Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 3, 543–562
DOI: 10.24193/subbmath.2023.3.07

Non-instantaneous impulsive fractional integro-differential equations with proportional fractional derivatives with respect to another function

Mohamed I. Abbas

Abstract. This paper concerns the existence and uniqueness of solutions of non-instantaneous impulsive fractional integro-differential equations with proportional fractional derivatives with respect to another function. By the aid of the nonlinear alternative of Leray-Schauder type and the Banach contraction mapping principle, the main results are demonstrated. Two examples are inserted to illustrate the applicability of the theoretical results.

Mathematics Subject Classification (2010): 34K37, 34A37, 47H10.

Keywords: Non-instantaneous impulses, proportional fractional derivatives, Leray-Schauder alternative.

1. Introduction

The theory of fractional differential equations has recently acquired plentiful circulation and great significance because of its rife applications in fields of science and engineering, see, for example [10, 17, 18, 19] and references cited therein. The field of fractional differential equations with instantaneous impulses has become a valuable tool for the description of sudden changes or discontinuous jumps in the evolution progress of dynamical systems such as the shocks, disturbance and natural disasters, see [1, 2] and references cited therein. In the instantaneous impulses the duration of impulsive effect is relatively short as compared to the overall duration of the whole process, see [15]. But many times it has been observed that some certain dynamics of

Received 23 August 2020; Accepted 27 September 2020.

evolution processes cannot be described by instantaneous impulsive dynamic systems. For example, the injecting drugs in the bloodstream, and the consequent absorption for the body are gradual and continuous process. In this case the impulsive action begins at any arbitrary fixed point and continues with a finite time interval. Such types of systems are known as non-instantaneous impulsive systems which are more suitable to study the dynamics of evolution processes. Hernándaz et al. [6] introduced a new class of evolution equations with non-instantaneous impulses of the form

$$\begin{cases} x'(t) = Ax(t) + f\left(t, x(t)\right), \ t \in (s_k, t_{k+1}], \ k = 0, 1, \cdots, m, \\ y(t) = g_k\left(t, x(t)\right), \ t \in (t_k, s_k], \ k = 1, \cdots, m, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq E \to E$, is the generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ on a Banach space E.

Recently, Agarwal et al. in [3] constructed monotone successive approximations for solutions to initial value problems for a scalar nonlinear Caputo fractional differential equation with non-instantaneous impulses of the form

$$\begin{cases} {}_0^C D^q x(t) = f\left(t, x(t)\right), \ t \in (t_k, s_k], \ k = 0, 1, \cdots, p, p+1 \\ x(t) = \phi_k\left(t, x(t), x(s_k - 0)\right), \ t \in (s_k, t_{k+1}], \ k = 1, \cdots, p, \\ x(0) = x_0, \end{cases} \tag{1.2}$$

where ${}_0^C D^q$ is the Caputo fractional derivative of order $0 < q < 1$.

In [12], Kumar et al. studied the sufficient conditions for the existence of mild solution of Atangana-Baleanu fractional differential system with non-instantaneous impulses of the form

$$\begin{cases} {}^{ABC}D^\rho x(t) = Ax(t) + f\left(t, x(t)\right), \ t \in \bigcup_{k=0}^m (s_k, t_{k+1}], \\ x(t) = \gamma_k\left(t, x(t)\right), \ t \in \bigcup_{k=1}^m (t_k, s_k], \\ x(0) = x_0 - g(x), \end{cases} \tag{1.3}$$

where ${}^{ABC}D^\rho$ is the Atangana-Baleanu-Caputo fractional derivative of order $0 < \rho < 1$ and $A : D(A) \subseteq X \to X$, is the generator of ρ-resolvent operator $\{S_\rho(t) : t \geq 0\}$.

In [14], Luo et al. considered the existence of solutions for a kind of ψ-Hilfer fractional differential inclusions involving non-instantaneous impulses of the form

$$\begin{cases} {}^H D_{t_0^+}^{\alpha,\beta;\psi} x(t) \in A(t)x(t) + G\left(t, x(t)\right), \ t \in (s_k, t_{k+1}] \cap [t_0, T], k = 0, 1, \cdots, p, \\ x(t) = \dfrac{\phi_k\left(t, x(t), x(t_k - 0)\right)}{\Gamma(\gamma)\Gamma(2-\gamma)}, \ t \in \bigcup_{k=1}^m (t_k, s_k] \cap [t_0, T], k = 1, \cdots, p, \\ x(t_0) = x_0, \end{cases} \tag{1.4}$$

where ${}^{H}D^{\alpha,\beta;\psi}_{t_0^+}$ is the ψ-Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $0 < \beta \leq 1$, with respect to function ψ, $A(t) : D \subseteq X \to X$ is a bounded operator and $G : (s_k, t_{k+1}] \cap [t_0, T] \times X \to P(X)$ is a multi-valued mapping, $P(X)$ is the family of all nonempty subsets of a real separable Banach space X.
For more recent contributions relevant to non-instantaneous impulsive fractional differential equations, we refer the reader to the papers [11, 13, 16, 20, 21] and references cited therein.

Motivated by the above papers, we investigate the following non-instantaneous impulsive fractional integro-differential equation:

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t) = f\left(t, y(t), {}_aI^{\beta,\rho,g}y(t)\right), \ t \in (s_k, t_{k+1}] \subset J, \ k = 0, 1, \cdots, m, \\ y(t) = \psi_k\left(t, y\left(t_k^+\right)\right), \ t \in (t_k, s_k], \ k = 1, \cdots, m, \\ {}_aI^{1-\alpha,\rho,g}y(a) = y_0 \in \mathbb{R}, \end{cases} \tag{1.5}$$

where $J = [a, T], T > a$, $0 < \alpha \leq 1$, $\beta, \rho > 0$, ${}_aD^{\alpha,\rho,g}$ is the proportional fractional derivative with respect to another function g, ${}_aI^{\beta,\rho,g}$ is the proportional fractional integral with respect to another function g, and $f \in C(J \times \mathbb{R}^2, \mathbb{R})$.
Here, $a = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_{m-1} \leq s_m \leq t_m \leq t_{m+1} = T$ are fixed numbers, $y\left(t_k^+\right) = \lim_{\epsilon \to 0^+} y\left(t_k + \epsilon\right)$, and $\psi_k \in C([t_k, s_k], \mathbb{R})$, $k = 1, \cdots, m$.

Remark 1.1.

- For the non-instantaneous impulsive fractional integro-differential equation (1.5), the $(t_k, s_k]$, $k = 1, \cdots, m$ are called intervals of non-instantaneous impulses and $\psi_k(t, y)$, $k = 1, \cdots, m$ are called non-instantaneous impulsive functions.
- If $t_k = s_{k-1}$, $k = 1, \cdots, m$, then the non-instantaneous impulsive fractional integro-differential equation (1.5) reduces to an impulsive fractional integro-differential equation.

In recent years, there are various new definitions of fractional derivatives, among these new definitions the so-called fractional conformable derivative, which is introduced by Khalil et al. [9]. Unfortunately, this new definition has an obstacle that it does not tend to the original function as the order ρ tends to zero. Anderson et al. [4] were able to define the proportional (conformable) derivative of order ρ by

$${}^{P}D^{\rho}_{t}f(t) = \kappa_1(\rho, t)f(t) + \kappa_0(\rho, t)f'(t),$$

where f is differentiable function and $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ are continuous functions of the variable t and the parameter $\rho \in [0,1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$\lim_{\rho \to 0^+} \kappa_0(\rho, t) = 0, \ \lim_{\rho \to 1^-} \kappa_0(\rho, t) = 1, \ \kappa_0(\rho, t) \neq 0, \ \rho \in (0, 1], \tag{1.6}$$

$$\lim_{\rho \to 0^+} \kappa_1(\rho, t) = 1, \ \lim_{\rho \to 1^-} \kappa_1(\rho, t) = 0, \ \kappa_1(\kappa, t) \neq 0, \ \rho \in [0, 1). \tag{1.7}$$

This newly defined local derivative tends to the original function as the order ρ tends to zero and hence improved the conformable derivatives. In [7, 8], Jarad et al. proposed more general forms and properties of proportional derivative of a function f with respect to another function g. The kernel obtained in their investigation contains an exponential function and is function dependent (more details can be seen in Section 2).

The novelty of the current work is that, to the best knowledge of the author, no one has yet been treated with non-instantaneous impulsive fractional differential equations involving the proportional fractional derivative with respect to another function.

2. Preliminaries

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_C = \sup_{t\in J} |y(t)|.$$

We consider the Banach space

$$PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} :\ y \in C((t_k, t_{k+1}],\mathbb{R}), k = 0,1,\cdots,m \text{ and there exist } y\left(t_k^-\right)$$

$$\text{and } y\left(t_k^+\right), k = 1,\cdots,m \text{ with } y\left(t_k^-\right) = y\left(t_k\right)\},$$

with the norm

$$\|y\|_{PC} = \sup_{t\in J} |y(t)|.$$

Now, we recall some basic definitions and properties of fractional proportional derivative and integral of a function with respect to another function. The terms and notations are adopted from [7, 8].

Definition 2.1. (The proportional derivative of a function with respect to another function) For $\rho \in [0,1]$, let the functions $\kappa_0, \kappa_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ be continuous such that for all $t \in \mathbb{R}$ we have

$$\lim_{\rho\to 0^+} \kappa_1(\rho,t) = 1,\ \lim_{\rho\to 0^+} \kappa_0(\rho,t) = 0,\ \lim_{\rho\to 1^-} \kappa_1(\rho,t) = 0,\ \lim_{\rho\to 1^-} \kappa_0(\rho,t) = 1,$$

and $\kappa_1(\rho,t) \neq 0, \rho \in [0,1]$, $\kappa_0(\rho,t) \neq 0, \rho \in [0,1]$. Let $g(t)$ be a strictly increasing continuous function. Then the proportional differential operator of order ρ of f with respect to g is defined by

$$D^{\rho,g} f(t) = \kappa_1(\rho,t) f(t) + \kappa_0(\rho,t)\frac{f'(t)}{g'(t)}. \tag{2.1}$$

For the restricted case when $\kappa_1(\rho,t) = 1-\rho$ and $\kappa_0(\rho,t) = \rho$, (2.1) becomes

$$D^{\rho,g} f(t) = (1-\rho) f(t) + \rho\frac{f'(t)}{g'(t)}. \tag{2.2}$$

Definition 2.2. (The proportional integral of a function with respect to another function) For $\rho \in (0,1]$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $g \in C[a,b]$, $g'(t) > 0$, we define the left and right fractional integrals of f with respect to g by

$$ {}_aI^{\alpha,\rho,g} f(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t)-g(s))^{\alpha-1} f(s) g'(s) ds, \quad (2.3)$$

$$ I_b^{\alpha,\rho,g} f(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_t^b e^{\frac{\rho-1}{\rho}(g(s)-g(t))} (g(s)-g(t))^{\alpha-1} f(s) g'(s) ds, \quad (2.4)$$

respectively.

Definition 2.3. For $\rho \in (0,1]$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, we define the left fractional derivative of f with respect to g as

$$ \begin{aligned} {}_aD^{\alpha,\rho,g} f(t) &= D^{n,\rho,g} {}_aI^{n-\alpha,\rho,g} f(t) \\ &= \frac{D_t^{n,\rho,g}}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t)-g(s))^{n-\alpha-1} f(s) g'(s) ds, \quad (2.5) \end{aligned}$$

and the right fractional derivative of f with respect to g as

$$ \begin{aligned} D_b^{\alpha,\rho,g} f(t) &= {}_\ominus D^{n,\rho,g} I_b^{n-\alpha,\rho,g} f(t) \\ &= \frac{{}_\ominus D^{n,\rho,g}}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_t^b e^{\frac{\rho-1}{\rho}(g(s)-g(t))} (g(s)-g(t))^{n-\alpha-1} f(s) g'(s) ds, \quad (2.6) \end{aligned}$$

where $n = [\Re(\alpha)] + 1$, $D^{n,\rho,g} = \underbrace{D^{\rho,g} D^{\rho,g} \cdots D^{\rho,g}}_{n \text{ times}}$ and

$$ {}_\ominus D^{\rho,g} := (1-\rho) f(t) - \rho \frac{f'(t)}{g'(t)}, \quad {}_\ominus D^{n,\rho,g} = \underbrace{{}_\ominus D^{\rho,g} {}_\ominus D^{\rho,g} \cdots {}_\ominus D^{\rho,g}}_{n \text{ times}}. $$

Lemma 2.4. ([8]) *If $\rho \in (0,1]$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$. Then, for f is continuous and defined for $t \geq a$, we have*

$$ {}_aI^{\alpha,\rho,g} \left({}_aI^{\beta,\rho,g} f\right)(t) = {}_aI^{\beta,\rho,g} \left({}_aI^{\alpha,\rho,g} f\right)(t) = \left({}_aI^{\alpha+\beta,\rho,g} f\right)(t), \quad (2.7)$$

$$ I_b^{\alpha,\rho,g} \left(I_b^{\beta,\rho,g} f\right)(t) = I_b^{\beta,\rho,g} \left(I_b^{\alpha,\rho,g} f\right)(t) = \left(I_b^{\alpha+\beta,\rho,g} f\right)(t). \quad (2.8)$$

Lemma 2.5. ([7]) *Let $\Re[\alpha] > 0$, $n = -[-\Re(\alpha)]$, $f \in L_1(a,b)$ and $({}_aI^{\alpha,\rho,g} f)(t) \in AC^n[a,b]$. Then*

$$ {}_aI^{\alpha,\rho,g} {}_aD^{\alpha,\rho,g} f(t) = f(t) - e^{\frac{\rho-1}{\rho}(g(t)-g(a))} \sum_{j=1}^{n} ({}_aI^{j-\alpha,\rho,g} f)(a^+) \frac{(g(t)-g(a))^{\alpha-j}}{\rho^{\alpha-j}\Gamma(\alpha+1-j)}. \quad (2.9)$$

For $0 < \alpha \leq 1$, we have

$$ {}_aI^{\alpha,\rho,g} {}_aD^{\alpha,\rho,g} f(t) = f(t) - e^{\frac{\rho-1}{\rho}(g(t)-g(a))} ({}_aI^{1-\alpha,\rho,g} f)(a^+) \frac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}. \quad (2.10)$$

Lemma 2.6. *Let $\alpha, \beta > 0$. Then, for any $a, b \in \mathbb{R}$, we get*

$$ I_g := \int_a^b (g(b)-g(s))^{\beta-1} (g(s)-g(a))^{\alpha-1} g'(s) ds = (g(b)-g(a))^{\alpha+\beta-1} \mathbf{B}(\alpha,\beta), $$

where $\mathbf{B}(\cdot,\cdot)$ *is the well-known beta function defined as*

$$\mathbf{B}(m,n)=\int_0^1 (1-s)^{m-1}s^{n-1}ds,\ \ m>0, n>0.$$

Proof. By the substitution $g(s)=g(b)z$, the integral I_g becomes

$$I_g=(g(b))^{\alpha+\beta-1}\int_{\frac{g(a)}{g(b)}}^1 (1-z)^{\beta-1}\left(z-\frac{g(a)}{g(b)}\right)^{\alpha-1}dz.$$

Using the following well-known integral

$$\int_a^b (s-a)^{m-1}(b-s)^{n-1}ds=(b-a)^{m+n-1}\mathbf{B}(m,n)=(b-a)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

$m>0$, $n>0$, we get

$$I_g=(g(b))^{\alpha+\beta-1}\left(1-\frac{g(a)}{g(b)}\right)^{\alpha+\beta-1}\mathbf{B}(\alpha,\beta).$$

The proof is finished. □

3. Existence and uniqueness results

In order to investigate the existence of solution for (1.5), we consider the following auxiliary lemma

Lemma 3.1. *Let* $0<\alpha\leq 1$ *and let* $h:J\to\mathbb{R}$ *be an integrable function. Then the linear problem*

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t)=h(t),\ \ t\in(s_k,t_{k+1}]\subset J,\ \ k=0,1,\cdots,m, \\ y(t)=\psi_k\left(t,y\left(t_k^+\right)\right),\ \ t\in(t_k,s_k],\ \ k=1,\cdots,m, \\ {}_aI^{1-\alpha,\rho,g}y(a)=y_0\in\mathbb{R}, \end{cases} \tag{3.1}$$

has a solution given by

$$y(t)=\begin{cases} e^{\frac{\rho-1}{\rho}(g(t)-g(a))}\dfrac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}y_0 \\ +\dfrac{1}{\rho^\alpha\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\left(g(t)-g(s)\right)^{\alpha-1}h(s)g'(s)ds\ ,t\in[a,t_1], \\ \psi_k\left(t,y\left(t_k^+\right)\right),\ \ t\in(t_k,s_k],\ \ k=1,\cdots,m, \\ e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))}\left(\dfrac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} \\ \times\Big[\psi_k\left(s_k,y\left(t_k^+\right)\right)-\dfrac{1}{\rho^\alpha\Gamma(\alpha)}\displaystyle\int_a^{s_k}e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))}\left(g(s_k)-g(s)\right)^{\alpha-1}h(s)g'(s)ds\Big] \\ +\dfrac{1}{\rho^\alpha\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\left(g(t)-g(s)\right)^{\alpha-1}h(s)g'(s)ds\ ,t\in(s_k,t_{k+1}], \\ k=1,\cdots,m. \end{cases} \tag{3.2}$$

Proof. Let $t \in (0, t_1]$. Then, using Lemma 2.5, the problem

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t) = h(t), \ t \in (a, t_1], \\ {}_aI^{1-\alpha,\rho,g}y(a) = y_0 \in \mathbb{R}, \end{cases} \tag{3.3}$$

has a solution given by

$$y(t) = e^{\frac{\rho-1}{\rho}(g(t)-g(a))} \frac{(g(t) - g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} y_0 + \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t) - g(s))^{\alpha-1} h(s)g'(s)ds \ , t \in [0, t_1].$$

For $t \in (t_1, s_1]$, $y(t) = \psi_1\left(t, y\left(t_1^+\right)\right)$. Again, using Lemma 2.5 and applying the proportional fractional integral ${}_aI^{\alpha,\rho,g}$ over $(a, t_2]$ to both sides of the problem

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t) = h(t), \ t \in (s_1, t_2], \\ y(s_1) = \psi_1\left(s_1, y\left(t_1^+\right)\right), \end{cases} \tag{3.4}$$

we get

$$y(t) = e^{\frac{\rho-1}{\rho}(g(t)-g(a))} \frac{(g(t) - g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} {}_aI^{1-\alpha,\rho,g}y(a) + \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t) - g(s))^{\alpha-1} h(s)g'(s)ds. \tag{3.5}$$

Substituting $t = s_1$ in (3.5), we get

$$y(s_1) = e^{\frac{\rho-1}{\rho}(g(s_1)-g(a))} \frac{(g(s_1) - g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} {}_aI^{1-\alpha,\rho,g}y(a) + \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^{s_1} e^{\frac{\rho-1}{\rho}(g(s_1)-g(s))} (g(s_1) - g(s))^{\alpha-1} h(s)g'(s)ds. \tag{3.6}$$

From the second equation of (3.3), we get

$${}_aI^{1-\alpha,\rho,g}y(a) = e^{-\frac{\rho-1}{\rho}(g(s_1)-g(a))} \frac{\Gamma(\alpha)\left(g(s_1) - g(a)\right)^{1-\alpha}}{\rho^{1-\alpha}} \times \left[\psi_1\left(s_1, y\left(t_k^+\right)\right) - \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^{s_1} e^{\frac{\rho-1}{\rho}(g(s_1)-g(s))} (g(s_1) - g(s))^{\alpha-1} h(s)g'(s)ds\right]. \tag{3.7}$$

Therefore, by substituting (3.7) in (3.5), we get

$$y(t) = e^{\frac{\rho-1}{\rho}(g(t)-g(s_1))} \left(\frac{(g(t) - g(a))}{(g(s_1) - g(a))}\right)^{\alpha-1} \times \left[\psi_1\left(s_1, y\left(t_1^+\right)\right) - \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^{s_1} e^{\frac{\rho-1}{\rho}(g(s_1)-g(s))} (g(s_1) - g(s))^{\alpha-1} h(s)g'(s)ds\right] + \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t) - g(s))^{\alpha-1} h(s)g'(s)ds.$$

For $t \in (t_2, s_2]$,

$$y(t) = \psi_2\left(t, y\left(t_2^+\right)\right).$$

Performing the same process, we deduce when $t \in (s_2, t_3]$ that the solution of the problem

$$\begin{cases} {}_aD^{\alpha,\rho,g} y(t) = h(t), \ t \in (s_2, t_3], \\ y(s_2) = \psi_2\left(s_2, y\left(t_2^+\right)\right), \end{cases} \tag{3.8}$$

is given by

$$\begin{aligned} y(t) &= e^{\frac{\rho-1}{\rho}(g(t)-g(s_2))} \left(\frac{(g(t)-g(a))}{(g(s_2)-g(a))} \right)^{\alpha-1} \\ &\times \left[\psi_2\left(s_2, y\left(t_2^+\right)\right) - \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{s_2} e^{\frac{\rho-1}{\rho}(g(s_2)-g(s))} \left(g(s_2)-g(s)\right)^{\alpha-1} h(s) g'(s) ds \right] \\ &+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} \left(g(t)-g(s)\right)^{\alpha-1} h(s) g'(s) ds. \end{aligned}$$

In general, when $t \in (s_k, t_{k+1}]$, the solution of the problem

$$\begin{cases} {}_aD^{\alpha,\rho,g} y(t) = h(t), \ t \in (s_k, t_{k+1}], \\ y(s_k) = \psi_k\left(s_k, y\left(t_k^+\right)\right), \end{cases} \tag{3.9}$$

is given by

$$\begin{aligned} y(t) &= e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))} \left(\frac{(g(t)-g(a))}{(g(s_k)-g(a))} \right)^{\alpha-1} \\ &\times \left[\psi_k\left(s_k, y\left(t_k^+\right)\right) - \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{s_k} e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))} \left(g(s_k)-g(s)\right)^{\alpha-1} h(s) g'(s) ds \right] \\ &+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} \left(g(t)-g(s)\right)^{\alpha-1} h(s) g'(s) ds. \end{aligned}$$

This shows that $y(t)$ satisfies (3.2). This completes the proof. □

By virtue of Lemma 3.1, we deduce that the solution of the non-instantaneous impulsive fractional integro-differential equation (1.5) is given by

$$y(t)=\begin{cases} e^{\frac{\rho-1}{\rho}(g(t)-g(a))}\dfrac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}y_0 \\ +\dfrac{1}{\rho^{\alpha}\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}(g(t)-g(s))^{\alpha-1}f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)g'(s)ds, \\ t\in[a,t_1], \\ \psi_k\left(t,y\left(t_k^+\right)\right),\ t\in(t_k,s_k],\ k=1,\cdots,m, \\ e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))}\left(\dfrac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\Big[\psi_k\left(s_k,y\left(t_k^+\right)\right) \\ -\dfrac{1}{\rho^{\alpha}\Gamma(\alpha)}\displaystyle\int_a^{s_k} e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))}(g(s_k)-g(s))^{\alpha-1}f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)g'(s)ds\Big] \\ +\dfrac{1}{\rho^{\alpha}\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}(g(t)-g(s))^{\alpha-1}f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)g'(s)ds, \\ t\in(s_k,t_{k+1}],\ k=1,\cdots,m. \end{cases} \tag{3.10}$$

For ease of handling later, we will use the following brief constants:

$$\begin{cases} \Theta_1:=(g(t_1)-g(a))^{\alpha},\ \Theta_2:=\max\left\{(g(t_{k+1})-g(a))^{\alpha},\ k=1,\cdots,m\right\}, \\ \Theta_3:=\dfrac{(g(t_1)-g(a))^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)},\ \Theta_4:=\dfrac{(g(t_1)-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}, \\ \Xi_1:=\dfrac{(g(t_1)-g(a))^{\alpha+\beta}}{\rho^{\beta}\Gamma(\beta+1)}, \\ \Xi_2:=\max\left\{\left(\dfrac{(g(t_{k+1})-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1},\ k=1,\cdots,m\right\}, \\ \Xi_3:=\max\left\{\dfrac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^{\beta}\Gamma(\beta+1)},\ k=1,\cdots,m\right\}, \\ \Xi_4:=\max\left\{\dfrac{(g(t_{k+1})-g(a))^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)},\ k=1,\cdots,m\right\}, \\ \Xi_5:=\max\left\{\dfrac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)},\ k=1,\cdots,m\right\}. \end{cases} \tag{3.11}$$

In order to investigate the main results, the following hypotheses will be imposed.

(H1). The function $f:J\times\mathbb{R}^2\to\mathbb{R}$ is continuous and $\psi_k\in C([t_k,s_k],\mathbb{R})$, $k=1,\cdots,m$.

(H2). There exists a constant $L_f>0$ such that

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\le L_f\left(|u_1-u_2|+|v_1-v_2|\right),$$

for each $t\in[s_k,t_{k+1}], k=0,1,\cdots,m$, for all $u_i,v_i\in\mathbb{R},\ i=1,2$.

(H3). There exist constants $L_k > 0, k = 1, \cdots, m$ such that

$$|\psi_k(t, u_1) - \psi_k(t, u_2)| \leq L_k |u_1 - u_2|,$$

for each $t \in [t_k, s_k], k = 1, \cdots, m$, for all $u_1, u_2 \in \mathbb{R}$.

(H4). There exist positive constants ℓ_0, ℓ_1 and ℓ_2 such that

$$|f(t, u, v)| \leq \ell_0 + \ell_1 |u| + \ell_2 |v|,$$

for each $t \in [s_k, t_{k+1}], k = 0, 1, \cdots, m$, for all $u, v \in \mathbb{R}$.

(H5). There exist positive constants $\aleph_0$ and $\aleph_1 0$ such that

$$|\psi_k(t, u)| \leq \aleph_0 + \aleph_1 |u|,$$

for each $t \in [t_k, s_k], k = 1, \cdots, m$, for all u.

(H6). There exists a constant $M > 0$ such that

$$\max\left\{ \frac{M}{\Theta_3(\ell_0 + \ell_1 M) + \Theta_4 \ell_2 M}, \frac{M}{\aleph_0 + \aleph_1 M}, \right.$$
$$\left. \frac{M}{\Xi_2 \left[\aleph_0 + \aleph_1 M + \Xi_4(\ell_0 + \ell_1 M) + \Xi_5 \ell_2 M \right] + \Xi_4(\ell_0 + \ell_1 M) + \Xi_5 \ell_2 M} \right\} > 1.$$

For the purpose of convenience, for each $t \in [a, T]$ and each $y_1, y_2 \in PC(J, \mathbb{R})$, we have

$$\begin{aligned}
&|{}_aI^{\beta,\rho,g} y_1(t) - {}_aI^{\beta,\rho,g} y_2(t)| \\
&\leq \frac{1}{\rho^\beta \Gamma(\beta)} \int_a^t \left| e^{\frac{\rho-1}{\rho}(g(t)-g(s))} \right| (g(t) - g(s))^{\beta-1} |y_1(s) - y_2(s)| g'(s) ds \\
&\leq \frac{(g(T) - g(a))^\beta}{\rho^\beta \Gamma(\beta + 1)} \|y_1 - y_2\|_{PC}. \qquad (3.12)
\end{aligned}$$

Also, since g is monotonic increasing, then $\forall t > a,\ \rho \in (0, 1)$, we have

$$\left| e^{\frac{\rho-1}{\rho}(g(t)-g(a))} \right| < 1.$$

The following result is based on the Banach contraction mapping principle.

Theorem 3.2. *Assume that the hypotheses* **(H1)**-**(H3)** *are satisfied. If*

$$\Omega := \max\left\{ \frac{L_f(\Theta_1 + \Xi_1)}{\rho^\alpha \Gamma(\alpha + 1)}, \Xi_2 \left(L_k + \frac{L_f(\Theta_2 + \Xi_3)}{\rho^\alpha \Gamma(\alpha + 1)} \right) + \frac{L_f(\Theta_2 + \Xi_3)}{\rho^\alpha \Gamma(\alpha + 1)} \right\} < 1, \qquad (3.13)$$

then the non-instantaneous impulsive fractional integro-differential equation (1.5) has a unique solution on J.

Proof. We transform the problem of non-instantaneous impulsive fractional integro-differential equation (1.5) into a fixed point problem.

Define an operator $\mathcal{N} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$(\mathcal{N}y)(t)=\begin{cases} e^{\frac{\rho-1}{\rho}(g(t)-g(a))}\dfrac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}y_0 \\ +\dfrac{1}{\rho^{\alpha}\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}(g(t)-g(s))^{\alpha-1} f(s,y(s),{}_aI^{\beta,\rho,g}y(s))g'(s)ds, \\ t\in[a,t_1]; \\ \psi_k\left(t,y\left(t_k^+\right)\right),\ t\in(t_k,s_k],\ k=1,\cdots,m; \\ e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))}\left(\dfrac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\Big[\psi_k\left(s_k,y\left(t_k^+\right)\right) \\ -\dfrac{1}{\rho^{\alpha}\Gamma(\alpha)}\displaystyle\int_a^{s_k} e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))}(g(s_k)-g(s))^{\alpha-1}f(s,y(s),{}_aI^{\beta,\rho,g}y(s))g'(s)ds\Big] \\ +\dfrac{1}{\rho^{\alpha}\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}(g(t)-g(s))^{\alpha-1}f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)g'(s)ds, \\ t\in(s_k,t_{k+1}],k=1,\cdots,m. \end{cases} \tag{3.14}$$

Obviously, it is easy to see that the operator $\mathcal{N}$ is well defined according to the continuity hypotheses of f and ψ_k. Next, we shall show that $\mathcal{N}$ is a contraction.

Case I. For each $t \in [a,t_1]$ and each $y_1,y_2 \in PC(J,\mathbb{R})$, using (3.11) and (3.12), we have

$$\begin{aligned} &|(\mathcal{N}y_1)(t)-(\mathcal{N}y_2)(t)| \\ \le\ & \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t\left|e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\right|(g(t)-g(s))^{\alpha-1}\left|f\left(s,y_1(s),{}_aI^{\beta,\rho,g}y_1(s)\right)\right. \\ & \left.-f\left(s,y_2(s),{}_aI^{\beta,\rho,g}y_2(s)\right)\right|g'(s)ds \\ \le\ & \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t(g(t)-g(s))^{\alpha-1}L_f\left(|y_1(s)-y_2(s)|\right. \\ & \left.+|{}_aI^{\beta,\rho,g}y_1(s)-{}_aI^{\beta,\rho,g}y_2(s)|\right)g'(s)ds \\ \le\ & \frac{L_f}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t(g(t)-g(s))^{\alpha-1}\left(\|y_1-y_2\|_{PC}+\frac{(g(t_1)-g(a))^{\beta}}{\rho^{\beta}\Gamma(\beta+1)}\|y_1-y_2\|_{PC}\right)g'(s)ds \\ \le\ & \frac{L_f}{\rho^{\alpha}\Gamma(\alpha+1)}(\Theta_1+\Xi_1)\|y_1-y_2\|_{PC}. \end{aligned}$$

Case II. For each $t\in(t_k,s_k], k=1,\cdots,m$ and each $y_1,y_2\in PC(J,\mathbb{R})$, we obtain

$$|(\mathcal{N}y_1)(t)-(\mathcal{N}y_2)(t)|\le L_k\|y_1-y_2\|_{PC}.$$

Case III. For each $t \in (s_k, t_{k+1}], k = 1, \cdots, m$ and each $y_1, y_2 \in PC(J, \mathbb{R})$, using (3.12), we get

$$
\begin{aligned}
&\left|(\mathcal{N}y_1)(t) - (\mathcal{N}y_2)(t)\right| \\
&\leq \left|e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))}\left(\frac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\right| \Bigg[\left|\psi_k\left(s_k, y_1\left(t_k^+\right)\right) - \psi_k\left(s_k, y_2\left(t_k^+\right)\right)\right| \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{s_k} \left|e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))}\right| (g(s_k)-g(s))^{\alpha-1} \left|f\left(s, y_1(s), {}_aI^{\beta,\rho,g} y_1(s)\right)\right. \\
&\left.-f\left(s, y_2(s), {}_aI^{\beta,\rho,g} y_2(s)\right)\right| g'(s)ds\Bigg] \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{t} \left|e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\right| (g(t)-g(s))^{\alpha-1} \left|f\left(s, y_1(s), {}_aI^{\beta,\rho,g} y_1(s)\right)\right. \\
&\left.-f\left(s, y_2(s), {}_aI^{\beta,\rho,g} y_2(s)\right)\right| g'(s)ds \\
&\leq \Bigg[\left(\frac{(g(t_{k+1})-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} \\
&\times \left(L_k + \frac{L_f}{\rho^\alpha \Gamma(\alpha+1)}\left[(g(t_{k+1})-g(a))^\alpha + \frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^\beta \Gamma(\beta+1)}\right]\right) \\
&+ \frac{L_f}{\rho^\alpha \Gamma(\alpha+1)}\left[(g(t_{k+1})-g(a))^\alpha + \frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^\beta \Gamma(\beta+1)}\right]\Bigg] \|y_1 - y_2\|_{PC} \\
&\leq \left[\Xi_2\left(L_k + \frac{L_f(\Theta_2+\Xi_3)}{\rho^\alpha \Gamma(\alpha+1)}\right) + \frac{L_f(\Theta_2+\Xi_3)}{\rho^\alpha \Gamma(\alpha+1)}\right] \|y_1 - y_2\|_{PC}
\end{aligned}
$$

Therefore, one has

$$\|\mathcal{N}y_1 - \mathcal{N}y_2\|_{PC} \leq \Omega \|y_1 - y_2\|_{PC}.$$

Since, by (3.13), $\Omega < 1$. Then, the operator $\mathcal{N}$ is a contraction and there exists a unique solution $y \in PC(J, \mathbb{R})$ of the non-instantaneous impulsive fractional integro-differential equation (1.5). This completes the proof. □

Now, we prove the existence of solutions of the non-instantaneous impulsive fractional integro-differential equation (1.5) by applying the following Leray-Schauder nonlinear alternative.

Theorem 3.3. [5] **(Leray-Schauder nonlinear alternative)** *Let $\mathbb{E}$ be a Banach space, D a closed convex subset of $\mathbb{E}$ and $\mathcal{S} \subset D$ an open subset with $0 \in \mathcal{S}$. Then each continuous compact mapping $\mathcal{N} : \overline{\mathcal{S}} \to D$ has at least one of the following properties:*

i. *$\mathcal{N}$ has a fixed point in $\overline{\mathcal{S}}$, or*

ii. *there exists $w \in \partial\mathcal{S}$ (the boundary of $\mathcal{S}$ in D) and $\xi \in (0,1)$ with $w = \xi\mathcal{N}(w)$.*

Theorem 3.4. *Assume that the hypotheses* **(H4)**-**(H6)** *are satisfied.If*

$$\max\left\{\Theta_3\ell_1 + \Theta_4\ell_2,\ \aleph_1,\ \Xi_2\left[\aleph_1 + \Xi_4\ell_1 + \Xi_5\ell_2\right] + \Xi_4\ell_1 + \Xi_5\ell_2\right\} < 1. \tag{3.15}$$

Then the non-instantaneous impulsive fractional integro-differential equation (1.5) has at least one solution on J.

Proof. Let $\mathcal{N}$ be defined by (3.14) and $\mathcal{B}_r = \{y \in PC(J,\mathbb{R}) : \|y\|_{PC} \leq r\}$ be a closed convex subset of $PC(J,\mathbb{R})$, where

$$r \geq \max\left\{\frac{\Theta_3\ell_0}{1-(\Theta_3\ell_1+\Theta_4\ell_2)},\ \frac{\aleph_0}{1-\aleph_1},\ \frac{\Xi_2\left[\aleph_0+\Xi_4\ell_0\right]+\Xi_4\ell_0}{1-\left(\Xi_2\left[\aleph_1+\Xi_4\ell_1+\Xi_5\ell_2\right]+\Xi_4\ell_1+\Xi_5\ell_2\right)}\right\}. \tag{3.16}$$

The proof will be given in several steps.

Step 1. $\mathcal{N}$ is continuous.

Let y_n be a sequence such that $y_n \to y$ in $PC(J,\mathbb{R})$.

Case I. For each $t \in [a,t_1]$, we have

$$\begin{aligned}
&|(\mathcal{N}y_n)(t)-(\mathcal{N}y)(t)| \\
&\leq \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t \left|e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\right|(g(t)-g(s))^{\alpha-1}\left|f\left(s,y_n(s),{}_aI^{\beta,\rho,g}y_n(s)\right)\right. \\
&\left.-f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)\right|g'(s)ds \\
&\leq \frac{(g(t_1)-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\left\|f\left(\cdot,y_n(\cdot),{}_aI^{\beta,\rho,g}y_n(\cdot)\right)-f\left(\cdot,y(\cdot),{}_aI^{\beta,\rho,g}y(\cdot)\right)\right\|_{PC}.
\end{aligned}$$

Case II. For each $t \in (t_k,s_k], k=1,\cdots,m$, we get

$$|(\mathcal{N}y_n)(t)-(\mathcal{N}y)(t)| \leq \|\psi_k\left(\cdot,y_n(\cdot)\right)-\psi_k\left(\cdot,y(\cdot)\right)\|_{PC}.$$

Case III. For each $t \in (s_k,t_{k+1}], k=1,\cdots,m$, we obtain that

$$\begin{aligned}
&|(\mathcal{N}y_n)(t)-(\mathcal{N}y)(t)| \\
&\leq \left(\frac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\left[\left|\psi_k\left(s_k,y_n\left(t_k^+\right)\right)-\psi_k\left(s_k,y\left(t_k^+\right)\right)\right|\right. \\
&+\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^{s_k}(g(s_k)-g(s))^{\alpha-1}\left|f\left(s,y_n(s),{}_aI^{\beta,\rho,g}y_n(s)\right)\right. \\
&\left.\left.-f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)\right|g'(s)ds\right] \\
&+\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t(g(t)-g(s))^{\alpha-1}\left|f\left(s,y_n(s),{}_aI^{\beta,\rho,g}y_n(s)\right)\right. \\
&\left.-f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)\right|g'(s)ds \\
&\leq \left(\frac{(g(t_{k+1})-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\left[\|\psi_k\left(\cdot,y_n(\cdot)\right)-\psi_k\left(\cdot,y(\cdot)\right)\|_{PC}\right. \\
&\left.+\frac{(g(s_k)-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\left\|f\left(\cdot,y_n(\cdot),{}_aI^{\beta,\rho,g}y_n(\cdot)\right)-f\left(\cdot,y(\cdot),{}_aI^{\beta,\rho,g}y(\cdot)\right)\right\|_{PC}\right] \\
&+\frac{(g(t_{k+1})-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\left\|f\left(\cdot,y_n(\cdot),{}_aI^{\beta,\rho,g}y_n(\cdot)\right)-f\left(\cdot,y(\cdot),{}_aI^{\beta,\rho,g}y(\cdot)\right)\right\|_{PC}.
\end{aligned}$$

Since the functions f and ψ_k are continuous. Then, from above inequalities, we deduce that $\|\mathcal{N}y_n - \mathcal{N}y\|_{PC} \to 0$ as $n \to \infty$.
Step 2. $\mathcal{N}$ is uniformly bounded.
Case I. For each $t \in [a, t_1]$ and for any $y \in \mathcal{B}_r$, using (3.11) and Lemma 2.6, we have

$$
\begin{aligned}
|(\mathcal{N}y)(t)| &\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left| f\left(s, y(s), {}_aI^{\beta,\rho,g} y(s)\right)\right| g'(s)ds \\
&\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1 |y(s)| + \ell_2 |{}_aI^{\beta,\rho,g} y(s)|\right) g'(s)ds \\
&\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1 |y(s)| + \ell_2 \, {}_aI^{\beta,\rho,g} |y(s)|\right) g'(s)ds \\
&\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \Bigg(\ell_0 + \ell_1 \|y\|_{PC} \\
&+ \frac{\ell_2}{\rho^\beta \Gamma(\beta)} \int_a^s (g(s)-g(\tau))^{\beta-1} \|y\|_{PC} \; g'(\tau)d\tau \Bigg) g'(s)ds \\
&\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1 r + \frac{\ell_2 r}{\rho^\beta \Gamma(\beta+1)} (g(s)-g(a))^\beta\right) g'(s)ds \\
&\leq \Theta_3(\ell_0 + \ell_1 r) + \Theta_4 \ell_2 r \leq r.
\end{aligned}
$$

Case II. For each $t \in (t_k, s_k], k = 1, \cdots, m$, and for any $y \in \mathcal{B}_r$, we get

$$
\begin{aligned}
|(\mathcal{N}y)(t)| &\leq \left|\psi_k\left(t, y\left(t_k^+\right)\right)\right| \\
&\leq \aleph_0 + \aleph_1 |y(t_k^+)| \\
&\leq \aleph_0 + \aleph_1 \|y\|_{PC} \\
&\leq \aleph_0 + \aleph_1 r \leq r.
\end{aligned}
$$

Case III. For each $t \in (s_k, t_{k+1}], k = 1, \cdots, m$, and for any $y \in \mathcal{B}_r$, using (3.11) and Lemma 2.6, we obtain

$$
\begin{aligned}
|(\mathcal{N}y)(t)| &\leq \left(\frac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} \Bigg[\aleph_0 + \aleph_1 |y(t_k^+)| \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{s_k} (g(s_k)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1 |y(s)| + \ell_2 |{}_aI^{\beta,\rho,g} y(s)|\right) g'(s)ds \Bigg] \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1 |y(s)| + \ell_2 |{}_aI^{\beta,\rho,g} y(s)|\right) g'(s)ds \\
&\leq \left(\frac{(g(t_{k+1})-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} \left[\aleph_0 + \aleph_1 r + \frac{(g(t_{k+1})-g(a))^\alpha}{\rho^\alpha \Gamma(\alpha+1)}(\ell_0 + \ell_1 r) + \frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \ell_2 r\right] \\
&+ \frac{(g(t_{k+1})-g(a))^\alpha}{\rho^\alpha \Gamma(\alpha+1)} (\ell_0 + \ell_1 r) + \frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \ell_2 r \\
&\leq \Xi_2 \left[\aleph_0 + \aleph_1 r + \Xi_4(\ell_0 + \ell_1 r) + \Xi_5 \ell_2 r\right] + \Xi_4(\ell_0 + \ell_1 r) + \Xi_5 \ell_2 r \leq r.
\end{aligned}
$$

From the above three inequalities, using (3.16), we infer that $\|\mathcal{N}y\|_{PC} \le r$. Hence, the operator $\mathcal{N}$ maps bounded sets into bounded sets of $PC(J,\mathbb{R})$.

Step 3. $\mathcal{N}$ maps bounded sets into equicontinuous sets.

Case I. For the interval $t \in [a, t_1]$, $a \le \vartheta_1 < \vartheta_2 \le t_1$ and for any $y \in \mathcal{B}_r$, we have

$$\begin{aligned}
&|(\mathcal{N}y)(\vartheta_2) - (\mathcal{N}y)(\vartheta_1)| \\
&\le \left| e^{\frac{\rho-1}{\rho}(g(\vartheta_2)-g(a))} \frac{(g(\vartheta_2)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} - e^{\frac{\rho-1}{\rho}(g(\vartheta_1)-g(a))} \frac{(g(\vartheta_1)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} \right| |y_0| \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{\vartheta_1} \left| (g(\vartheta_2)-g(s))^{\alpha-1} - (g(\vartheta_1)-g(s))^{\alpha-1} \right| \\
&\times \left(\ell_0 + \ell_1|y(s)| + \ell_2\, {}_aI^{\beta,\rho,g}|y(s)|\right) g'(s)ds \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{\vartheta_1}^{\vartheta_2} (g(\vartheta_2)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1|y(s)| + \ell_2\, {}_aI^{\beta,\rho,g}|y(s)|\right) g'(s)ds \\
&\le \left| e^{\frac{\rho-1}{\rho}(g(\vartheta_2)-g(a))} \frac{(g(\vartheta_2)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} - e^{\frac{\rho-1}{\rho}(g(\vartheta_1)-g(a))} \frac{(g(\vartheta_1)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} \right| |y_0| \\
&+ \frac{\ell_0 + \ell_1 r}{\rho^\alpha \Gamma(\alpha+1)} \left(2\,(g(\vartheta_2)-g(\vartheta_1))^\alpha + |(g(\vartheta_2)-g(a))^\alpha - (g(\vartheta_1)-g(a))^\alpha|\right) \\
&+ \frac{\ell_2 r}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta+1)} \left(\int_a^{\vartheta_1} \left| (g(\vartheta_2)-g(s))^{\alpha-1} - (g(\vartheta_1)-g(s))^{\alpha-1} \right| \right. \\
&\times (g(s)-g(a))^\beta g'(s)ds \\
&\left. + \int_{\vartheta_1}^{\vartheta_2} (g(\vartheta_2)-g(s))^{\alpha-1} (g(s)-g(a))^\beta g'(s)ds \right) \to 0, \ as\ \vartheta_2 \to \vartheta_1.
\end{aligned}$$

Case II. For each $t \in (t_k, s_k], k = 1, \cdots, m$, $a \le \vartheta_1 < \vartheta_2 \le t_1$ and for any $y \in \mathcal{B}_r$, one has

$$|(\mathcal{N}y)(\vartheta_2) - (\mathcal{N}y)(\vartheta_1)| \ \le \ \left|\psi_k\left(\vartheta_2, y\left(t_k^+\right)\right) - \psi_k\left(\vartheta_1, y\left(t_k^+\right)\right)\right| \to 0, \ as\ \vartheta_2 \to \vartheta_1.$$

Case III. For each $t \in (s_k, t_{k+1}], k = 1, \cdots, m$, $a \le \vartheta_1 < \vartheta_2 \le t_1$ and for any $y \in \mathcal{B}_r$, using Lemma 2.6, one has

$$\begin{aligned}
&|(\mathcal{N}y)(\vartheta_2) - (\mathcal{N}y)(\vartheta_1)| \\
&\le \left| e^{\frac{\rho-1}{\rho}(g(\vartheta_2)-g(s_k))} \left(\frac{(g(\vartheta_2)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(g(\vartheta_1)-g(s_k))} \left(\frac{(g(\vartheta_1)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} \right| \\
&\times \Big[\left|\psi_k\left(s_k, y\left(t_k^+\right)\right)\right| + \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^{s_k} (g(s_k)-g(s))^{\alpha-1} \\
&\times \left(\ell_0 + \ell_1|y(s)| + \ell_2\, {}_aI^{\beta,\rho,g}|y(s)|\right) g'(s)ds \Big] \\
&+ \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^{\vartheta_1} \left| (g(\vartheta_2)-g(s))^{\alpha-1} - (g(\vartheta_1)-g(s))^{\alpha-1} \right| \\
&\times \left(\ell_0 + \ell_1|y(s)| + \ell_2\, {}_aI^{\beta,\rho,g}|y(s)|\right) g'(s)ds
\end{aligned}$$

$$+\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_{\vartheta_1}^{\vartheta_2}(g(\vartheta_2)-g(s))^{\alpha-1}\left(\ell_0+\ell_1|y(s)|+\ell_2\ {}_aI^{\beta,\rho,g}|y(s)|\right)g'(s)ds$$

$$\leq\left|e^{\frac{\rho-1}{\rho}(g(\vartheta_2)-g(s_k))}\left(\frac{(g(\vartheta_2)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}-e^{\frac{\rho-1}{\rho}(g(\vartheta_1)-g(s_k))}\left(\frac{(g(\vartheta_1)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\right|$$

$$\times\left[|\psi_k\left(s_k,y\left(t_k^+\right)\right)|+\frac{(g(s_k)-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}(\ell_0+\ell_1 r)+\frac{(g(s_k)-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2 r\right]$$

$$+\frac{\ell_0+\ell_1 r}{\rho^\alpha\Gamma(\alpha+1)}\left(2\left(g(\vartheta_2)-g(\vartheta_1)\right)^\alpha+|(g(\vartheta_2)-g(a))^\alpha-(g(\vartheta_1)-g(a))^\alpha|\right)$$

$$+\frac{\ell_2 r}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta+1)}\left(\int_a^{\vartheta_1}|(g(\vartheta_2)-g(s))^{\alpha-1}-(g(\vartheta_1)-g(s))^{\alpha-1}|(g(s)-g(a))^\beta g'(s)ds\right.$$

$$\left.+\int_{\vartheta_1}^{\vartheta_2}(g(\vartheta_2)-g(s))^{\alpha-1}(g(s)-g(a))^\beta g'(s)ds\right)\to 0,\ as\ \vartheta_2\to\vartheta_1.$$

In view of the above three inequalities, we infer that $\|(\mathcal{N}y)(\vartheta_2)-(\mathcal{N}y)(\vartheta_1)\|_{PC}\to 0$ independently of $y\in\mathcal{B}_r$, as $\vartheta_2\to\vartheta_1$. Consequently, the operator $\mathcal{N}$ is equicontinuous and uniformly bounded. Hence, by Arzelà-Ascoli Theorem, the operator $\mathcal{N}:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is is completely continuous.

Step 4. We show that there exists an open set $\mathcal{S}\subset PC(J,\mathbb{R})$ with $y\neq\xi\mathcal{N}y$ for $\xi\in(0,1)$ and $y\in\partial\mathcal{S}$.

In other words, we shall show that the part **(i)** in Theorem 3.3 is verified.

Consider the equation $y=\xi\ \mathcal{N}y$, for $\xi\in(0,1)$. Then, in view of **Step 2**, we have the following cases:

Case I. For the interval $t\in[a,t_1]$, we have

$$\begin{aligned}|y(t)| &= |\xi\ \mathcal{N}y(t)|\\ &\leq \frac{(g(t_1)-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}(\ell_0+\ell_1\|y\|_{PC})+\frac{(g(t_1)-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2\|y\|_{PC},\end{aligned}$$

which implies that:

$$\frac{\|y\|_{PC}}{\Theta_3(\ell_0+\ell_1\|y\|_{PC})+\Theta_4\ell_2\|y\|_{PC}}\leq 1. \tag{3.17}$$

Case II. For each $t\in(t_k,s_k], k=1,\cdots,m$, one has

$$\begin{aligned}|y(t)| &= |\xi\ \mathcal{N}y(t)|\\ &\leq \aleph_0+\aleph_1\|y\|_{PC},\end{aligned}$$

which implies that:

$$\frac{\|y\|_{PC}}{\aleph_0+\aleph_1\|y\|_{PC}}\leq 1. \tag{3.18}$$

Case III. For each $t \in (s_k, t_{k+1}], k = 1, \cdots, m$, one has

$$\begin{aligned}
|y(t)| &= |\xi\, \mathcal{N}y(t)| \\
&\leq \left(\frac{(g(t_{k+1}) - g(a))}{(g(s_k) - g(a))}\right)^{\alpha-1} \Bigg[\aleph_0 + \aleph_1\|y\|_{PC} + \frac{(g(t_{k+1}) - g(a))^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}(\ell_0 + \ell_1\|y\|_{PC}) \\
&+ \frac{(g(t_{k+1}) - g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2\|y\|_{PC}\Bigg] \\
&+ \frac{(g(t_{k+1}) - g(a))^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}(\ell_0 + \ell_1\|y\|_{PC}) + \frac{(g(t_{k+1}) - g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2\|y\|_{PC},
\end{aligned}$$

which implies, by (3.11), that:

$$\frac{\|y\|_{PC}}{\Xi_2[\aleph_0+\aleph_1\|y\|_{PC}+\Xi_4(\ell_0+\ell_1\|y\|_{PC})+\Xi_5\ell_2\|y\|_{PC}]+\Xi_4(\ell_0+\ell_1\|y\|_{PC})+\Xi_5\ell_2\|y\|_{PC}} \leq 1. \tag{3.19}$$

By combining (3.17),(3.18) and (3.19) together with **(H6)**, there exists M such that:

$$M \neq \|y\|_{PC}.$$

Let us set

$$\mathcal{S} = \{y \in PC(J, \mathbb{R} : \|y\|_{PC} < M)\}.$$

Note that the operator $\mathcal{N} : \overline{\mathcal{S}} \to PC(J, \mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{S}$, there is no $y \in \partial\mathcal{S}$ such that $y = \xi\mathcal{N}y$ for $\xi \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 3.3), we deduce that $\mathcal{N}$ has a fixed point $y \in \overline{\mathcal{S}}$ which is a solution of (1.5). This completes the proof. □

4. Illustrative examples

Example 4.1. Consider the following non-instantaneous impulsive fractional problem:

$$\begin{cases}
{}_aD^{\frac{1}{2},2,t^2}y(t) = \dfrac{e^{-2t}\left(|y(t)|+\left|{}_{0^+}I^{\frac{3}{4},2,t^2}y(t)\right|\right)}{(1+7e^t)\left(1+|y(t)|+\left|{}_{0^+}I^{\frac{3}{4},2,t^2}y(t)\right|\right)}, \ t \in (0, \frac{1}{3}] \cup (\frac{2}{3}, 1], \\[2ex]
y(t) = \dfrac{|y(\frac{1}{3}^+)|}{(3+7e^{2t})(1+|y(\frac{1}{3}^+)|)}, \ t \in (\frac{1}{3}, \frac{2}{3}], \\[2ex]
{}_{0^+}I^{\frac{1}{2},2,t^2}y(0) = 0,
\end{cases} \tag{4.1}$$

Here, $J = [0,1]$, $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$, and $\alpha = \frac{1}{2}, \beta = \frac{3}{4}, \rho = 1, m = 1$. Set

$$g(t) = t^2, \quad f(t,u,v) = \frac{e^{-2t}\,(|u| + |v|)}{(1+7e^t)\,(1+|u|+|v|)}$$

and

$$\psi_1(t,u) = \frac{|u|}{(3+7e^{2t})(1+|u|)}.$$

Let $u_i, v_i \in \mathbb{R}, i = 1, 2$ and $t \in [0, \frac{1}{3}] \cup (\frac{2}{3}, 1]$. Then, we get

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{1}{8}\left(|u_1 - u_2| + |v_1 - v_2|\right).$$

Let $u_1, u_2 \in \mathbb{R}$ and $t \in (\frac{1}{3}, \frac{2}{3}]$. Then, we obtain

$$|\psi_1(t, u_1) - \psi_1(t, u_2)| \leq \frac{1}{10}|u_1 - u_2|.$$

Thus, the hypotheses **(H1)**, **(H2)** and **(H3)** in Theorem 3.2 are satisfied with $L_f = \frac{1}{8}$ and $L_k = L_1 = \frac{1}{10}$. Therefore, by (3.13), one can deduce that:

$$\Omega = \max\{0.0568660825, 0.55752695\} = 0.55752695 < 1.$$

Hence, the non-instantaneous impulsive fractional problem (4.1) has a unique solution on $[0, 1]$.

Example 4.2. Consider

$$\begin{cases} {}_aD^{\frac{1}{2},2,t^2}y(t) = \frac{\sin t}{5\sqrt{9+t^2}} + \frac{|y(t)|}{10e^t(1+|y(t)|)} + \frac{\left|{}_{0^+}I^{\frac{3}{4},2,t^2}y(t)\right|}{25+t^2}, \ t \in (0, \frac{1}{3}] \cup (\frac{2}{3}, 1], \\ y(t) = \frac{e^{-t}}{16+t^4} + \frac{\cos y\left(\frac{1}{3}^+\right)}{4\sqrt{49+t^2}}, \ t \in (\frac{1}{3}, \frac{2}{3}], \\ {}_{0^+}I^{\frac{1}{2},2,t^2}y(0) = 0, \end{cases} \tag{4.2}$$

Here, $J = [0, 1]$, $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$, and $\alpha = \frac{1}{2}, \beta = \frac{3}{4}, \rho = 1, m = 1$. Set

$$g(t) = t^2, \quad f(t, u, v) = \frac{\sin t}{5\sqrt{9+t^2}} + \frac{|u|}{10e^t(1+|u|)} + \frac{|v|}{25+t^2}$$

and

$$\psi_1(t, u) = \frac{e^{-t}}{16+t^4} + \frac{\cos u}{4\sqrt{49+t^2}}.$$

For all $u, v \in \mathbb{R}$ and each $t \in [0, \frac{1}{3}] \cup (\frac{2}{3}, 1]$, we get

$$|f(t, u, v)| \leq \frac{1}{15} + \frac{1}{10}|u| + \frac{1}{25}|v|.$$

For all $u \in \mathbb{R}$ and each $t \in (\frac{1}{3}, \frac{2}{3}]$, we get

$$|\psi_1(t, u)| \leq \frac{1}{16} + \frac{1}{28}|u|.$$

Thus, the hypotheses **(H4)** and **(H5)** hold with $\ell_0 = \frac{1}{15}, \ell_1 = \frac{1}{10}, \ell_2 = \frac{1}{25}, \aleph_0 = \frac{1}{16}$ and $\aleph_1 = \frac{1}{28}$. Moreover, from (3.15), we get

$$\begin{aligned} &\max\left\{\Theta_3\ell_1 + \Theta_4\ell_2, \ \aleph_1, \ \Xi_2\left[\aleph_1 + \Xi_4\ell_1 + \Xi_5\ell_2\right] + \Xi_4\ell_1 + \Xi_5\ell_2\right\} \\ &= \{0.039877417, 0.03571428571, 0.500165849\} \\ &= 0.0.500165849 < 1. \end{aligned}$$

By Theorem 3.4, we conclude that our theoretical results are applicable to the problem (4.2).

References

[1] Abbas, M.I., *On the initial value problems for the Caputo-Fabrizio impulsive fractional differential equations*, Asian-Eur. J. Math., (2021), 2150073, DOI: 10.1142/S179355712150073X.

[2] Abbas, M.I., *Ulam stability of fractional impulsive differential equations with Riemann-Liouville integral boundary conditions*, J. Contemp. Math. Anal., **50**(5)(2015), 209-219.

[3] Agarwal, R., Hristova, S., O'Regan, D., *Iterative techniques for the initial value problem for Caputo fractional differential equations with non-instantaneous impulses*, Appl. Math. Comput., **334**(2018), 407-421.

[4] Anderson, D.R., Ulness, D.J., *Newly defined conformable derivatives*, Adv. Dyn. Syst. Appl., **10**(2)(2015), 109-137.

[5] Dugundji, J., Granas, A., *Fixed Point Theory*, Warsaw, 1982.

[6] Hernándaz, E., O'Regan, D., *On a new class of abstract impulsive differential equation*, Proc. Amer. Math. Soc., **141**(2013), 1641-1649.

[7] Jarad, F., Abdeljawad, T., Rashid, S., Hammouch, Z., *More properties of the proportional fractional integrals and derivatives of a function with respect to another function*, Adv. Differential Equations, (2020), 2020:303.

[8] Jarad, F., Alqudah, M.A., Abdeljawad, T., *On more general forms of proportional fractional operators*, Open Math., **18**(2020), 167-176.

[9] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., *A new definition of fractional derivative*, J. Comput. Appl. Math., **264**(2014), 65-70.

[10] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier B. V., Amsterdam, 2006.

[11] Kumar, A., Chauhan, H.V., Ravichandran, C., Nisar, K.S., Baleanu, D., *Existence of solutions of non-autonomous fractional differential equations with integral impulse condition*, Adv. Differential Equations, (2020), 2020:434.

[12] Kumar, A., Pandey, D.N., *Existence of mild solution of Atangana-Baleanu fractional differential equations with non-instantaneous impulses and with non-local conditions*, Chaos Solitons Fractals, **132**(2020), 109551.

[13] Liu, K., Wang, J., O'Regan, D., Fečkan, M., *A new class of (ω, c)-periodic non-instantaneous impulsive differential equations*, Mediterr. J. Math., (2020), **17**:155.

[14] Luo, D., Luo, Z., *Existence of solutions for fractional differential inclusions with initial value condition and non-instantaneous impulses*, Filomat, **33**(17)2019), 5499-5510.

[15] Myshkis, A.D., Samoilenko, A.M., *Systems with impulsive at fixed moments of time*, Mat. Sb., **74**(1967), 202-208.

[16] Ngo, V.H., O'Regan, D., *A remark on ψ-Hilfer fractional differential equations with non-instantaneous impulses*, Math. Methods Appl. Sci., (2020), 1-15.

[17] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[18] Samko, S., Kilbas, A., Marichev, O., *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, Longhorne, PA, 1993.

[19] Srivastava, H.M., Saad, K.M., *Some new models of the time-fractional gas dynamics equation*, Adv. Math. Models Appl., **3**(1)(2018), 5-17.

[20] Zada, A., Ali, N., Riaz, U., *Ulam's stability of multi-point implicit boundary value problems with non-instantaneous impulses*, Boll. Unione Mat. Ital., **13**(2020), 305-328.

[21] Zhao, Y., Luo, C., Chen, H., *Existence results for non-instantaneous impulsive nonlinear fractional differential equation via variational methods*, Bull. Malays. Math. Sci. Soc., **43**(2020), 2151-2169.

Mohamed I. Abbas
Department of Mathematics and Computer Science,
Faculty of Science, Alexandria University,
Alexandria 21511, Egypt
e-mail: `miabbas@alexu.edu.eg, miabbas77@gmail.com`