_2}^{\infty} \int_{I_n} \frac{dr}{r} = \sum_{n=n_2}^{\infty} \left(\log \frac{1}{1-r}\right)_{r_n}^{\left(1+\frac{1}{n}\right) r_n} = \sum_{n=n_2}^{\infty} \log \frac{n+1}{n} = \infty.$$

This proves the lemma. □

Lemma 3.8. *Let $f(z) = \sum_{n=1}^{\infty} C_{\lambda_n} z^{\lambda_n}$ be an entire function with $0 < \sigma(f,\varphi) < \infty$ where $\varphi(r)$ only satisfies $\lim\limits_{r\to+\infty} \frac{\log \varphi(\alpha r)}{\log \varphi(r)} = 1$ for some $\alpha > 1$. If the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap condition $\frac{\lambda_n}{n} \to \infty$ as $n \to \infty$. Then for any given $\beta < \sigma(f,\varphi)$, there exists a set $E_6 \subset (1,\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_6$ we have*

$$|f(z)| > \exp\left\{\varphi(r)^{\beta}\right\}.$$

Proof. By Lemma 3.6, for any $\varepsilon > 0$, there exists a set E_4 of finite logarithmic measure such that for all $|z| = r \notin E_4$, we have

$$\log L(r,f) > (1-\varepsilon) \log M(r,f),$$

which implies that

$$L(r,f) > [M(r,f)]^{(1-\varepsilon)}.$$

For any given $\beta < \sigma(f,\varphi)$, we can choose $\delta > 0$ such that $\beta < \delta < \sigma(f,\varphi)$ and sufficiently small ε satisfying $0 < \varepsilon < \frac{\delta-\beta}{2}$. Then by Lemma 3.7, there exists a set E_5 of infinite logarithmic measure such that for all $|z| = r \in E_5$, we have

$$|f(z)| > L(r,f) > [M(r,f)]^{(1-\varepsilon)} > \left(\exp\left\{\varphi(r)^{\beta}\right\}\right)^{(1-\varepsilon)} > \exp\left\{\varphi(r)^{\beta}\right\},$$

where $E_6 = E_5 \backslash E_4$ is a set with infinite logarithmic measure.

Thus the lemma is established. □

4. Proof of main results

Proof of Theorem 2.1. By Remark 2.2, we know that $\sigma(A_l,\varphi) = \sigma$. Let $f \not\equiv 0$ be a meromorphic solution of equation (1.1). Now let us suppose that $\sigma(f,\varphi) < \sigma(A_l,\varphi) + 1 = \sigma + 1 < \infty$. From the conditions of Theorem 2.1, there is a set H of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ such that for $z \in H$, we have (2.1) and (2.2) as $|z| = r \to \infty$. Set $H_1 = \{|z| = r : z \in H\}$, since $\overline{\log dens}\{|z| : z \in H\} > 0$, then by Proposition 1.1, H_1 is a set with $\int_{H_1} \frac{dr}{r} = \infty$.

We divide equation (1.1) by $f(z+l)$ to get

$$-A_l(z)=\sum_{\substack{j=0\\ i\neq l}}^{n} A_j(z)\frac{f(z+j)}{f(z+l)}. \tag{4.1}$$

Since $A_j(z)$ $(j=0,1,...,n)$ are entire functions, then by equation (4.1), we get that

$$\begin{aligned} m(r,A_l)=T(r,A_l) &\leq \sum_{\substack{j=0\\ i\neq l}}^{n} m(r,A_j)+\sum_{\substack{j=0\\ i\neq l}}^{n} m\left(r,\frac{f(z+j)}{f(z+l)}\right)+O(1)\\ &=\sum_{\substack{j=0\\ i\neq l}}^{n} T(r,A_j)+\sum_{\substack{j=0\\ i\neq l}}^{n} m\left(r,\frac{f(z+j)}{f(z+l)}\right)+O(1). \end{aligned} \tag{4.2}$$

Now by Lemma 3.5, for any ε $\left(0<\varepsilon<\frac{\sigma+1-\sigma(f,\varphi)}{2}\right)$, we have

$$m\left(r,\frac{f(z+j)}{f(z+l)}\right)=O\left((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\right). \tag{4.3}$$

Substituting (2.1), (2.2) and (4.3) into (4.2), we get for $|z|=r\to\infty,\ z\in H$ that

$$\begin{aligned} &\exp\left\{\alpha(\varphi(r))^{\sigma-\varepsilon}\right\}\leq n\exp\left\{\beta(\varphi(r))^{\sigma-\varepsilon}\right\}+O\left((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\right)\\ \Rightarrow\ &\exp\left\{(\varphi(r))^{\sigma-\varepsilon}\right\}\{\exp(\alpha)-\exp(\beta)\}\leq O(1)(\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}. \end{aligned}$$

Since, $(\exp(\alpha)-\exp(\beta))>0$, so it follows that

$$1\leq O(1)(\varphi(r))^{\sigma(f,\varphi)-1+2\varepsilon-\sigma}\to 0 \text{ as } r\to\infty, \tag{4.4}$$

which is a contradiction since $0<\varepsilon<\frac{\sigma+1-\sigma(f,\varphi)}{2}$.
Hence, we get $\sigma(f,\varphi)\geq\sigma(A_l,\varphi)+1$.
This completes the proof of the theorem. □

Proof of Theorem 2.2. If $\sigma(f,\varphi)=\infty$, then the result is trivial. Now let us suppose that $\sigma(f,\varphi)<\sigma(A_l,\varphi)+1<\infty$. We divide equation (1.2) by $f(z+l)$ to get

$$-A_l(z)=\sum_{\substack{j=0\\ i\neq l}}^{n} A_j(z)\frac{f(z+j)}{f(z+l)}-\frac{F(z)}{f(z)}\cdot\frac{f(z)}{f(z+l)},$$

which implies that

$$|A_l(z)|\leq\sum_{\substack{j=0\\ i\neq l}}^{n}|A_j(z)|\left|\frac{f(z+j)}{f(z+l)}\right|+\left|\frac{F(z)}{f(z)}\right|\cdot\left|\frac{f(z)}{f(z+l)}\right|. \tag{4.5}$$

By Lemma 3.4, for any given ε $\left(0<\varepsilon<\frac{\sigma(A_l,\varphi)+1-\sigma(f,\varphi)}{2}\right)$, there exists a subset $E_3\subset(1,\infty)$ of finite logarithmic measure such that for all $r\notin[0,1]\cup E_3$, we have

$$\left|\frac{f(z+j)}{f(z+l)}\right|\leq\exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\varepsilon}}{r}\right\},\quad (j=0,1,...,n,j\neq l). \tag{4.6}$$

Now by the assumption (2.4), we have that for sufficiently large r,

$$|A_j(z)| \leq \exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\}, \quad (j = 0, 1, ..., n, j \neq l), \tag{4.7}$$

and

$$|F(z)| \leq \exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\}.$$

Since $M(r, f) > 1$ for sufficiently large r, we have that

$$\frac{|F(z)|}{M(r,f)} \leq |F(z)| \leq \exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\}. \tag{4.8}$$

Now by the Definition 1.5 of φ-order and for above $\varepsilon > 0$, we get that

$$|A_l(z)| \geq \exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)-\varepsilon}\right\}. \tag{4.9}$$

Substituting (4.6)-(4.9) into (4.5) for all $r \notin [0,1] \cup E_3$ and $|f(z)| = M(r, f)$, we have

$$\exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)-\varepsilon}\right\} \leq |A_l(z)|$$

$$\leq (n+1)\exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\} \cdot \exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\varepsilon}}{r}\right\}. \tag{4.10}$$

Since $\varepsilon \left(0 < \varepsilon < \frac{\sigma(A_l,\varphi)+1-\sigma(f,\varphi)}{2}\right)$, so we obtain a contradiction from (4.10) by applying the same procedure we applied in (4.4). Hence we get that $\sigma(f,\varphi) \geq \sigma(A_l,\varphi)+1$.

This proves the theorem. □

Proof of Theorem 2.3. If $\sigma(f,\varphi) = \infty$, then the result is trivial. Now let us suppose that $\sigma(f,\varphi) < \sigma(A_l,\varphi) + 1 < \infty$. Now by Lemma 3.8, there exists a set $E_6 \subset (1,\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_6$ we have

$$|A_l(z)| > \exp\left\{(\varphi(r))^{\beta}\right\}. \tag{4.11}$$

Substituting (4.6)-(4.8) and (4.11) into (4.5) for all $r \in E_6 \setminus [0,1] \cup E_3$ and $|f(z)| = M(r, f)$, we have

$$\exp\left\{(\varphi(r))^{\beta}\right\} \leq |A_l(z)| \leq (n+1)\exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\} \cdot \exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\varepsilon}}{r}\right\}. \tag{4.12}$$

We we get a contradiction from (4.12) by applying the same procedure we applied in (4.4). Hence we get that $\sigma(f,\varphi) \geq \sigma(A_l,\varphi) + 1$.

This proves the theorem. □

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