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Some classes of Janowski functions associated with conic domain and a shell-like curve involving Ruscheweyh derivative

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Abstract. Making use of Ruscheweyh derivative, we define a new class of starlike functions of complex order subordinate to a conic domain impacted by Janowski functions. Coefficient estimates and Fekete-Szegö inequalities for the defined class are our main results. Some of our results generalize the related work of some authors.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f analytic in the open unit disk

$$\mathcal{U} = \{\ z \in \mathbb{C} : |z| < 1\}$$

and satisfying the normalization condition

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Thus, the functions in $\mathcal{A}$ are represented by the Taylor-Maclaurin series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U}). \tag{1.1}$$

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Let $\mathcal{S} \subset \mathcal{A}$ be the class of functions which are univalent. We let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ to denote the well known classes of starlike, convex and close-to-convex (normalized) function respectively. For $0 \le \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ symbolize the classes of starlike functions of order α and convex functions of order α respectively. Also let $\mathcal{P}$ denote the class of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ that are analytic in $\mathcal{U}$ and such that $Re\,(p(z)) > 0$ for all z in $\mathcal{U}$.

For arbitrary fixed numbers A, B, $-1 < A \le 1$, $-1 \le B < A$, we denote by $\mathcal{P}(A, B)$ the family of functions $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ analytic in the unit disc and $p(z) \in \mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)},$$

where $w(z)$ is the Schwartz function. Geometrically, $p(z) \in \mathcal{P}(A, B)$ if and only if $p(0) = 1$ and $p(U)$ lies inside an open disc centered with center $\frac{1-AB}{1-B^2}$ on the real axis having radius $\frac{A-B}{1-B^2}$ with diameter end points $p_1(-1) = \frac{1-A}{1-B}$ and $p_1(1) = \frac{1+A}{1+B}$. On observing that $w(z) = \frac{p(z)-1}{p(z)+1}$ for $p(z) \in \mathcal{P}$, we have $P(z) \in \mathcal{P}(A, B)$ if and only if for some $p(z) \in \mathcal{P}$

$$P(z) = \frac{(1 + A)p(z) + 1 - A}{(1 + B)p(z) + 1 - B}. \tag{1.2}$$

For detailed study on the class of Janowski functions, we refer [3].

The function $p_{k,\alpha}(z)$ plays the role of an extremal functions those related to these conic domain $\mathcal{D}_k = \left\{u + iv : \ u > k\sqrt{(u-1)^2 + v^2}\right\}$ and is given by

$$\hat{p}_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z}, & \text{if } k = 0,\\ 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \text{if } k = 1,\\ 1 + \frac{2(1-\alpha)}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right) arc\tanh\sqrt{z}\right], & \text{if } 0 < k < 1,\\ 1 + \frac{2(1-\alpha)}{1-k^2}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{t}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{k^2-1}, & \text{if } k > 1, \end{cases} \tag{1.3}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0, 1)$ and t is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, with $R(t)$ is Legendres complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. Clearly, $\hat{p}_{k,\alpha}(z)$ is in $\mathcal{P}$ with the expansion of the form

$$\hat{p}_{k,\alpha}(z) = 1 + \delta_1 z + \delta_2 z^2 + \cdots, \qquad (\delta_j = p_j(k, \alpha),\ j = 1, 2, 3, \ldots), \tag{1.4}$$

we get

$$\delta_1 = \begin{cases} \frac{8(1-\alpha)(\arccos k)^2}{\pi^2(1-k^2)}, & \text{if } 0 \le k < 1,\\ \frac{8(1-\alpha)}{\pi^2}, & \text{if } k = 1,\\ \frac{\pi^2(1-\alpha)}{4\sqrt{t}(k^2-1)R^2(t)(1+t)}, & \text{if } k > 1. \end{cases} \tag{1.5}$$

Noor in [8, 9] replaced $p(z)$ in (1.2) with $\hat{p}_{k,\alpha}(z)$ and studied the impact of Janowski function on conic regions.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an Schwartz function $w(z)$ in $\mathcal{U}$ such that

$|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Using the concept of subordination for holomorphic functions, Ma and Minda [6] introduced the classes

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf^{'}(z)}{f(z)} \prec \phi \right\} \quad \text{and} \quad \mathcal{C}(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi \right\}$$

where $\phi \in \mathcal{P}$ with $\phi^{'}(0) > 0$ maps $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis. By choosing ϕ to map unit disc on to some specific regions like parabolas, cardioid, lemniscate of Bernoulli, booth lemniscate in the right-half plane of the complex plane, various interesting subclasses of starlike and convex functions can be obtained. Raina and Sokół [10] studied the class $\mathcal{S}^*(\phi)$ for $\phi(z) = z + \sqrt{1+z^2}$ and found some interesting coefficient inequalities. The function $\phi(z) = z + \sqrt{1+z^2}$ maps the unit disc $\mathcal{U}$ onto a shell shaped region on the right half plane and it is analytic and univalent on $\mathcal{U}$. For detailed study of starlike functions related to shell shaped region, refer to a recent work of Murugusundaramoorthy and Bulboacă [7]. Khatter et al. [5] studied the convex combination of constant function $f(z) = 1$ with e^z and $\sqrt{1+z}$. Recently, Gandhi in [2] studied a class $\mathcal{S}^*(\phi)$ with $\phi = \beta e^z + (1-\beta)(1+z)$, $0 \le \delta \le 1$ a convex combination of two starlike functions.

Definition 1.1. [12] For $f \in \mathcal{A}$ of the form (1.1) and $\lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the operator R^λ is defined by $R^\lambda : \mathcal{A} \to \mathcal{A}$,

$$\begin{aligned} R^0 f(z) &= f(z), \\ R^1 f(z) &= zf'(z), \\ &\vdots \\ (\lambda+1) R^{\lambda+1} f(z) &= z\left(R^\lambda f(z)\right)' + \lambda R^\lambda f(z), \quad z \in \mathcal{U}. \end{aligned}$$

Remark 1.2. If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, then for $\lambda > -1$

$$R^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) = z + \sum_{n=2}^{\infty} \varphi_n(\lambda) a_n z^n,$$

where

$$\varphi_n(\lambda) = \frac{[\lambda+1]_{n-1}}{(n-1)!}, \tag{1.6}$$

$$[t]_n = \begin{cases} 1, & n = 0, \\ (t)(t+1)(t+2)\ldots(t+n-1), & n \in \mathbb{N}. \end{cases}$$

is a Pochhammer symbol,

$$\Gamma(t+1) = \begin{cases} 1, & t = 1, \\ [t]\Gamma(t), & t > 0. \end{cases}$$

is a gamma function. The symbol “ $*$ ” stands for Hadamard product.

Motivated by Gandhi [2], we introduce the following new subclasses of analytic functions using Ruscheweyh differential operator.

Definition 1.3. For $\hat{p}_{k,\alpha}(z)$, $(k \geq 0,\ 0 \leq \alpha < 1)$ is defined as in (1.3), $-1 \leq B < A \leq 1$, $\lambda > -1$, $\mid t \mid \leq 1, t \neq 1$ and for some $b \in \mathbb{C}\backslash\{0\}$, we let $k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$ to be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$1 + \frac{1}{b}\left(\frac{(1-t)R^{\lambda+1}f(z)}{R^{\lambda}f(z) - R^{\lambda}f(tz)} - 1\right) \prec \frac{(A+1)h(z) - (A-1)}{(B+1)h(z) - (B-1)}, \quad (z \in \mathcal{U}) \tag{1.7}$$

where

$$h(z) = \beta\left[\hat{p}_{k,\alpha}(z)\right] + (1-\beta)\left[z + \sqrt{1+z^2}\right],\ 0 \leq \beta \leq 1. \tag{1.8}$$

Remark 1.4. Note that $\hat{p}_{k,\alpha}(z)$ is not univalent but belongs to $\mathcal{P}$, whereas $z + \sqrt{1+z^2}$ is univalent in $\mathcal{U}$. Since the linear combination of two convex function is not convex in $|z| < 1$, $h(z)$ is not convex univalent in $\mathcal{U}$.

The following definition is motivated by the Alexander transform relationship between convex and starlike functions.

Definition 1.5. For $\hat{p}_{k,\alpha}(z)$, $(k \geq 0,\ 0 \leq \alpha < 1)$ is defined as in (1.3), $-1 \leq B < A \leq 1$, $\lambda > -1$, $\mid t \mid \leq 1, t \neq 1$ and for some $b \in \mathbb{C}\backslash\{0\}$, we let $k - \mathcal{CL}(A, B, \alpha, \beta, \lambda, t, b)$ to be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$1 + \frac{1}{b}\left(\frac{(1-t)\left(R^{\lambda+1}f(z)\right)'}{\left(R^{\lambda}f(z) - R^{\lambda}f(tz)\right)'} - 1\right) \prec \frac{(A+1)h(z) - (A-1)}{(B+1)h(z) - (B-1)}, \quad (z \in \mathcal{U}) \tag{1.9}$$

where $h(z)$ is defined as in (1.8).

We let $k - \mathcal{CL}(A, B, \lambda, t, b)$ and $k - \mathcal{CL}(A, B, \alpha, 1, \lambda, t, b)$ to denote the special cases of the function class $k - \mathcal{CL}(A, B, \alpha, \beta, \lambda, t, b)$ obtained by letting $\beta = 0$ and $\beta = 1$ respectively.

Remark 1.6. The versatility of classes $k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$ and $k - \mathcal{CL}(A, B, \alpha, \beta, \lambda, t, b)$ is that it unifies the study of starlike and convex functions with respect to symmetric points. Here we list just a few special cases.

1. If we let $b = 1$, $t = 0$, $\alpha = 0$, $\beta = 1$ and $\lambda = 0$ in the definition of the function class $k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$ and $k - \mathcal{CL}(A, B, \alpha, \beta, \lambda, t, b)$, we get the classes $k - \mathcal{SL}(A, B)$ and $k - \mathcal{CL}(A, B)$ introduced and studied by Noor and Malik in [9].
2. For $b = 1$, $\beta = 1$ and $\lambda = 0$, the class $k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$ reduces to the respective classes $k - \mathcal{SL}(A, B, \alpha, 1, 0, t, 1)$ studied by Arif et al. in [1].

Unless otherwise mentioned, we assume throughout this paper that the function $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$, $\lambda > -1$, $k \geq 0$, $-1 \leq B < A \leq 1$, $|t| \leq 1$, $t \neq 1$, $b \in \mathbb{C}\backslash\{0\}$ and $z \in \mathcal{U}$.

2. Fekete-Szegö inequalities for the starlike class $k-\mathcal{SL}(A,B,\alpha,\beta,\lambda,t,b)$

Many extremal problems within the class of univalent functions are solved by the Koebe function. On the other hand, the Koebe function satisfies

$$\left|a_3-\lambda a_2^2\right|=|3-4\lambda|$$

whereas Fekete and Szegö showed

$$\max_{f\in\mathcal{S}}\left|a_3-\lambda a_2^2\right|=|3-4\lambda|=1+2e^{-2\lambda/(1-\lambda)}$$

for $\lambda\in[0,1]$. In this section, we obtain the Fekete-Szegö for the class $k-\mathcal{SL}(A,B,\alpha,\beta,\lambda,t,b)$. We need the following lemma to establish our main result.

Lemma 2.1. [6] *Let $p(z)\in\mathcal{P}$ and also let v be a complex number, then*

$$|c_2-vc_1^2|\leq 2\ \max\left\{1,|2v-1|\right\}, \tag{2.1}$$

the result is sharp for functions given by

$$p(z)=\frac{1+z^2}{1-z^2},\qquad\qquad p(z)=\frac{1+z}{1-z}.$$

Theorem 2.2. *If $f(z)\in k-\mathcal{SL}(A,B,\alpha,\beta,\lambda,t,b)$ then for $\mu\in\mathbb{C}$ we have*

$$\mid a_3-\mu a_2^2\mid\leq\frac{\mid b\mid|\beta(\delta_1-1)+1|(A-B)}{2[\varphi_3(\lambda+1)-u_3\varphi_3(\lambda)]}\ \max\left\{1,|2v-1|\right\}, \tag{2.2}$$

where

$$\begin{aligned} v=&\frac{1}{2}-\frac{\beta(2\delta_2-1)+1}{4\left[\beta(\delta_1-1)+1\right]}+\frac{\left[\beta(\delta_1-1)+1\right](B+1)}{4}\\ &-\frac{b\varphi_2(\lambda)[\beta(\delta_1-1)+1](A-B)}{4[\varphi_2(\lambda+1)-u_2\varphi_2(\lambda)]}\left(u_2-\mu\frac{\varphi_3(\lambda+1)-u_3\varphi_3(\lambda)}{\varphi_2(\lambda)[\varphi_2(\lambda+1)-u_2\varphi_2(\lambda)]}\right)\end{aligned} \tag{2.3}$$

and $u_n=1+t+t^2+\cdots+t^{n-1}$. The result is sharp.

Proof. Let $p(z)\in\mathcal{P}$ be of the form $1+\sum\limits_{n=1}^{\infty}p_nz^n$, we consider

$$p(z)=\frac{1+w(z)}{1-w(z)},$$

where $w(z)$ is such that $w(0)=0$ and $\mid w(z)\mid<1$. On simple computation, we have

$$\begin{aligned} w(z)&=\frac{p(z)-1}{p(z)+1}=\frac{p_1z+p_2z^2+p_3z^3+\cdots}{2+p_1z+p_2z^2+p_3z^3+\cdots}\\ &=\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2\right)z^2+\frac{1}{2}\left(p_3-p_1p_2+\frac{1}{4}p_1^3\right)z^3+\cdots.\end{aligned} \tag{2.4}$$

Using (2.4) in $h(z) = 1 + [\beta(\delta_1 - 1) + 1]\, z + \frac{1}{2}\,[\beta(2\delta_2 - 1) + 1]\, z^2 + \cdots$, we have

$$\begin{aligned} h(w(z)) &= 1 + [\beta(\delta_1 - 1) + 1]\, w(z) + \frac{1}{2}\,[\beta(2\delta_2 - 1) + 1]\,[w(z)]^2 + \cdots \\ &= 1 + [\beta(\delta_1 - 1) + 1]\left[\frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right) z^2 + \frac{1}{2}\left(p_3 - p_1p_2 + \frac{1}{4}p_1^3\right) z^3 + \cdots\right] \\ &\quad + \frac{1}{2}\,[\beta(2\delta_2 - 1) + 1]\left[\frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right) z^2 + \frac{1}{2}\left(p_3 - p_1p_2 + \frac{1}{4}p_1^3\right) z^3 + \cdots\right]^2 + \cdots \\ &= 1 + \frac{[\beta(\delta_1 - 1) + 1]\, p_1}{2} z + \frac{[\beta(\delta_1 - 1) + 1]}{2}\left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\beta(2\delta_2 - 1) + 1}{2\,[\beta(\delta_1 - 1) + 1]}\right)\right] z^2 + \cdots . \end{aligned}$$

As $f(z) \in k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$, by (1.7) we have

$$1 + \frac{1}{b}\left(\frac{(1-t)R^{\lambda+1}f(z)}{R^\lambda f(z) - R^\lambda f(tz)} - 1\right) = p(z), \tag{2.5}$$

where

$$\begin{aligned} p(z) &= \frac{(A+1)h(w(z)) - (A-1)}{(B+1)h(w(z)) - (B-1)} \\ &= \frac{2 + \frac{(A+1)[\beta(\delta_1-1)+1]p_1}{2} z + \frac{(A+1)[\beta(\delta_1-1)+1]}{2}\left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\beta(2\delta_2-1)+1}{2[\beta(\delta_1-1)+1]}\right)\right] z^2 + \cdots}{2 + \frac{(B+1)[\beta(\delta_1-1)+1]p_1}{2} z + \frac{(B+1)[\beta(\delta_1-1)+1]}{2}\left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\beta(2\delta_2-1)+1}{2[\beta(\delta_1-1)+1]}\right)\right] z^2 + \cdots} \\ &= 1 + \frac{[\beta(\delta_1 - 1) + 1]\,(A - B)p_1}{4} z + \frac{[\beta(\delta_1 - 1) + 1]\,(A - B)}{4} \\ &\qquad \left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\beta(2\delta_2 - 1) + 1}{2\,[\beta(\delta_1 - 1) + 1]} + \frac{[\beta(\delta_1 - 1) + 1]\,(B + 1)}{2}\right)\right] z^2 + \cdots . \end{aligned} \tag{2.6}$$

From (2.5), we obtain

$$\begin{aligned} 1 + \frac{1}{b}\left(\frac{(1-t)R^{\lambda+1}f(z)}{R^\lambda f(z) - R^\lambda f(tz)} - 1\right) &= 1 + \frac{1}{b}\Bigg([\varphi_2(\lambda+1) - u_2\varphi_2(\lambda)]a_2 z \\ &\quad + \Big[[\varphi_3(\lambda+1) - u_3\varphi_3(\lambda)]a_3 - [\varphi_2(\lambda+1) - u_2\varphi_2(\lambda)]u_2\varphi_2(\lambda)a_2^2\Big] z^2 + \cdots\Bigg). \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), the coefficients of z and z^2 are given by

$$a_2 = \frac{b\,[\beta(\delta_1 - 1) + 1]\,(A - B)p_1}{4[\varphi_2(\lambda+1) - u_2\varphi_2(\lambda)]}$$

and

$$\begin{aligned} a_3 = \frac{b\,[\beta(\delta_1 - 1) + 1]\,(A - B)}{4[\varphi_3(\lambda+1) - u_3\varphi_3(\lambda)]}\Bigg[p_2 - \frac{p_1^2}{2}\Bigg(1 - \frac{\beta(2\delta_2 - 1) + 1}{2\,[\beta(\delta_1 - 1) + 1]} + \frac{[\beta(\delta_1 - 1) + 1]\,(B + 1)}{2} \\ - \frac{bu_2\varphi_2(\lambda)\,[\beta(\delta_1 - 1) + 1]\,(A - B)}{2[\varphi_2(\lambda+1) - u_2\varphi_2(\lambda)]}\Bigg)\Bigg]. \end{aligned}$$

Therefore, we have

$$| a_3 - \mu a_2^2 | \leq \frac{| b | |\beta(\delta_1 - 1) + 1|(A - B)}{2[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]} | p_2 - vp_1^2 |, \tag{2.8}$$

where

$$\begin{aligned} v = {} & \frac{1}{2} - \frac{\beta(2\delta_2 - 1) + 1}{4\,[\beta(\delta_1 - 1) + 1]} + \frac{[\beta(\delta_1 - 1) + 1]\,(B + 1)}{4} \\ & - \frac{b\varphi_2(\lambda)[\beta(\delta_1 - 1) + 1](A - B)}{4[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \left(u_2 - \mu \frac{\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)}{\varphi_2(\lambda)[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \right). \end{aligned} \tag{2.9}$$

Taking the modules for both sides of the above relation, with the aid of the inequality (2.1) of Lemma 2.1, we easily get the required estimate. The result is sharp for the functions

$$1 + \frac{1}{b} \left(\frac{(1 - t)R^{\lambda+1}f(z)}{R^\lambda f(z) - R^\lambda f(tz)} - 1 \right) = p(z)$$

and

$$1 + \frac{1}{b} \left(\frac{(1 - t)R^{\lambda+1}f(z)}{R^\lambda f(z) - R^\lambda f(tz)} - 1 \right) = p(z^2).$$

where $p(z)$ is given by the equation (2.6). Hence the proof of the Theorem 2.2 is complete. $\square$

If $\beta = 0$ in the Theorem 2.2, we get the following corollary.

Corollary 2.3. *If $f(z) \in k - \mathcal{CL}(A, B, \lambda, t, b)$ then for $\mu \in \mathbb{C}$ we have*

$$| a_3 - \mu a_2^2 | \leq \frac{| b | \,(A - B)}{12[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]} \max\{1, |2v - 1|\}, \tag{2.10}$$

where

$$v = \frac{B + 1}{4} - \frac{b\varphi_2(\lambda)(A - B)}{4[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \left(u_2 - \mu \frac{3[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]}{4\varphi_2(\lambda)[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \right) \tag{2.11}$$

and $u_n = 1 + t + t^2 + \cdots + t^{n-1}$. The result is sharp.

If $\beta = 1$ in the Theorem 2.2, we get the following corollary.

Corollary 2.4. *If $f(z) \in k - \mathcal{CL}(A, B, \alpha, 1, \lambda, t, b)$ then for $\mu \in \mathbb{C}$ we have*

$$| a_3 - \mu a_2^2 | \leq \frac{| b | \,|\delta_1|(A - B)}{12[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]} \max\{1, |2v - 1|\}, \tag{2.12}$$

where

$$\begin{aligned} v = {} & \frac{1}{2} - \frac{\delta_2}{2\delta_1} + \frac{\delta_1(B + 1)}{4} \\ & - \frac{b\varphi_2(\lambda)\delta_1(A - B)}{4[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \left(u_2 - \mu \frac{3[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]}{4\varphi_2(\lambda)[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \right), \end{aligned} \tag{2.13}$$

δ_1 is defined as in (1.5) and $u_n = 1 + t + t^2 + \cdots + t^{n-1}$. The result is sharp.

If $t = -1$ in the Theorem 2.2, we get the following corollary.

Corollary 2.5. *If $f \in k-\mathcal{SL}(A,B,\alpha,\beta,\lambda,-1,b)$ then for $\mu \in \mathbb{C}$ we have*

$$| a_3 - \mu a_2^2 | \leq \frac{| b | \, |\beta(\delta_1 - 1) + 1|(A-B)}{2[\varphi_3(\lambda+1) - \varphi_3(\lambda)]} \max\{1, |2v-1|\}, \tag{2.14}$$

where

$$\begin{aligned} v = \frac{1}{2} - \frac{\beta(2\delta_2 - 1) + 1}{4\,[\beta(\delta_1 - 1) + 1]} + \frac{[\beta(\delta_1 - 1) + 1]\,(B+1)}{4} \\ + \frac{\mu b[\varphi_3(\lambda+1) - \varphi_3(\lambda)]}{[\varphi_2(\lambda+1)]^2} \frac{[\beta(\delta_1 - 1) + 1]\,(A-B)}{4}. \end{aligned} \tag{2.15}$$

The result is sharp.

If $t=-1$, $A=1$, $B=-1$, $\alpha=0$, $\beta=1$, $\lambda=0$ and $b=1$ in the Theorem 2.2, we get the following corollary of [4].

Corollary 2.6. *If $f(z) \in \mathcal{M}_s(p_k)$ then we have*

$$a_2 = \frac{\delta_1 p_1}{4}, \quad a_3 = \frac{\delta_1}{4}\left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\delta_2}{\delta_1}\right)\right]$$

and for any complex number μ,

$$| a_3 - \mu a_2^2 | \leq \frac{\delta_1}{2} \max\left\{1, \left|\frac{\delta_2}{\delta_1} - \frac{\mu\delta_1}{2}\right|\right\}.$$

If $t=0$, $A=1$, $B=-1$, $\alpha=0$, $\beta=0$, $\lambda=0$ and $b=1$ in the Theorem 2.2, we get the following corollary.

Corollary 2.7. [10] *If $f(z) \in \mathcal{SL}_q$ then $|a_2| \leq 1$, $|a_3| \leq \frac{3}{4}$ and*

$$|a_3 - \mu a_2^2| \leq \max\left\{\frac{1}{2}, \left|\mu - \frac{3}{4}\right|\right\}.$$

3. Coefficient estimates for the convex classes $k-\mathcal{CL}(A,B,\lambda,t,b)$ and $k-\mathcal{CL}(A,B,\alpha,1,\lambda,t,b)$

To find the coefficient estimates, we need the following lemmas.

Lemma 3.1. [11] *Let $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$ be an analytic and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n$ is an analytic and convex in $\mathcal{U}$. If $f(z) \prec g(z)$, then $|a_n| \leq |b_1|$, for $n = 1, 2, \ldots$.*

Remark 3.2. Since Lemma 3.1 can be applied only if $g(z)$ is convex in $\mathcal{U}$. But the right hand side in (1.7) namely $\frac{(A+1)h(z)-(A-1)}{(B+1)h(z)-(B-1)}$ (where $h(z)$ is given as in (1.8)) is not convex in $\mathcal{U}$. So we find the coefficient inequalities for the fixed values of $\beta = 0$ and $\beta = 1$.

The following result was obtained by Noor and Malik in [9].

Lemma 3.3. [9] *Let the function $\hat{p}_{k,\alpha}(z)$ be defined as in (1.4) and let $p(z) \in \mathcal{P}$ satisfy the condition*

$$p(z) \prec \frac{(A+1)\hat{p}_{k,\alpha}(z) - (A-1)}{(B+1)\hat{p}_{k,\alpha}(z) - (B-1)}. \tag{3.1}$$

Then

$$| p_n | \leq \frac{|\delta_1|(A-B)}{2}, \quad (n \geq 1). \tag{3.2}$$

Remark 3.4. Similar result fails if $\frac{(A+1)k(z)-(A-1)}{(B+1)k(z)-(B-1)}$, as $k(z) = z + \sqrt{1+z^2}$ is starlike but not convex.

Theorem 3.5. *Let $k - \mathcal{CL}(A, B, \alpha, 1, \lambda, t, b)$, then for $n \geq 2$*

$$| a_n | \leq \prod_{j=1}^{n-1} \frac{\left| b\, j\, u_j \varphi_j(\lambda)\delta_1(A-B) - 2j\left[\varphi_j(\lambda+1) - u_j\varphi_j(\lambda)\right] B \right|}{2(j+1)[\varphi_{j+1}(\lambda+1) - u_{j+1}\varphi_{j+1}(\lambda)]}, \tag{3.3}$$

where δ_1 is defined as in (1.5) and $u_n = 1 + t + t^2 + \cdots + t^{n-1}$.

Proof. By the definition of $k - \mathcal{CL}(A, B, \alpha, \lambda, t, b)$, we have

$$1 + \frac{1}{b}\left(\frac{(1-t)\left(R^{\lambda+1}f(z)\right)'}{\left(R^{\lambda}f(z) - R^{\lambda}f(tz)\right)'} - 1 \right) = p(z), \tag{3.4}$$

where $p(z) \in \mathcal{P}$ and satisfies the subordination condition

$$p(z) \prec \frac{(A+1)\hat{p}_{k,\alpha}(z) - (A-1)}{(B+1)\hat{p}_{k,\alpha}(z) - (B-1)}.$$

Equivalently (3.4) can be rewritten as

$$\begin{aligned} 1 + \frac{1}{b}&\left(\frac{\sum\limits_{n=2}^{\infty} n\left[\varphi_n(\lambda+1) - u_n\varphi_n(\lambda)\right] a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} n u_n \varphi_n(\lambda) a_n z^{n-1}} - 1 \right) \\ &= 1 + \sum_{n=1}^{\infty} p_n z^n \sum_{n=2}^{\infty} n\left[\varphi_n(\lambda+1) - u_n\varphi_n(\lambda)\right] a_n z^{n-1} \\ &= b\left(1 + \sum_{n=2}^{\infty} n u_n \varphi_n(\lambda) a_n z^{n-1}\right) \sum_{n=1}^{\infty} p_n z^n. \end{aligned}$$

Equating the coefficients of z^{n-1} on both sides of the above equation, we have

$$n\left[\varphi_n(\lambda+1) - u_n\varphi_n(\lambda)\right] a_n = b\sum_{j=1}^{n-1}(n-j)u_{n-j}\varphi_{n-j}(\lambda)\, a_{n-j} p_j$$

which implies that

$$n\left[\varphi_n(\lambda+1) - u_n\varphi_n(\lambda)\right] | a_n | \leq b\sum_{j=1}^{n-1}(n-j)u_{n-j}\varphi_{n-j}(\lambda) \mid a_{n-j} \mid \mid p_j \mid . \tag{3.5}$$

Since $p \in \mathcal{P}$, by Lemma 3.3, we obtain

$$| p_j | \leq \frac{|\delta_1| A - B}{2}.$$

Following the steps as in Theorem 2.6 of Noor and Malik [9], we can establish the assertion of the Theorem 3.5. □

If $b = 1$, $t = 0$, $\alpha = 0$ and $\lambda = 0$ in the Theorem 3.5, we get the following result.

Corollary 3.6. [9] *Let $f \in k - \mathcal{CL}(A, B)$, then*

$$| a_n | \leq \frac{1}{n} \prod_{j=0}^{n-2} \frac{|\delta_1(A - B) - 2jB|}{2(j + 1)} \quad (n \geq 2).$$

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