Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 237–247
DOI: 10.24193/subbmath.2023.2.01

On a generalization of the Wirtinger inequality and some its applications

Latifa Agamalieva, Yusif S. Gasimov and Juan E. Nápoles-Valdes

Abstract. In this paper, we present generalized versions of the Wirtinger inequality, which contains as particular cases many of the well-known versions of this classic isoperimetric inequality. Some applications and open problems are also presented in the work.

Mathematics Subject Classification (2010): 26A33, 26Dxx, 35A23.

Keywords: Integral operator, fractional calculus, Wirtinger inequality.

1. Introduction

It is known that Fractional Calculus has a history practically similar to that of Ordinary Calculus, however only in the last 40 years has it become one of the most dynamic areas of Mathematics. Not only the development of the "classic" (global to be more precise) Fractional Calculus has contributed to this, but since the 1960s generalized differential operators, called local fractional ones, began to appear, which have shown their usefulness in different problems of application. However, until 2014 (see [22]) it is that a formalization of these operators is not achieved with the appearance of what is called Conformable Derivative, on the other hand, in 2018, a local derivative of a new type is presented, called Non conformable [13, 34], which comes to consolidate this area as one in constant development.

As we said, between the theoretical development and the multiplicity of applications, a multitude of operators, fractional and generalized, have appeared, making it practically impossible to follow these new operators. In [5] suggests and justifies the idea of a fairly complete classification of the known operators of the Fractional Calculus (global or local), on the other hand, in the work [6] some reasons are presented

Received 27 August 2020; Accepted 22 September 2020.
© Studia UBB MATHEMATICA. Published by Babeş-Bolyai University
This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.

why new operators linked to applications and developments theorists appear every day. These operators had been developed by numerous mathematicians with a barely specific formulation, for instance, the Riemann-Liouville (RL), the Weyl, Erdelyi-Kober, Hadamard integrals, and the Liouville and Katugampola fractional operators and many authors have introduced new fractional operators generated from general classical local derivatives.

In addition, Chapter 1 of [2] presents a history of differential operators, both local and global, from Newton to Caputo and presents a definition of local derivative with new parameter, providing a large number of applications, with a difference qualitative between both types of operators, local and global. Most importantly, Section 1.4 LIMITATIONS AND STRENGTH ..... concludes "We can therefore conclude that both the Riemann-Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result [38] that, the local fractional operator is not a fractional derivative" (p.24). As we said before, they are new tools that have proved their usefulness and potential in the modeling of different processes and phenomena.

Wirtinger's inequality for real functions was an inequality used in Fourier analysis. It was named after Wilhelm Wirtinger. It was used in 1904 to prove the isoperimetric inequality, one of the versions of this inequality is the following

$$\pi^2 \int_0^b |f(x)|^2 dx \leq b^2 \int_0^b |f'(x)|^2 dx,$$

whenever f is a C^1 function such that $f(0) = f(b) = 0$. In this form, Wirtinger's inequality is seen as the one-dimensional version of Friedrichs' inequality. If in the proof of the previous result, the well-known Schwarz Inequality is used, it is reduced to

$$\int_0^b |f(x)|^2 dx \leq b^2 \int_0^b |f'(x)|^2 dx, \tag{1.1}$$

where condition $y(b) = 0$ is not needed. It is worth noting that this inequality is relevant because it gives an estimate of the function f through its derivative.

An interesting survey on Wirtinger's and related inequalities can be found in a recently published monograph [28], which represents numerous extensions, refinements, variants, discrete analogues and applications of (1.1), and provides more than 200 references on this subject. Some interesting applications of this inequality can be found in [4, 7, 14, 23, 24, 27, 35, 36, 39].

The word calculus comes from the Latin calculus, which means having stones. The Integral Calculus is a branch of mathematics with so many ramifications and applications, that the sole intention of enumerating them makes the task practically impossible. It was used initially by, Aristoteles, Descartes, Newton and Barrow with the contributions of Newton, if we refer only to the case of integral inequalities present in the alliterature, there are different types of these, which involve certain properties of the functions involved, from generalizations of the known Mean Value Theorem of classical Integral Calculus, to varied inequalities in norm, going through the inequality of Wirtinger presented above.

In this article, a Wirtinger-type inequality is studied, in the context of the generalized derivative that we will define in the following section, some remarks will be presented that will show the strength of our results, having as particular cases, several of those reported in the literature. In particular we will deal with real integral operators defined on $\mathbb{R}$.

2. A general integral operator

We assume that the reader is familiar with the classic definition of the Riemann Integral, so we will not present it. In [15] was presented an integral operator generalized, whcih contain as particular cases, many of the well-known integral operators, both integer order and not. First we will present the definition of generalized derivative (see [30]) which was defined in the following way.

Definition 2.1. Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by

$$N_F^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon F(x, \alpha)) - f(x)}{\varepsilon} \tag{2.1}$$

for all $x > 0$, $\alpha \in (0, 1)$ being $F(\alpha, t)$ is some function. Here we will use some cases of F defined in function of $E_{a,b}(.)$ the classic definition of Mittag-Leffler function with $Re(a), Re(b) > 0$. Also we consider $E_{a,b}(x^{-\alpha})_k$ is the k-nth term of $E_{a,b}(.)$.

If f is $\alpha-$differentiable in some $(0, \alpha)$, and $\lim_{t \to 0^+} N_F^\alpha f(x)$ exists, then define $N_F^\alpha f(0) = \lim_{t \to 0^+} N_F^\alpha f(x)$. Note that if f is differentiable, then $N_F^\alpha f(x) = F(x, \alpha) f'(x)$, where $f'(x)$ is the ordinary derivative.

The function $E_{a,b}(z)$ was defined and studied by Mittag-Leffler in the year 1903. It is a direct generalization of the exponential function. This generalization was studied by Wiman in 1905, Agarwal in 1953 and Humbert and Agarwal in 1953, and others. In this address the reader can check [3, 8, 9, 10, 12, 16-18, 20, 21] where several fractional calculus operators have been introduced and investigated

We consider the following examples:

I) $F(x, \alpha) \equiv 1$, in this case we have the ordinary derivative.

II) $F(x, \alpha) = E_{1,1}(x^{-\alpha})$. In this case we obtain, from Definition 2.1, the non conformable derivative $N_1^\alpha f(x)$ defined in [13] (see also [29]).

III) $F(x, \alpha) = E_{1,1}((1 - \alpha)x) = e^{(1-\alpha)x}$, this kernel satisfies that $F(x, \alpha) \to 1$ as $\alpha \to 1$, a conformable derivative used in [11].

IV)$F(x, \alpha) = E_{1,1}(x^{1-\alpha})_1 = x^{1-\alpha}$ with this kernel we have $F(x, \alpha) \to 0$ as $\alpha \to 1$ (see [22]), a conformable derivative.

V)$F(x, \alpha) = E_{1,1}(x^{-\alpha})_1 = x^\alpha$ with this kernel we have $F(x, \alpha) \to x$ as $\alpha \to 1$ (see [31]). It is clear that since it is a non-conformable derivative, the results will differ from those obtained previously, which enhances the study of these cases.

VI) $F(x, \alpha) = E_{1,1}(x^{-\alpha})_1 = x^{-\alpha}$ with this kernel we have $F(x, \alpha) \to x^{-1}$ as $\alpha \to 1$ This is the derivative N_3^α studied in [25]. As in the previous case, the results obtained have not been reported in the literature.

Now, we give the definition of a general fractional integral. Throughout the work we will consider that the integral operator kernel T defined below is an absolutely continuous function.

Let I be an interval $I \subseteq \mathbb{R}$, $a, x \in I$ and $\alpha \in \mathbb{R}$. The integral operator $J^{\alpha}_{T,a}$, right and left, is defined for every locally integrable function f on I as

$$J^{\alpha}_{T,a+}(f)(x) = \int_a^x \frac{f(s)}{T(s,\alpha)} ds, x > a, \tag{2.2}$$

$$J^{\alpha}_{T,b-}(f)(x) = \int_x^b \frac{f(s)}{T(s,\alpha)} ds, b < x. \tag{2.3}$$

Sometimes the kernel of the integral operator is exactly the kernel of the generalized derivative.

Remark 2.2. It is easy to see that the case of the J^{α}_{T} operator defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. However, we will see that it goes much further by containing the cases listed at the beginning of the work. So, we have
1) If $F(x,\alpha) = x^{1-\alpha}$, $\quad T(x,\alpha) = \Gamma(\alpha)F(x-x,\alpha)$, from (2.2) we have the right side Riemann-Liouville fractional integrals $(R^{\alpha}_{a+}f)(x)$, similarly from (2.3) we obtain the left derivative of Riemann-Liouville. Then its corresponding right differential operator is

$$({}^{RL}D^{\alpha}_{a^+}f)(x) = \frac{d}{dx}(R^{1-\alpha}_{a+}f)(x),$$

analogously we obtain the left.
2) With $F(x,\alpha) = x^{1-\alpha}$, $\quad T(x-x,\alpha) = \Gamma(\alpha)F(lnt - lnx, \alpha)t$, we obtain the right Hadamard integral from (2.2), the left Hadamard integral is obtained similarly from (2.3). The right derivative is

$$({}^{H}D^{\alpha}_{a^+}f)(x) = x\frac{d}{dx}(H^{1-\alpha}_{a+}f)(x),$$

in a similar way we can obtain the left.
3) The right Katugampola integral is obtained from (2.2) making

$$F(x,\alpha) = x^{1-\alpha}, \quad e(x) = x^{\varrho}, \quad T(x,\alpha) = \frac{\Gamma(\alpha)}{F(\rho,\alpha)}\frac{F(e(x)-e(x),\alpha)}{e'(x)},$$

analogously for the integral left fractional. In this case, the right derivative is

$$({}^{K}D^{\alpha,\rho}_{a^+}f)(x) = x^{1-\rho}\frac{d}{dx}K^{1-\alpha,\rho}_{a^+}f(x) = F(x,\rho)\frac{d}{dx}K^{1-\alpha,\rho}_{a^+}f(x),$$

and we can obtain the left derivative in the same way.
4) The solution of equation $(-\Delta)^{-\frac{\alpha}{2}}\phi(u) = -f(u)$ called Riesz potential, is given by the expression

$$\phi = C^{\alpha}_{n}\int_{R^n}\frac{f(v)}{|u-v|^{n-\alpha}}dv,$$

where C^{α}_{n} is a constant (see [7, 18, 27]). Obviously, this solution can be expressed in terms of the operator (2.2) very easily.
5) Obviously, we can define the lateral derivative operators (right and left) in the

case of our generalized derivative, for this it is sufficient to consider them from the corresponding integral operator. To do this, just make use of the fact that if f is differentiable, then $N_F^\alpha f(x) = F(x,\alpha)f'(x)$ where $f'(x)$ is the ordinary derivative. For the right derivative we have

$$\left(N_{F,a+}^\alpha f\right)(x) = N_F^\alpha\left[J_{T,a+}^\alpha(f)(x)\right] = \frac{d}{dx}\left[J_{T,a+}^\alpha(f)(x)\right]F(x,\alpha),$$

similarly to the left.
6) It is clear then, that from our definition, new extensions and generalizations of known integral operators can be defined.
7) We can define the function space $L_\alpha^p[a,b]$ as the set of functions over $[a,b]$ such that $(J_{F,a+}^\alpha[f(x)]^p(b)) < +\infty$.

The following results are generalizations of the known results of the integer order Calculus.

Proposition 2.3. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$, $0 < \alpha \leq 1$ and f a α-differentiable function on I such that f' is a locally integrable function on I. Then, we have for all $x \in I$,*

$$J_{F,a+}^\alpha\big(N_F^\alpha(f)\big)(x) = f(x) - f(a).$$

Proposition 2.4. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$ and $\alpha \in (0,1]$.*

$$N_F^\alpha\big(J_{F,a+}^\alpha(f)\big)(x) = f(x),$$

for every continuous function f on I and $a, t \in I$.

In [22] it is defined the integral operator $J_{F,a}^\alpha$ for the choice of the function F given by $F(x,\alpha) = x^{1-\alpha}$, and [22, Theorem 3.1] shows

$$N^\alpha J_{x^{1-\alpha},\,a}^\alpha(f)(x) = f(x),$$

for every continuous function f on I, $a, x \in I$ and $\alpha \in (0,1]$. Hence, Proposition 2.4 extends to any F this important equality.

Theorem 2.5. *Let I be an interval $I \subseteq \mathbb{R}$, $a, b \in I$ and $\alpha \in \mathbb{R}$. Suppose that f, g are locally integrable functions on I, and $k_1, k_2 \in \mathbb{R}$. Then we have*

(1) $J_{T,a}^\alpha\big(k_1 f + k_2 g\big)(x) = k_1 J_{T,a}^\alpha f(x) + k_2 J_{T,a}^\alpha g(x)$,
(2) *if $f \geq g$, then $J_{T,a}^\alpha f(x) \geq J_{T,a}^\alpha g(x)$ for every $t \in I$ with $t \geq a$,*
(3) $\left|J_{T,a}^\alpha f(x)\right| \leq J_{T,a}^\alpha \left|f\right|(x)$ *for every $t \in I$ with $t \geq a$,*
(4) $\int_a^b \frac{f(s)}{T(s,\alpha)} ds = J_{T,a}^\alpha f(x) - J_{T,b}^\alpha f(x) = J_{T,a}^\alpha f(x)(b)$ *for every $t \in I$.*

Let $C^1[a,b]$ be the set of functions f with first ordinary derivative continuous on $[a,b]$, we consider the following norms on $C^1[a,b]$:

$$\|F\|_C = \max_{[a,b]}|f(x)|, \quad \|F\|_{C^1} = \left\{\max_{[a,b]}|f(x)| + \max_{[a,b]}|f'(x)|\right\}.$$

Theorem 2.6. *The fractional derivatives $N_{F,a+}^\alpha f(x)$ and $N_{F,b-}^\alpha f(x)$ are bounded operators from $C^1[a,b]$ to $C[a,b]$ with*

$$\left|N_{F,a+}^\alpha f(x)\right| \leq K\|F\|_C\|f\|_{C^1}, \tag{2.4}$$

$$\left|N^{\alpha}_{F,b-}f(x)\right| \leq K\|F\|_{C}\|f\|_{C^1}, \tag{2.5}$$

where the constant K, may be depend of derivative frame considered.

Remark 2.7. From previous results we obtain that the derivatives $N^{\alpha}_{F,a+}f(x)$ and $N^{\alpha}_{F,b-}f(x)$ are well defined.

Theorem 2.8. *(Integration by parts) Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions and $\alpha \in (0, 1]$. Then, the following property hold*

$$J^{\alpha}_{F,a+}((f)(N^{\alpha}_{F,a+}g(x))) = [f(x)g(x)]^b_a - J^{\alpha}_{F,a+}((g)(N^{\alpha}_{F,a+}f(x))). \tag{2.6}$$

3. The generalized Wirtinger inequality

In this section, we will state Wirtinger type inequalities using generalized integral operator defined in the previous section.

First, we will give a generalized version of the inequality (1.1).

Theorem 3.1. *For any function f, N-differentiable on $[a, b]$ with $a < b$ such that $f(a) = 0$, $\alpha \in (0, 1]$, we have*

$$J^{\alpha}_{F,a+}(f^2(x))(b) \leq [\mathcal{F}(b)]^2 \quad J^{\alpha}_{F,a+}\left(N^{\alpha}_{F}f(x)\right)(b) \tag{3.1}$$

with $\mathcal{F}(b) = J^{\alpha}_{F,a+}\left(1\right)(b)$ and $F(x, \alpha) > 0$.

Proof. From Proposition 2.3 we have $J^{\alpha}_{F,a+}\big(N^{\alpha}_{F}(f)\big)(x) = f(x) - f(a)$, and using the fact that $f(a) = 0$ we have $f(x) = J^{\alpha}_{F,a+}\big(N^{\alpha}_{F}(f)\big)(x)$ and so, from Property (3) of Theorem 2.5

$$|f(x)| = \left|J^{\alpha}_{F,a+}N^{\alpha}_{F}(f)(x)\right| \leq J^{\alpha}_{F,a+}\left|N^{\alpha}_{F}(f)\right|(x) \tag{3.2}$$

for every $x \in [a, b]$. Applying Schwarz inequality to the right side of (3.2) we obtain

$$\begin{aligned} |f(x)| &\leq \big(J^{\alpha}_{F,a+}(1)(x)\big)^{\frac{1}{2}}\Big(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^2(x)\Big)^{\frac{1}{2}} \\ &\leq (\mathcal{F}(x))^{\frac{1}{2}}\Big(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^2(x)\Big)^{\frac{1}{2}} \\ &\leq (\mathcal{F}(b))^{\frac{1}{2}}\Big(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^2(b)\Big)^{\frac{1}{2}}. \end{aligned}$$

From this last inequality, squaring and N-integrating between a and b, we obtain the desired result. □

Remark 3.2. If in the previous result we have $a \equiv 0$, the kernel $F(x, \alpha) \equiv 1$, that is, the classic Riemann integral, we get the inequality (1.1).

Remark 3.3. Similarly, if we use the kernel $F(x, \alpha) = x^{1-\alpha}$, that is, in the case of the conformable derivative T_α of [22], we obtain the inequality of Theorem 3.1 of [1].

Remark 3.4. It's easy to get new versions of the classic Wirtinger Inequality using other kernels, for example, if we take $F(x, \alpha) = x^{\alpha}$ we get, from (3.1) the following

$$J^{\alpha}_{F,a+}f(x)(b) \leq \frac{\left(b^{1-\alpha} - a^{1-\alpha}\right)^2}{(1-\alpha)^2} J^{\alpha}_{F,a+}\left(N^{\alpha}_{F}f(x)\right)(b). \tag{3.3}$$

4. Some applications

In [19] the authors gave a generalized Wirtinger type inequality using an auxiliary function. Thus, the following result is a generalization of this result.

Theorem 4.1. *For any positive continuous funtion $M(x)$ on $[a,b]$ with $N_F^\alpha M(x) > 0$ or $N_F^\alpha M(x) < 0$ on $[a,b]$, $\alpha \in (0,1]$, we have*

$$J_{F,a+}^\alpha \left((N_F^\alpha M(x))\right)(y^2(x))(b) \leq 4J_{F,a+}^\alpha \left[\left(\frac{M^2(x)}{N_F^\alpha M(x)}\right)(N_F^\alpha y(x))^2\right](b), \tag{4.1}$$

for all continuous function $y(x)$ defined on $[a,b]$ with $y(a) = y(b) = 0$.

Proof. We consider the case $N_F^\alpha M(x) > 0$ then, N-integrating by parts (see Theorem 2.8), we have

$$\begin{aligned} J_1 &= J_{F,a+}^\alpha \left((N_F^\alpha M(x))\right)(y^2(x))(b) \\ &= M(b)y^2(b) - M(a)y^2(a) - 2J_{F,a+}^\alpha(M(x))(N_F^\alpha y(x))(y(x))(b). \end{aligned}$$

From here, using the Schwarz inequality we get

$$\begin{aligned} J_1 &= J_{F,a+}^\alpha \left((N_F^\alpha M(x))\right)(y^2(x))(b) \\ &= -2J_{F,a+}^\alpha(M(x))(N_F^\alpha y(x))y(x))(b) \\ &\leq 2J_{F,a+}^\alpha \left[\sqrt{\frac{(M(x))^2}{N_F^\alpha M(x)}}\right] |N_F^\alpha y(x)| \sqrt{N_F^\alpha M(x)}\, |y(x)|(b) \\ &\leq 2\sqrt{J_1 J_2} \end{aligned}$$

with

$$J_2 = J_{F,a+}^\alpha \left[\left(\frac{M^2(x)}{N_F^\alpha M(x)}\right)(N_F^\alpha y(x))^2\right].$$

From the above inequality we have then $J_1 = 2\sqrt{J_1 J_2}$, where the desired conclusion is reached. □

Remark 4.2. If in the previous result we have $a \equiv 0$, the kernel $F(x,\alpha) \equiv 1$, that is, the classic Riemann integral, we get the Lemma 1 of [19].

Remark 4.3. Similarly, if we use the kernel $F(x,\alpha) = x^{1-\alpha}$, that is, in the case of the conformable derivative T_α of [22], we obtain the inequality of Theorem 3.2 of [1].

Remark 4.4. Of course we can also generate new generalizations considering other kernels, a simple matter that we leave to the reader.

Theorem 4.5. *For any function f, N-differentiable on $[a,b]$ with $a < b$ such that $f(a) = 0$, $p \geq 1$, $\alpha \in (0,1]$, we have*

$$J_{F,a+}^\alpha |f(x)|^p(b) \leq \frac{[\mathcal{F}(b)]^p}{p} \quad J_{F,a+}^\alpha \left(|N_F^\alpha f(x)|^p\right)(b) \tag{4.2}$$

with $\mathcal{F}(b) = J_{F,a+}^\alpha(1)(b)$ and $F(x,\alpha) > 0$.

Proof. As before, from Proposition 2.3 we have $J^{\alpha}_{F,a+}\big(N^{\alpha}_{F}(f)\big)(x) = f(x) - f(a)$, and using the fact that $f(a) = 0$ we have $f(x) = J^{\alpha}_{F,a+}\big(N^{\alpha}_{F}(f)\big)(x)$ and so, from Property (3) of Theorem 2.5

$$|f(x)| = \left|J^{\alpha}_{F,a+}N^{\alpha}_{F}(f)(x)\right| \leq J^{\alpha}_{F,a+}\left|N^{\alpha}_{F}(f)\right|(x)$$

for every $x \in [a, b]$. Using the Holder inequality with p and $\frac{p}{p-1}$ we obtain

$$|f(x)| \leq \big(J^{\alpha}_{F,a+}(1)(x)\big)^{\frac{p-1}{p}}\big(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^{p}(x)\big)^{\frac{1}{p}},$$

so we have

$$|f(x)|^{p} \leq \big(J^{\alpha}_{F,a+}(1)(x)\big)^{p-1}\big(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^{p}(x)\big),$$

N-integrating by parts (see Theorem 2.8), the conclusion of the Theorem is obtained. □

Remark 4.6. If in the previous result we have $a \equiv 0$, the kernel $F(x, \alpha) \equiv 1$, that is, the classic Riemann integral, we get a light variant of result of [6].

Remark 4.7. Similarly, if we use the kernel $F(x, \alpha) = x^{1-\alpha}$, that is, in the case of the conformable derivative T_{α} of [22], we obtain the inequality of Theorem 6 of [34] (see also Theorem 2.2 of [33]). Theorem 2.2 is obtained in a different way, and its conclusion differs from that presented here, it is necessary that $f(b) = 0$, however the interested reader can obtain it without any difficulty and have, instead of (4.2) the following expression

$$J^{\alpha}_{F,a+}|f(x)|^{p}(b) \leq \frac{[\mathcal{F}(b)]^{p}}{2^{p-1}p} \quad J^{\alpha}_{F,a+}\left(|N^{\alpha}_{F}f(x)|^{p}\right)(b).$$

Remark 4.8. New inequalities of type Wirtinger, generalizations of (4.2), can be obtained using other kernels.

Following the ideas of Theorem 3.1 we can obtain a weighted version of Wirtinger's Inequality as follows.

Theorem 4.9. *Let f and g N-differentiables functions on $[a, b]$ with $f(a) = g(a) = 0$ and $f, g \in L^{2}_{\alpha}[a, b]$ then, we have the following inequality*

$$J^{\alpha}_{F,a+}(|f|\,|g|)(b) \leq \frac{\mathcal{F}^{2}(b)}{2} J^{\alpha}_{F,a+}\left(\left(N^{\alpha}_{F}f\right)^{2} + \left(N^{\alpha}_{F}g\right)^{2}\right).$$

Proof. From assumptions and proceeding as in Theorem 3.1 we get easily

$$J^{\alpha}_{F,a+}(|f|\,|g|)(b) \leq \mathcal{F}^{2}(b)\left[J^{\alpha}_{F,a+}\left(\left(N^{\alpha}_{F}f\right)^{2}\right)\right]^{\frac{1}{2}}\left[J^{\alpha}_{F,a+}\left(\left(N^{\alpha}_{F}g\right)^{2}\right)\right]^{\frac{1}{2}}$$

Applying the known inequality $\sqrt{AB} \leq \frac{A+B}{2}$ with $A, B > 0$, the desired conclusion is obtained. □

5. Conclusion

In this paper, some generalized extensions of classical Wirtinger type inequality are obtained, using less restrictive conditions on the function f, for example, the condition $f(b) = 0$ is not used. In addition, several known results in this topic are obtained as particular cases of our results, in addition to raising several possibilities of future work in this address.

Taking into account the ideas of [26] we can define generalized partial derivatives as follows.

Definition 5.1. Given a real function $f : \mathbb{R}^n \to \mathbb{R}$ and $\overrightarrow{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ a point whose i-th component is positive. Then the generalized partial N-derivative of order α of f en el punto $\overrightarrow{a} = (a_1, \ldots, a_n)$ is defined by

$$N^{\alpha}_{F_i,t_i} f(\overrightarrow{a}) = \lim_{\varepsilon \to 0} \frac{f(a_1, .., a_i + \varepsilon F_i(a_i, \alpha), \ldots, a_n) - f(a_1, \ldots, a_i, ..., a_n))}{\varepsilon} \tag{5.1}$$

if it exists, it is denoted by $N^{\alpha}_{F_i,t_i} f(\overrightarrow{a})$, and is called the i-th generalized partial derivative of order $\alpha \in (0, 1]$ of f in $\overrightarrow{a}$.

Remark 5.2. If a real function f multivariable, has all the partial derivatives of order $\alpha \in (0, 1]$ in $\overrightarrow{a}$, with $a_i > 0$, then the generalized α-gradient of f of order $\alpha \in (0, 1]$ in $\overrightarrow{a}$ is

$$\nabla^{\alpha}_{N} f(\overrightarrow{a}) = (N^{\alpha}_{t_1} f(\overrightarrow{a}), \ldots, N^{\alpha}_{t_n} f(\overrightarrow{a})) \tag{5.2}$$

On this basis, the results of [32] can be generalized to a generalized context without much difficulty.

References

[1] Asliyüce, S., *Wirtinger type inequalities via fractional integral operators*, Stud. Univ. Babes-Bolyai Math., **64**(2019), no. 1, 35-42.

[2] Atangana, A., *Derivative with a New Parameter Theory, Methods and Applications*, Academic Press, 2016.

[3] Aziz, R., Kumawat, Y., *Marichev-Saigo-Maeda fractional operators with extended Mittag-Leffler function and generalized Galue type Struve function*, Adv. Math. Models Appl., **4**(2019), no. 3, 210-223.

[4] Baleanu, D., *Comments On: Ortigueira M., Martynyuk V., Fedula M., Machado J.A.T., The failure of certain fractional calculus operators in two physical models*, in Fract. Calc. Appl. Anal., **22**(2019), no. 2, Fract. Calc. Appl. Anal., **23**(2020), no. 1.

[5] Baleanu, D., Fernandez, A., *On fractional operators and their classifications*, Mathematics, **7**(2019), 830.

[6] Beesack, P.R., *Integral inequalities of the Wirtinger type*, Duke Math. J., **25**(1958), 477-498.

[7] Cınar, I., *On some properties of generalized Riesz potentials*, Intern. Math. J., **3**(2003), no. 12, 1393-1397.

[8] Capelas de Oliveira, E., Tenreiro Machado, J.A. *A review of definitions for fractional derivatives and integral*, Math. Probl. Eng., Volume 2014, Article ID 238459, 6 pages http://dx.doi.org/10.1155/2014/238459

[9] Diaz, J.B., Metcalf, F.T., *Variations of Wirtinger's Inequality, Inequalities,* Academic Press, New York, 1967, 79-103.

[10] Ertugral, F., Sarikaya, M.Z., *Some trapezoid type inequalities for generalized fractional integral*, 2018, preprint.

[11] Fleitas, A., Méndez, J.A., Nápoles, J.E., Sigarreta, J.M., *On the some classical systems of Liénard in general context*, Rev. Mexicana Fís., **65**(2019), no. 6, 618-625.

[12] Fleitas, A., Nápoles Valdés, J.E., Rodríguez, J.M., Sigarreta, J.M., *On some fractional differential equations*, (Submitted).

[13] Guzman, P.M., Langton, G., Lugo, L.M., Medina J., Nápoles Valdés, J.E., *A new definition of a fractional derivative of local type*, J. Math. Anal. **9**(2018), no. 2, 88-98.

[14] Guzman, P.M., Lugo Motta Bittencurt, L.M., Nápoles Valdes, J.E., *A note on stability of certain Lienard fractional equation,* Int. J. Math. Comput. Sci., **14**(2019), no. 2, 301-315.

[15] Guzman, P.M., Lugo, L.M., Nápoles, J.E., Vivas-Cortez, M., *On a new generalized integral operator and certain operating properties*, Axioms, **9**(2020), no. 2, 69.

[16] Gorenflo, R., Mainardi, F., *Fractional Calculus: Integral and Differential Equations of Fractional Order*, Springer, Wien, 1997, 223-276.

[17] Hardy, G.H., Littlewood, J.E., Polya, G., *Inequalities*, 2nd ed., Cambridge University Press, London and New York, 1959.

[18] Helms, L.L., *Introduction To Potential Theory*, New York, Wiley-Interscience, 1969.

[19] Hinton, D.B., Lewis, R.T., *Discrete spectra criteria for singular differential operators with middle terms*, Math. Proc. Cambridge Philos. Soc., **77**(1975), 337-347.

[20] Katugampola, U.N., *New approach to a generalized fractional integral*, Appl. Math. Comput., **218**(2011), 860-865.

[21] Katugampola, U.N., *A new approach to generalized fractional derivatives*, Bull. Math. Anal. App., **6**(2014), 1-15.

[22] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., *A new definition of fractional derivative*, J. Comput. Appl. Math., **264**(2014), 65-70.

[23] Ma, M., Baleanu, D., Gasimov, Y.S., Yang, X.-J., *New results for multidimensional diffusion equations in fractal dimensional space*, Rom. J. Phys., **61**(2016), no. 5-6, 784-794.

[24] Machado, J.T., Kiryakova, V., Mainardi, F., *Recent history of fractional calculus*, Commun. Nonlinear Sci. Numer. Simul., **16**(2011), 1140-1153.

[25] Martínez, F., Mohammed, P.O., Nápoles, V J.E., *Non conformable fractional Laplace transform*, Kragujevac J. Math., **46**(2022), no. 3, 341-354.

[26] Martínez, F., Nápoles V., J.E., *Towards a Non-Conformable Fractional Calculus of N-variables*, J. Math. Appl. (to appear).

[27] Martinez, C., Sanz, M., Periogo, F., *Distributional fractional powers of Laplacian, Riesz potential*, Studia Math., **135**(1999), no. 3, 253-271.

[28] Mitrinović, D.S., Pecarić, J.E., Fink, A.M., *Inequalities Involving Functions and Their Integrals and Derivatives*, Dordrecht, 1991.

[29] Nápoles Valdés, J.E., Guzman, P.M., Lugo, L.M., *Some new results on nonconformable fractional calculus*, Adv. Dyn. Syst. Appl., **13**(2018), no. 2, 167-175.

[30] Nápoles Valdés, J.E., Guzman, P.M., Lugo, L.M., Kashuri, A., *The local non conformable derivative and Mittag Leffler function*, Sigma J. Eng. Nat. Sci., **38**(2020), no. 2, 1007-1017.

[31] Nápoles Valdés, J.E., Rodríguez, J.M., Sigarreta, J.M., *On Hermite-Hadamard type inequalities for non-conformable integral operators*, Symmetry, **11**(2019), no. 9, 1108.

[32] Reis, J.C., Da Silva, P.N., Boldrini, J.L., *Uma desigualdade do tipo Wirtinger*, Cadernos do IME - Serie Matematica, **8**(2014).

[33] Sarikaya, M.Z., *On the new Wirtinger type inequalities*, Konuralp J. Math., **7**(2019), no. 1, 112-116.

[34] Sarikaya, M.Z., Bilisik, C.C., *Wirtinger type inequalities for conformable fractional integrals,* preprint.

[35] Sarikaya, M.Z., Ertugral, F., *On the generalized Hermite-Hadamard inequalities,* 2017 (submitted).

[36] Sarikaya, M.Z., Yildirim, H., *On generalization of the Riesz potential*, Indian J. Math. Math. Sci., **3**(2007), no. 2, 231-235.

[37] Swanson, C.A., *Wirtinger's inequality,* SIAM J. Math. Anal., **9**(1978), no. 3, 484-491.

[38] Umarov, S., Steinberg, S., *Variable order differential equations with piecewise constant order-function and diffusion with changing modes*, Z. Anal. Anwend., **28**(2009), no. 4, 431-450.

[39] Yang, X.-J., Gasimov, Y.S., Gao, F., Allahverdiyeva, N.A., *Travelling-wave solutions for Klein-Gordon and Helmholtz equations on Cantor sets,* Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **43**(2017), no. 1, 123-131.

Latifa Agamalieva
Azerbaijan University, J. Hajibeyli, 71, AZ1007, Baku, Azerbaijan,
Baku State University, Institute for Physical Problems,
Z.Khalilov, 23, AZ1148, Baku, Azerbaijan
e-mail: latifa.agamalieva@gmail.com

Yusif S. Gasimov
Azerbaijan University, J. Hajibeyli, 71, AZ1007, Baku, Azerbaijan,
Baku State University, Institute for Physical Problems,
Z.Khalilov, 23, AZ1148, Baku, Azerbaijan
Institute of Mathematics and Mechanics, ANAS
B.Vahabzade, 9, AZ1148, Baku, Azerbaijan
e-mail: gasimov.yusif@gmail.com

Juan E. Nápoles-Valdes
UNNE, FaCENA, Ave. Libertad 5450, Corrientes, 3400, Argentina
UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina
e-mail: jnapoles@exa.unne.edu.ar