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Fixed point theorems for maps on cones in Fréchet spaces via the projective limit approach

Marlène Frigon

Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. We present fixed point results for admissibly compact maps on cones in Fréchet spaces. We first extend the Krasnosel'skiĭ fixed point theorem with order type cone-compression and cone-expansion conditions. Then, we extend the monotone iterative method to this context. Finally, we present fixed point results under a combination of the assumptions of the previous results. More precisely, we combine a cone-compressing or cone-extending condition only on one side of the boundary of an annulus with an assumption on the existence of an upper fixed point. In addition, we show that the usual monotonicity condition can be weaken.

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1. Introduction

The classical Krasnosel'skiĭ fixed point theorem is very well known and useful, see [13, 14]. Assuming cone-compression and cone-expansion conditions on the boundary of two nested bounded, neighborhoods of the origin relative to a cone, it establishes the existence of nontrivial fixed points of maps on cones in Banach spaces. Two types of cone-compression and cone-expansion conditions were considered: one involving the norm and the other involving the order on the space induced by the cone. This result was extended to Fréchet spaces in [1, 2, 12] using the fact that a Fréchet space is the projective limit of a sequence of Banach spaces. All those generalizations rely on at least one cone-compression condition involving the norm of the values of maps on the relative boundary of suitable bounded, open sets in those Banach spaces.

On the other hand, the monotone iterative method is often applied to deduce the existence of fixed points of nondecreasing maps f defined on closed intervals $[\alpha, \beta]$ in ordered Banach spaces, where α is a *lower fixed point of* f (i.e. $\alpha \leq f(\alpha)$) and β is an *upper fixed point of* f (i.e. $f(\beta) \leq \beta$). The fixed points are obtained as the limits of iterative sequences. This method was introduced by Amann [3] for single-valued maps and extended to multivalued maps in [7].

In a series of papers, Cabada, Cid, Infante and their collaborators (see [4, 5, 6, 8, 10]) obtained many fixed point theorems on cones in Banach spaces by imposing cone-compression or cone-extension conditions on the boundary relative to a cone of only one bounded, neighborhood of the origin instead of two. The usual second condition was replaced by assuming that the map f is nondecreasing (or nonincreasing) on a suitable shell and by assuming the existence of an upper fixed point (or a lower fixed point) instead of assuming the existence of both as in the monotone iterative method.

In this paper, we present fixed point results for maps on cones in Fréchet spaces. In section 3, we extend the Krasnosel'skiĭ fixed point theorem with order type cone-compression and cone-expansion conditions instead of norm-type conditions. Our results will rely on the fixed point index theory for multivalued mapping in cones obtained by Fitzpatrick and Petryshyn [9].

In section 4, we extend the monotone iterative method to Fréchet spaces. In addition, we show that the monotonicity condition can be dropped. In that case, the existence of a fixed point is still insured but some precision on its localization is lost.

Finally, in the last section, existence results are presented relying on one cone-compression or cone-expansion condition combined with one condition of the type upper fixed point or lower fixed point. It is not assumed that the cone is normal or solid. Also, a condition weaker than monotonicity is imposed. Therefore, even in the particular case where the Fréchet space is a Banach space, our results generalize theorems due to Cabada, Cid and Infante [6].

Using the fact that a Fréchet space is the projective limit of a sequence of Banach spaces, our results are presented for admissibly compact maps. This notion was introduced in [11]. It is worthwhile to mention that our results could have been presented for admissibly condensing maps or admissible maps satisfying a Leggett-William type condition as in [1]. We first present some preliminaries on the fixed point index for multivalued maps on closed, convex sets, then on Fréchet spaces, and finally on admissibly compact maps.

2. Preliminaries

2.1. Fixed point index

In all this text, E denotes a Fréchet space endowed with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let X, Y be subsets of E and $F : X \to Y$ a multivalued map with non-empty closed values. The map F is *compact* if $F(X)$ is relatively compact in Y; it is *completely continuous* if $F(B)$ is relatively compact in Y for every $B \subset X$ bounded. It is *upper semi-continuous* (u.s.c.) if $\{x \in X : F(x) \cap A \neq \emptyset\}$ is closed in Y for every A closed in X.

Let C be a closed, convex set in E. For U a nonempty, open set in E, we denote $U_C = U \cap C$, $\overline{U}_C = \overline{U} \cap C$ and $\partial_C U = \overline{U}_C \backslash U_C$ the boundary of U in C.

In [9], Fitzpatrick and Petryshyn defined a fixed point index for upper semi-continuous, condensing, multivalued maps $F : \overline{U}_C \to C$ with nonempty, convex, compact values such that F has no fixed point on $\partial_C U$. This fixed point index is denoted $i_C(F, U)$. Here is their Theorem 2.1 in the particular case of compact maps.

Theorem 2.1 ([9]). *Let $F : \overline{U}_C \to C$ be a compact, u.s.c., multivalued map with nonempty, convex, compact values and such that $x \notin F(x)$ for all $x \in \partial_C U$. Then, the following statements hold:*

(1) *If $i_C(F, U) \neq 0$, then F has a fixed point.*
(2) *If $x_0 \in U_C$, then $i_C(\{x_0\}, U) = 1$, where $\{x_0\}$ denotes the constant map.*
(3) *If $U = U_1 \cup U_2$, where U_1 and U_2 are disjoint open sets and are such that $x \notin F(x)$ if $x \in \partial_C U_1 \cup \partial_C U_2$, then*

$$i_C(F, U) = i_C(F, U_1) + i_C(F, U_2).$$

(4) *If $H : [0,1] \times \overline{U}_C \to C$ is a compact, u.s.c., multivalued map with nonempty, convex, compact values and such that $x \notin H(t,x)$ for all $t \in [0,1]$ and $x \in \partial_C U$, then*

$$i_C(H(1,\cdot), U) = i_C(H(0,\cdot), U).$$

By K, we denote a *cone* in E; that is a closed set such that, for every $x, y \in K$ and every $\lambda, \delta \geq 0$, $\lambda x + \delta y \in K$ and $K \cap (-K) = \{0\}$. A cone K is called *normal* if, for every $n \in \mathbb{N}$, there exists $c_n \geq 1$ such that

$$\|x\|_n \leq c_n \|y\|_n \quad \text{for every } x, y \in K \text{ such that } y - x \in K.$$

Fitzpatrick and Petryshyn [9] obtained the following Krasnosel'skiĭ type fixed point result which relied on the previous theorem in the particular case where the closed, convex set is a cone. Using the fact that a Fréchet space is metrizable, they considered d a metric on E generating the same topology. For $r > 0$, let

$$B_d(x_0, r) = \{x \in E : d(x, x_0) < r\} \quad \text{and} \quad \overline{B_d(x_0, r)} = \{x \in E : d(x, x_0) \leq r\}.$$

Again, their theorem is stated for compact maps instead of condensing maps.

Theorem 2.2 ([9]). *Let $r_1, r_2 \in (0, \infty)$, $r = \min\{r_1, r_2\}$ and $R = \max\{r_1, r_2\}$. Let K be a cone in E and $F : \overline{B_d(0,R)} \cap K \to K$ a compact, u.s.c., multivalued map with nonempty, convex, compact values satisfying the following conditions:*

(i) *$\big(F(x) - x\big) \subset K$ if $x \in \partial_K B_d(0, r_1)$;*
(ii) *$\big(x - F(x)\big) \subset K$ if $x \in \partial_K B_d(0, r_2)$;*
(iii) *there exists a continuous semi-norm p, non-vanishing on K, such that $(I - F)(\overline{B_d(0, r_1)} \cap K)$ is p-bounded.*

Then, F has a fixed point $x_0 \in \overline{B_d(0,R)} \backslash B_d(0, r)$.

It could be difficult to apply this result to deduce the existence of solutions to differential or integral equations on unbounded intervals. Indeed, in general, the operator associated to the problem will not be compact on open sets. The problem is that open sets in Fréchet spaces are big.

Let us give an example. Let $C(\mathbb{R})$ be the space of continuous functions on the real line and, for $n \in \mathbb{N}$, the semi-norm

$$\|x\|_n = \max_{t\in[-n,n]} |x(t)|.$$

Endowed with the family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$, $C(\mathbb{R})$ is a Fréchet space. Let $U \subset C(\mathbb{R})$ be a neighborhood of 0. Then, there exist $n_0 \in \mathbb{N}$ and $r > 0$ such that

$$\{x \in C(\mathbb{R}) : \|x\|_{n_0} < r\} \subset U.$$

Also, in this context, it could be more difficult to get non trivial fixed points. For example, let

$$B(0,r) = \{x \in C(\mathbb{R}) : |x(t)| < r \ \forall t \in \mathbb{R}\}.$$

From the previous remark, $B(0,r)$ has empty interior. Therefore, there exists a sequence $\{x_n\}$ in $C(\mathbb{R})$ such that $x_n \to 0$ and $\|x_n\|_n \geq r$ for every $n \in \mathbb{N}$.

2.2. Fréchet spaces and projective limits

For sake of completeness, we recall some notations and properties of Fréchet spaces presented in [11].

Let E be a Fréchet space with the topology generated by a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. In what follows, we will always assume that the following condition is satisfied:

$$\|x\|_1 \leq \|x\|_2 \leq \cdots \quad \text{for every } x \in E. \tag{2.1}$$

For $\hat{x} \in E$, $r > 0$, $R = (r_1, r_2, \dots) \in (0,\infty)^{\mathbb{N}}$ and $n \in \mathbb{N}$, we denote

$$\begin{aligned}
B_n(\hat{x}, r) &= \{x \in E : \|x - \hat{x}\|_n < r\},\\
\overline{B_n(\hat{x}, r)} &= \{x \in E : \|x - \hat{x}\|_n \leq r\},\\
B(\hat{x}, R) &= \{x \in E : \|x - \hat{x}\|_n < r_n \ \forall n \in \mathbb{N}\},\\
\overline{B(\hat{x}, R)} &= \{x \in E : \|x - \hat{x}\|_n \leq r_n \ \forall n \in \mathbb{N}\}.
\end{aligned}$$

For $X \subset E$ and $n \in \mathbb{N}$, we denote by diam_n, the n-diameter of X induced by $\|\cdot\|_n$; that is,

$$\operatorname{diam}_n(X) = \sup\{\|x - y\|_n : x, y \in X\} \in [0,\infty) \cup \{\infty\}.$$

We say that X is *bounded* if there exists $R \in (0,\infty)^{\mathbb{N}}$ such that $X \subset B(0,R)$; so, $\operatorname{diam}_n(X) < \infty$ for every $n \in \mathbb{N}$.

Remark 2.3. Observe that if E is not a Banach space, then

(1) an open set in E is never bounded;
(2) a bounded set in E has empty interior.

The space E is the projective limit of a sequence of Banach spaces $\{E_n\}$. Indeed, for each $n \in \mathbb{N}$, we write

$$x \sim_n y \quad \text{if and only if} \quad \|x - y\|_n = 0. \tag{2.2}$$

This defines an equivalence relation on E. We denote by E_n the completion of the quotient space $E/{\sim_n}$ with respect to $\|\cdot\|_n$ (the norm on $E/{\sim_n}$ induced by $\|\cdot\|_n$ and its extension to E_n are still denoted by $\|\cdot\|_n$). This construction defines a continuous map $\mu_n : E \to E_n$ such that

$$\mu_n(x) = [x]_n, \quad \text{(i.e. } \mu_n(x) = \mu_n(y) \quad \Longleftrightarrow \quad x \sim_n y).$$

Similarly, for every $m \geq n$, we can define an equivalence relation on E_m, still noted $\sim_n$, which defines a continuous map $\mu_{n,m} : E_m \to E_n$ since $E_m/{\sim_n}$ can be regarded as a subset of E_n. So, E is the projective limit of $\{E_n\}$.

For each subset $X \subset E$ and each $n \in \mathbb{N}$, we set $X_n = \mu_n(X)$, and we denote $\overline{X}_n$, and $\partial_n X_n$, respectively the closure and the boundary of X_n with respect to $\|\cdot\|_n$ in E_n.

The following lemma gives an important property of closed subsets of E.

Lemma 2.4 ([11]). *Let E be a Fréchet space endowed with a family of semi-norms satisfying* (2.1)*, and let X be a closed subset of E. Then, for every sequence $\{z_n\}$ with $z_n \in \overline{X}_n$, such that for every $n \in \mathbb{N}$, $\{\mu_{n,m}(z_m)\}_{m\geq n}$ is a Cauchy sequence in $\overline{X}_n$, there exists $x \in X$ such that $\{\mu_{n,m}(z_m)\}_{m\geq n}$ converges to $\mu_n(x) \in X_n$ for every $n \in \mathbb{N}$.*

For every $n \in \mathbb{N}$, let $A(n) \subset E_n$. We define

$$\begin{aligned}\operatorname*{Lim}_{n\to\infty} A(n) = \{x \in E :\ & \exists N_0 \subset \mathbb{N} \text{ infinite and } z_n \in A(n) \text{ for } n \in N_0 \\ & \text{such that } \ \forall\, n \in \mathbb{N}, \quad \mu_{n,m}(z_m) \to \mu_n(x) \\ & \text{as } m \to \infty \text{ with } m \in N_0 \text{ and } m \geq n\}.\end{aligned} \tag{2.3}$$

Notice that if X is closed, then

$$X = \operatorname*{Lim}_{n\to\infty} \overline{X}_n.$$

Taking into account the fact that many applications in Fréchet spaces lead to look for solutions in a closed set with empty interior, the notion of pseudo-interior was introduced in [11].

Definition 2.5. Let X be a subset of E. The *pseudo-interior* of X is defined by

$$\text{pseudo-int}(X) = \{x \in X : \mu_n(x) \in \overline{X}_n \backslash \partial X_n \ \text{ for every } n \in \mathbb{N}\}.$$

The set X is *pseudo-open* if $X = \text{pseudo-int}(X)$.

For $n \in \mathbb{N}$, let C_n be a closed, convex set in E_n. In what follows, the topology in C_n induced by $\|\cdot\|_n$ will play a key role. So, we introduce the following notation. Let U be a nonempty pseudo-open set in E, we denote

$$U_{C_n} = U_n \cap C_n, \quad \overline{U}_{C_n} = \overline{U}_n \cap C_n \quad \text{and} \quad \partial_{C_n} U_n = \overline{U}_{C_n} \backslash U_{C_n} = \big(\overline{U}_n \backslash U_n\big) \cap C_n.$$

2.3. Admissibly compact maps

Here is the notion of admissibly compact maps introduced in [11].

Definition 2.6. Let $X \subset E$ and C closed and convex in E. A map $f : X \to C$ is called *admissibly compact* if it satisfies the following properties for every $n \in \mathbb{N}$:

(i) The multivalued map $\widehat{F}_n : X_n \to \overline{C}_n$ defined by

$$\widehat{F}_n(\mu_n(x)) = \overline{\mathrm{co}}\Big(\mu_n\big(f(\{x\}_{n,X})\big)\Big),$$

admits an upper semi-continuous compact extension $F_n : \overline{X}_n \to \overline{C}_n$ with convex, compact values, where

$$\{x\}_{n,X} = \{y \in X : \mu_n(y) = \mu_n(x)\} = \mu_n^{-1}\big([x]_n\big) \cap X.$$

(ii) For every $\varepsilon > 0$, there exists $m \geq n$ such that, for every $x \in X$,

$$\mathrm{diam}_n\Big(f\big(\{x\}_{m,X}\big)\Big) < \varepsilon.$$

A map $f : X \to C$ is called *admissibly completely continuous* if it is admissibly compact on every bounded sets in X.

The following proposition will play a key role in the proof of the forthcoming fixed point theorems.

Proposition 2.7. *Let $X \subset E$ be closed, $C \subset E$ closed, convex, and $f : X \to C$ an admissibly compact map. Assume that there exists $N_0 \subset \mathbb{N}$ infinite such that, for every $n \in N_0$, there exists $z_n \in \overline{X}_n$ such that $z_n \in F_n(z_n)$. Then, f has a fixed point.*

Proof. For $m \in N_0$, F_m has a fixed point $z_m \in \overline{X}_m$. From the definition of F_n, one sees that

$$\mu_{n,m}(z_m) \in F_n(\mu_{n,m}(z_m)) \quad \text{for every } n \leq m.$$

Thus, without lost of generality, we can assume that $N_0 = \mathbb{N}$.

The compactness of F_1 implies that the sequence $\{\mu_{1,k}(z_k)\}_{k\geq 1}$ has a subsequence $\{\mu_{1,k}(z_k)\}_{k\in N_1}$ converging to some $x_1 \in \overline{X}_1$. It follows from the upper semi-continuity of F_1 that $x_1 \in F_1(x_1)$.

Similarly, the sequence $\{\mu_{2,k}(z_k)\}_{k\in N_1}$ has a subsequence $\{\mu_{2,k}(z_k)\}_{k\in N_2}$ converging to $x_2 \in \overline{X}_2$, with $x_2 \in F_2(x_2)$. The uniqueness of the limit implies that $\mu_{1,2}(x_2) = x_1$.

Repeating this argument gives, for every $n \in \mathbb{N}$, the existence of $x_n \in \overline{X}_n$ such that $x_n \in F_n(x_n)$ and $\mu_{n,m}(x_m) = x_n$ for every $m \geq n$. It follows from Lemma 2.4 that there exists $x \in X$ such that $\mu_n(x) \in F_n(\mu_n(x))$ for every $n \in \mathbb{N}$.

We have to show that $x = f(x)$. If this is false, there exist $n \in \mathbb{N}$ and $r > 0$ such that $\|x - f(x)\|_n = r$. Let $\varepsilon < r/2$. By Definition 2.6(ii), there exists $m \geq n$ such that $\mathrm{diam}_n\Big(f\big(\{x\}_{m,X}\big)\Big) < \varepsilon$. Observe that

$$\mathrm{diam}_n\Big(f\big(\{x\}_{m,X}\big)\Big) = \mathrm{diam}_n\Big(\mathrm{co}\Big(f\big(\{x\}_{m,X}\big)\Big)\Big).$$

On the other hand, since $\mu_m(x) \in F_m(\mu_m(x))$, there is $w \in \mathrm{co}\big(f\big(\{x\}_{m,X}\big)\big)$ such that $\|x - w\|_m < \varepsilon$. Thus,

$$\begin{aligned} r = \|x - f(x)\|_n &\leq \|x - w\|_n + \|w - f(x)\|_n \\ &< \|x - w\|_m + \mathrm{diam}_n\Big(\mathrm{co}\Big(f\big(\{x\}_{m,X}\big)\Big)\Big) < 2\varepsilon < r. \end{aligned}$$

Thus, $x = f(x)$. □

3. Krasnosel'skiĭ type fixed point results

In this section, we present Krasnosel'skiĭ type fixed point results with order-type cone-compressing and cone-extending conditions on the pseudo-boundary of bounded sets in E.

Let us first recall the following two fixed point results obtained in [12] for admissibly completely continuous maps in Fréchet spaces satisfying norm-type cone-compressing and cone-extending type conditions. Notice that, for K a cone in E, one has that $\overline{K}_n$ is a cone in E_n for every $n \in \mathbb{N}$.

Theorem 3.1 ([12]). *Let $f : K \to K$ be an admissibly completely continuous map. Assume that there exist U, V two bounded, pseudo-open subsets of E satisfying the following conditions for every $n \in \mathbb{N}$:*

(i) $\|y\|_n \geq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} U_n$
(resp. $\|y\|_n \leq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} U_n$*);*
(ii) $\|y\|_n \leq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} V_n$
(resp. $\|y\|_n \geq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} V_n$*);*
(iii) $0 \in \overline{U}_n \backslash \partial_n U_n \subset \overline{U}_n \subset \overline{V}_n \backslash \partial_n V_n$ *for every* $n \in \mathbb{N}$.

Then, there exists x a fixed point of f such that

$$x \in \operatorname*{Lim}_{n\to\infty} A(n),$$

where $A(n) = \overline{K}_n \cap \overline{V_n \backslash U_n}$ *and* $\mathrm{Lim}_{n\to\infty} A(n)$ *is defined in* (2.3).

In the particular case where U and V are pseudo-balls, the previous result can be stated as follows.

Corollary 3.2 ([12]). *Let $f : K \to K$ be an admissibly completely continuous map. Assume that there exist $\{r_{1,n}\}$ and $\{r_{2,n}\}$ nondecreasing sequences in $(0, \infty)$ such that, for every $n \in \mathbb{N}$,*

(i) $\|y\|_n \geq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} B_n(0, r_{1,n})$*;*
(ii) $\|y\|_n \leq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} B_n(0, r_{2,n})$*;*
(iii) $r_{1,n} \neq r_{2,n}$.

Then, there exists x a fixed point of f such that

$$x \in \operatorname*{Lim}_{n\to\infty} \overline{K}_n \cap \overline{B_n(0, R_n) \backslash B_n(0, r_n)},$$

where $R_n = \max\{r_{1,n}, r_{2,n}\}$ *and* $r_n = \min\{r_{1,n}, r_{2,n}\}$.

Analogous results can be obtained if the norm-type cone-compressing and cone-extending conditions are replaced by order-type conditions.

Theorem 3.3. *Let $f : K \to K$ be an admissibly completely continuous map. Assume that there exist U, V two bounded, pseudo-open subsets of E satisfying the following conditions for every $n \in \mathbb{N}$:*

(i) $\big(F_n(x) - x\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ *for all* $x \in \partial_{\overline{K}_n} U_n$
(resp. $\big(x - F_n(x)\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ *for all* $x \in \partial_{\overline{K}_n} U_n$*);*

(ii) $\big(x - F_n(x)\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ *for all* $x \in \partial_{\overline{K}_n} V_n$
(resp. $\big(F_n(x) - x\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ *for all* $x \in \partial_{\overline{K}_n} V_n$*);*

(iii) $0 \in \overline{U}_n \backslash \partial_n U_n \subset \overline{U}_n \subset \overline{V}_n \backslash \partial_n V_n$ *for every* $n \in \mathbb{N}$.

Then, there exists x *a fixed point of* f *such that*

$$x \in \operatorname*{Lim}_{n\to\infty} A(n),$$

where $A(n) = \overline{K}_n \cap \overline{V_n \backslash U_n}$.

Proof. For every $n \in \mathbb{N}$, we claim that

$$\exists z_n \in F_n(z_n) \quad \text{such that } z_n \in A(n). \tag{3.1}$$

If this is false, we define

$$H_n : [0,1] \times \overline{U}_{\overline{K}_n} \to \overline{K}_n \quad \text{by} \quad H_n(t,x) = tF_n(x).$$

For $x \in \partial_{\overline{K}_n} U_n$ and $t \in (0,1]$, $x \notin H_n(t,x)$. Otherwise,

$$\left(\frac{1}{t} - 1\right) x \in \big(F_n(x) - x\big) \cap \overline{K}_n,$$

which contradicts (i). It follows from (iii) and Theorem 2.1(2), (4) that

$$i_{\overline{K}_n}(F_n, U_n) = i_{\overline{K}_n}(0, U_n) = 1. \tag{3.2}$$

On the other hand, choose $\hat{u} \in \overline{K}_n$ such that

$$\|\hat{u}\|_n > \max\{\|x - y\|_n : x \in \overline{V}_{\overline{K}_n},\ y \in F_n(x)\}. \tag{3.3}$$

Such $\hat{u}$ exists since V_n and $F_n(\overline{V}_{\overline{K}_n})$ are bounded. Let us define

$$\widehat{H}_n : [0,1] \times \overline{V}_{\overline{K}_n} \to \overline{K}_n \quad \text{by} \quad \widehat{H}_n(t,x) = t\hat{u} + F_n(x).$$

By (ii), $x \notin \widehat{H}_n(t,x)$ for all $t \in [0,1]$ and $x \in \partial_{\overline{K}_n} V_n$. It follows from (3.3) that $x \notin \widehat{H}_n(1,x)$ for every $x \in \overline{V}_{\overline{K}_n}$. Theorem 2.1(1), (4) implies that

$$i_{\overline{K}_n}(F_n, V_n) = i_{\overline{K}_n}(\widehat{H}_n(1,\cdot), V_n) = 0. \tag{3.4}$$

Combining (3.2) and (3.4) and applying Theorem 2.1(3) permit us to deduce that

$$i_{\overline{K}_n}(F_n, V_n \backslash \overline{U}_n) = i_{\overline{K}_n}(F_n, V_n) - i_{\overline{K}_n}(F_n, U_n) = -1.$$

Therefore, (3.1) holds.

The conclusion follows from Proposition 2.7. □

Here is a corollary of the previous theorem in the particular case where U and V are pseudo-balls.

Corollary 3.4. *Let* $f : K \to K$ *be an admissibly completely continuous map. Assume that there exist* $\{r_{1,n}\}$ *and* $\{r_{2,n}\}$ *nondecreasing sequences in* $(0,\infty)$ *such that, for every* $n \in \mathbb{N}$,

(i) $x - F_n(x) \subset \overline{K}_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} B_n(0, r_{1,n})$;

(ii) $F_n(x) - x \subset \overline{K}_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} B_n(0, r_{2,n})$.

Then, there exists x a fixed point of f such that

$$x \in \operatorname*{Lim}_{n\to\infty} \overline{K}_n \cap \overline{B_n(0,R_n)} \backslash B_n(0,r_n),$$

where $R_n = \max\{r_{1,n}, r_{2,n}\}$ *and* $r_n = \min\{r_{1,n}, r_{2,n}\}$.

4. Monotone iterative method in cones

Let K be a cone in E and $\overline{K}_n$ the associated cone in E_n for every $n \in \mathbb{N}$. The cone K defines the partial orderings in E and in E_n given by

$$\begin{aligned} &\text{for } x,y \in E, && x \preceq y && \text{if and only if} && y - x \in K, \\ &\text{for } n \in \mathbb{N} \text{ and } x,y \in E_n, && x \preceq_n y && \text{if and only if} && y - x \in \overline{K}_n. \end{aligned} \tag{4.1}$$

For $x, y \in E$ such that $x \preceq y$ (resp. $x, y \in E_n$ such that $x \preceq_n y$ for some $n \in \mathbb{N}$) we denote

$$\begin{aligned} [x,y] &= \{z \in E : x \preceq z \preceq y\} && (\text{resp.} \quad [x,y]_n = \{z \in E_n : x \preceq_n z \preceq_n y\}), \\ [x,\infty) &= \{z \in E : x \preceq z\} && (\text{resp.} \quad [x,\infty)_n = \{z \in E_n : x \preceq_n z\}). \end{aligned}$$

Arguing as in [3], the well-known monotone iterative method permits to get the following fixed point result in Fréchet space.

Theorem 4.1. *Let $\alpha \preceq \beta$ be in E and $f : [\alpha,\beta] \to E$ a compact map. Assume the following conditions are satisfied:*

(i) *$\alpha \preceq f(\alpha)$ and $f(\beta) \preceq \beta$;*
(ii) *f is* nondecreasing*; that is, for every $x, y \in [\alpha,\beta]$ such that $x \preceq y$, one has $f(x) \preceq f(y)$.*

Then, f has a fixed point and the iterative sequences $\{f^k(\alpha)\}$ and $\{f^k(\beta)\}$ converge respectively to the smallest and the greatest fixed point of f in $[\alpha,\beta]$.

For some $\alpha \in E$ (resp. $\beta \in E$) such that $\alpha \not\preceq f(\alpha)$ (resp. $f(\beta) \not\preceq \beta$), there could exist some $n \in \mathbb{N}$ such that $\mu_n(\alpha) \preceq_n \mu_n(f(\alpha))$ (resp. $\mu_n(f(\beta) \preceq_n \mu_n(\beta))$. This remark leads us to consider admissibly compact maps. Since they involve multivalued maps, different notions of monotonicity can be defined.

Definition 4.2. Let Y be a space endowed with a partial order $\le$, $X \subset Y$ and $T : X \to Y$ a multivalued map. Let $x^-, x^+ \in X$ and $y^-, y^+ \in Y$ be such that $x^- \le x^+$ and $y^- \le y^+$.

(i) The map T is *right-nondecreasing on* $[x^-, x^+]$ *and in* $[y^-, y^+]$ if $y^- \in T(x^-)$ and, for every $x_1, x_2 \in X$ and every $y_1 \in T(x_1)$ such that

$$x^- \le x_1 \le x_2 \le x^+ \quad \text{and} \quad y^- \le y_1 \le y^+,$$

there exists $y_2 \in T(x_2)$ such that $y_1 \le y_2 \le y^+$.

(ii) The map T is *left-nondecreasing on* $[x^-, x^+]$ *and in* $[y^-, y^+]$ if $y^+ \in T(x^+)$ and, for every $x_1, x_2 \in X$ and every $y_2 \in T(x_2)$ such that

$$x^- \le x_1 \le x_2 \le x^+ \quad \text{and} \quad y^- \le y_2 \le y^+,$$

there exists $y_1 \in T(x_1)$ such that $y^- \le y_1 \le y_2$.

Similarly, one can define that T is *right-nonincreasing* (resp. *left-nonincreasing*) *on* $[x^-, x^+]$ *and in* $[y^-, y^+]$.

The following fixed point result concerns admissibly compact maps which are nondecreasing in the sense of the previous definition.

Theorem 4.3. *Let $X \subset E$ be closed and $f : X \to E$ an admissibly compact map. Assume the following conditions are satisfied:*

- (i) *there exists $N_0 \subset \mathbb{N}$ infinite such that, for every $n \in N_0$, there exist $\alpha_n, \beta_n \in \overline{X}_n$ such that $\alpha_n \preceq_n \beta_n$ in E_n and $[\alpha_n, \beta_n]_n \subset \overline{X}_n$;*
- (ii) *for every $n \in N_0$, there exists $\xi_n \in F_n(\alpha_n) \cap [\alpha_n, \beta_n]_n$ (resp. $\zeta_n \in F_n(\beta_n) \cap [\alpha_n, \beta_n]_n$);*
- (iii) *for every $n \in N_0$, F_n is right-nondecreasing on $[\alpha_n, \beta_n]_n$ and in $[\xi_n, \beta_n]_n$ (resp. F_n is left-nondecreasing on $[\alpha_n, \beta_n]_n$ and in $[\alpha_n, \zeta_n]_n$).*

Then, f has a fixed point

$$x \in \operatorname*{Lim}_{\substack{n\to\infty \\ n\in N_0}} A(n),$$

where

$$A(n) = \Big\{ z \in [\alpha_n, \beta_n]_n : z = \lim_{k\to\infty} u_k \text{ with } u_{k+1} \in F_n(u_k) \text{ and } \alpha_n \preceq_n \xi_n = u_1 \preceq_n u_2 \preceq_n \cdots \preceq_n \beta_n \Big\},$$

$$\Big(\textit{resp.}\quad A(n) = \Big\{ z \in [\alpha_n, \beta_n]_n : z = \lim_{k\to\infty} v_k \text{ with } v_{k+1} \in F_n(v_k) \text{ and } \alpha_n \preceq_n \cdots \preceq_n v_2 \preceq_n v_1 = \zeta_n \preceq_n \beta_n \Big\}\Big).$$

Proof. For $n \in N_0$, $F_n : [\alpha_n, \beta_n]_n \to E_n$ is compact, u.s.c. with compact, convex values. From (i)-(iii), one can construct a sequence $\{u_k^n\}$ in $[\alpha_n, \beta_n]_n$ such that $u_1^n = \xi_n$, $u_{k+1}^n \in F_n(u_k^n)$ and $u_k^n \preceq_n u_{k+1}^n$ for every $k \in \mathbb{N}$. Arguing as in the proof of Theorem 3.4 in [7], one deduces that there exists $z_n = \lim_{k\to\infty} u_k^n \in A(n)$ such that $z_n \in F_n(z_n)$. The conclusion follows from Proposition 2.7. □

Observe that assumption (iii) of the previous theorem implies that

$$F_n(x) \cap [\alpha_n, \beta_n]_n \neq \emptyset \quad \forall x \in [\alpha_n, \beta_n]_n, \quad \forall n \in N_0.$$

In fact, this is sufficient to insure that f has a fixed point. However, we loose some precision on its localization.

Theorem 4.4. *Let $X \subset E$ be closed and $f : X \to E$ an admissibly compact map. Assume the following conditions are satisfied:*

- (i) *there exists $N_0 \subset \mathbb{N}$ infinite such that, for every $n \in N_0$, there exist $\alpha_n, \beta_n \in \overline{X}_n$ such that $\alpha_n \preceq_n \beta_n$ in E_n and $[\alpha_n, \beta_n]_n \subset \overline{X}_n$;*
- (ii) *for every $n \in N_0$, $x \in [\alpha_n, \beta_n]_n$, there exists $u \in F_n(x) \cap [\alpha_n, \beta_n]_n$.*

Then, f has a fixed point

$$x \in \operatorname*{Lim}_{\substack{n\to\infty \\ n\in N_0}} [\alpha_n, \beta_n]_n.$$

Proof. For $n \in N_0$, let us define $\widetilde{F}_n : [\alpha_n, \beta_n]_n \to [\alpha_n, \beta_n]_n$ by

$$\widetilde{F}_n(x) = F_n(x) \cap [\alpha_n, \beta_n]_n.$$

The assumptions imply that $\widetilde{F}_n$ is a compact, u.s.c., multivalued map with nonempty, compact, convex values and defined on a closed, convex subset of the Banach space E_n. The Kakutani fixed point theorem insures the existence of $z_n \in \widetilde{F}_n(z_n)$. The conclusion follows from Proposition 2.7. □

Remark 4.5. In the results of this section, one can replace the compactness assumption by the complete continuity if, in addition, we assume that K is normal. Indeed, in a normal cone, an interval $[\alpha, \beta]$ (resp. $[\alpha_n, \beta_n]_n$) is bounded.

5. Fixed point results in cones with mixed type conditions

In this section, we present fixed point results relying on a combination of conditions imposed in the theorems obtained in the two previous sections. In particular, the existence of suitable pairs (α_n, β_n) is not assumed. More precisely, the assumption on the existence of a suitable $\{\alpha_n\}$ in Theorem 4.4 is removed and replaced by some conditions on the pseudo-boundary of a suitable pseudo-open set. As before, $\overline{K}_n$ is the cone in E_n associated to a cone K in E.

Theorem 5.1. *Let $\beta \in K$ and $f : [0, \beta] \to K$ an admissibly compact map. Assume the following conditions are satisfied:*

(i) *there exists U a bounded, pseudo-open set in E such that, for every $n \in \mathbb{N}$, $0 \in \overline{U}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} U_n \subset \overline{U}_{\overline{K}_n} \subset [0, \mu_n(\beta)]_n$;*

(ii) *the set*

$$N_0 = \{n \in \mathbb{N} : \forall x \in \partial_{\overline{K}_n} U_n,\ (F_n(x) - x) \cap \overline{K}_n = \emptyset \quad \text{or} \quad F_n(x) \cap [x, \mu_n(\beta)]_n \neq \emptyset\}$$

is infinite;

(iii) *for every $n \in N_0$ and every $\hat{x} \in \partial_{\overline{K}_n} U_n$ such that $F_n(\hat{x}) \cap [\hat{x}, \infty)_n \neq \emptyset$, one has that $F_n(x) \cap [\hat{x}, \mu_n(\beta)]_n \neq \emptyset$ for every $x \in [\hat{x}, \mu_n(\beta)]_n$.*

Then, f has a fixed point x^* such that

$$x^* \in \operatorname*{Lim}_{\substack{n\to\infty \\ n \in N_0}} A(n),$$

where

$$A(n) = \left(\overline{U}_{\overline{K}_n}\right) \cup \Big(\bigcup_{x \in \partial_{\overline{K}_n} U_n} [x, \mu_n(\beta)]_n\Big).$$

Proof. It follows from (i) that, for every $n \in N_0$ and every $x \in \partial_{\overline{K}_n} U_n$, one has $x \preceq_n \mu_n(\beta)$.

Let

$$N_1 = \Big\{n \in N_0 : \exists \alpha_n \in \partial_{\overline{K}_n} U_n \text{ such that } F_n(\alpha_n) \cap [\alpha_n, \infty)_n \neq \emptyset\}.$$

If N_1 is infinite, then the assumptions of Theorem 4.4 are satisfied with α_n and $\beta_n = \mu_n(\beta)$. Therefore, f has a fixed point

$$x \in \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_1}} [\alpha_n, \mu_n(\beta)]_n \subset \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}} A(n).$$

On the other hand, if N_1 is empty or finite, then $N_2 = N_0 \backslash N_1$ is infinite and, for every $n \in N_2$, $(F_n(z) - z) \cap \overline{K}_n = \emptyset$ for every $z \in \partial_{\overline{K}_n} U_n$. Arguing as in the proof of Theorem 3.3, one deduces that the fixed point index

$$i_{\overline{K}_n}(F_n, U_n) = 1 \quad \forall n \in N_2.$$

Hence, there exists $z_n \in U_{\overline{K}_n}$ such that $z_n \in F_n(z_n)$. Proposition 2.7 permits to conclude that f has a fixed point

$$x \in \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_2}} U_{\overline{K}_n} \subset \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}} A(n).$$

□

We obtain the following corollary by adding a monotonicity condition.

Corollary 5.2. *Let $\beta \in K$ and $f : [0, \beta] \to K$ an admissibly compact map satisfying conditions* (i) *and* (ii) *of Theorem* 5.1*. In addition, assume that*

(iii') *for every $n \in N_0$ and every $\hat{x} \in \partial_{\overline{K}_n} U_n$ such that $F_n(\hat{x}) \cap [\hat{x}, \infty)_n \neq \emptyset$, there exists $\hat{y} \in F_n(\hat{x})$ such that F_n is right-nondecreasing on $[\hat{x}, \mu_n(\beta)]_n$ and in $[\hat{y}, \mu_n(\beta)]_n$.*

Then, f has a fixed point.

In [6], Cabada, Cid and Infante consided completely continuous maps defined on a solid, normal cone in a Banach space and which are nondecreasing on $[0, \beta] \backslash B(0, r/c)$. Here is a corollary of Theorem 5.1 for admissibly completely continuous maps satisfying a monotonicity condition analogous to the condition imposed in [6].

Corollary 5.3. *Let K be a normal cone and $f : K \to K$ an admissibly completely continuous map. Assume there exist $\beta \in K$ and $\{r_n\}$ a nondecreasing sequence in $(0, \infty)$ satisfying the following conditions:*

(i) $\overline{B_n(0, r_n)} \cap \overline{K}_n \subset [0, \mu_n(\beta)[_n$;

(ii) *the set*

$$N_0 = \{n \in \mathbb{N} : \forall x \in \partial_{\overline{K}_n} B_n(0, r_n),\ (F_n(x) - x) \cap \overline{K}_n = \emptyset \\ \text{or} \quad F_n(x) \cap [x, \mu_n(\beta)]_n \neq \emptyset\}$$

is infinite;

(iii) *for every $n \in N_0$ and every $\hat{x} \in \overline{K}_n \backslash B_n(0, r_n/c_n)$ such that $F_n(\hat{x}) \cap [0, \mu_n(\beta)]_n \neq \emptyset$, one has that F_n is right-nondecreasing on $[\hat{x}, \mu_n(\beta)]_n$ and in $[\hat{y}, \mu_n(\beta)]_n$ for every $\hat{y} \in F_n(\hat{x}) \cap [0, \mu_n(\beta)]_n$.*

Then, f has a fixed point.

Proof. Since K is normal, $[0,\beta]$ is bounded and hence, $f:[0,\beta]\to K$ is admissibly compact. Moreover,

$$[\hat{x},\mu_n(\beta)]_n \subset \overline{K}_n\backslash B_n(0,r_n/c_n) \quad \forall \hat{x}\in \partial_{\overline{K}_n} B_n(0,r_n).$$

So, (iii) insures that condition (iii') of Corollary 5.2 is satisfied. □

Adding extra assumptions to Theorem 5.1 permits to obtain more precision on the localization of the fixed point.

Theorem 5.4. *Let $\beta\in K$, $X\subset K$ closed such that $[0,\beta]\subset X$ and let $f:X\to K$ be an admissibly compact map satisfying conditions* (i)-(iii) *of Theorem* 5.1. *In addition, assume that the following conditions are satisfied:*

(iv) *there exists V a pseudo-open set in E such that, for every $n\in N_0$,*

$$0\in \overline{V}_{\overline{K}_n}\backslash \partial_{\overline{K}_n} V_n \subset \overline{V}_{\overline{K}_n} \subset \overline{U}_{\overline{K}_n}\backslash \partial_{\overline{K}_n} U_n,$$
$$\Big(\textit{resp.}\quad 0\in \overline{U}_{\overline{K}_n}\backslash \partial_{\overline{K}_n} U_n \subset \overline{U}_{\overline{K}_n} \subset \overline{V}_{\overline{K}_n}\backslash \partial_{\overline{K}_n} V_n \subset \overline{V}_{\overline{K}_n} \subset \overline{X}_n\Big);$$

(v) *for every $n\in N_0$, the fixed point index*

$$i_{\overline{K}_n}(F_n,V_n)=0.$$

Then, f has a fixed point x^* such that

$$x^*\in \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}} \widehat{A}(n),$$

where

$$\widehat{A}(n)=\Big(\overline{U}_{\overline{K}_n}\backslash V_{\overline{K}_n}\Big)\cup\Big(\bigcup_{x\in\partial_{\overline{K}_n}U_n}[x,\mu_n(\beta)]_n\Big),$$
$$\Big(\textit{resp.}\quad \widehat{A}(n)=\Big(\overline{V}_{\overline{K}_n}\backslash U_{\overline{K}_n}\Big)\cup\Big(\bigcup_{x\in\partial_{\overline{K}_n}U_n}[x,\mu_n(\beta)]_n\Big)\Big).$$

Proof. It follows from the proof of Theorem 5.1 that f has a fixed point

$$x\in \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}}\Big(\bigcup_{x\in\partial_{\overline{K}_n}U_n}[x,\mu_n(\beta)]_n\Big),$$

or, there exists $N_2\subset N_0$ infinite such that $i_{\overline{K}_n}(F_n,U_n)=1$ for every $n\in N_2$. Theorem 2.1(1), (3), and assumptions (iv) and (v) imply that, for every $n\in N_2$,

$$i_{\overline{K}_n}(F_n,U_n\backslash\overline{V}_n)=-1 \quad (\text{resp. } i_{\overline{K}_n}(F_n,V_n\backslash\overline{U}_n)=-1).$$

So, there exists

$$z_n\in F_n(z_n)\cap U_{\overline{K}_n}\backslash\overline{V}_{\overline{K}_n} \quad (\text{resp. } z_n\in F_n(z_n)\cap V_{\overline{K}_n}\backslash\overline{U}_{\overline{K}_n}).$$

The conclusion follows from Proposition 2.7. □

Remark 5.5. The fixed point obtained in the previous theorem is non trivial if

$$0\notin \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}} \widehat{A}(n).$$

Remark 5.6. Even in the particular case where E is a Banach space, Theorem 5.4 generalizes Theorem 2.3 in [6]. In particular, the cone is not assumed to be normal and solid, and no monotonicity condition is imposed on f.

Corollary 5.7. *Let $\beta \in K$, $X \subset K$ closed such that $[0,\beta] \subset X$ and let $f : X \to K$ be an admissibly compact map satisfying conditions* (i)-(iv) *of Theorem* 5.4. *In addition, assume that*

(v') *for every $n \in N_0$, $\big(x - F_n(x)\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ for every $x \in \partial_{\overline{K}_n} V_n$.*

Then, f has a fixed point x^* such that

$$x^* \in \operatorname*{Lim}_{\substack{n\to\infty \\ n\in N_0}} \widehat{A}(n),$$

where $\widehat{A}(n)$ is defined in Theorem 5.4.

Proof. Arguing as in the proof of Theorem 3.3, one can show that, for every $n \in N_0$, F_n has a fixed point in $\partial_{\overline{K}_n} V_n$ or

$$i_{\overline{K}_n}(F_n, V_n) = 0.$$

The conclusion follows from Theorem 5.4. □

Condition (iii) in Theorem 5.1 insured that, for suitable x, there exists $y \in F_n(x)$ such that $y \le \mu_n(\beta)$. In the next result, we assume the opposite inequality.

Theorem 5.8. *Let $\alpha \in K$, $X \subset K$ closed such that $[0,\alpha] \subset X$ and $f : X \to K$ an admissibly compact map. Assume the following conditions are satisfied:*

(i) *there exists U a bounded pseudo-open set in E such that, for every $n \in \mathbb{N}$, $0 \in \overline{U}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} U_n \subset \overline{U}_{\overline{K}_n} \subset [0, \mu_n(\alpha)]_n$;*

(ii) *the set*

$$N_0 = \{n \in \mathbb{N} : \forall x \in \partial_{\overline{K}_n} U_n,\ (x - F_n(x)) \cap \overline{K}_n = \emptyset \quad \textit{or} \quad F_n(x) \subset [x, \infty)_n$$

is infinite;

(iii) *there exists V a pseudo-open set in E such that, for every $n \in N_0$,*

$$0 \in \overline{V}_n \backslash \partial_{\overline{K}_n} V_n \subset \overline{V}_{\overline{K}_n} \subset \overline{U}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} U_n,$$
$$\Big(\textit{resp.}\quad 0 \in \overline{U}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} U_n \subset \overline{U}_{\overline{K}_n} \subset \overline{V}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} V_n \subset \overline{V}_{\overline{K}_n} \subset \overline{X}_n\Big);$$

(iv) *for every $n \in N_0$, the fixed point index*

$$i_{\overline{K}_n}(F_n, V_n) = 1.$$

Then, f has a fixed point x^* such that

$$x^* \in \operatorname*{Lim}_{\substack{n\to\infty \\ n\in N_0}} \widetilde{A}(n),$$

where

$$\widetilde{A}(n) = \Big(\overline{U}_{\overline{K}_n} \backslash V_{\overline{K}_n}\Big) \cup \Big(\bigcup_{x \in \partial_{\overline{K}_n} U_n} [x, \mu_n(\alpha)]_n \Big),$$

$$\Big(\textit{resp.} \quad \widetilde{A}(n) = \Big(\overline{V}_{\overline{K}_n} \backslash U_{\overline{K}_n}\Big) \cup \Big(\bigcup_{x \in \partial_{\overline{K}_n} U_n} [x, \mu_n(\alpha)]_n \Big)\Big).$$

Proof. It follows from (i) that, for every $n \in N_0$ and every $x \in \partial_{\overline{K}_n} U_n$, one has $x \preceq_n \mu_n(\alpha)$.

Let

$$N_1 = \Big\{ n \in N_0 : \exists x \in \partial_{\overline{K}_n} U_n \text{ such that } \big(x - F_n(x)\big) \cap \overline{K}_n \neq \emptyset \}.$$

If N_1 is infinite, then, for $z_n \in \partial_{\overline{K}_n} U_n$ such that there exists $u \in F_n(z_n)$ with $z_n - u \in \overline{K}_n$, one has $F_n(z_n) \subset [z_n, \infty)_n$. So, $u \preceq_n z_n \preceq_n u$. Thus, $z_n \in F_n(z_n)$ and f has a fixed point

$$x \in \operatorname*{Lim}_{\substack{n \to \infty \\ n \in N_1}} \Big(\bigcup_{x \in \partial_{\overline{K}_n} U_n} [x, \mu_n(\alpha)]_n \Big) \subset \operatorname*{Lim}_{\substack{n \to \infty \\ n \in N_0}} \widetilde{A}(n).$$

On the other hand, if N_1 is empty or finite, then $N_2 = N_0 \backslash N_1$ is infinite and, for every $n \in N_2$, $(z - F_n(z)) \cap \overline{K}_n = \emptyset$ for every $z \in \partial_{\overline{K}_n} U_n$. Arguing as in the proof of Theorem 3.3, one deduces that the fixed point index

$$i_{\overline{K}_n}(F_n, U_n) = 0 \quad \forall n \in N_2.$$

Theorem 2.1(3), and assumptions (iii) and (iv) imply that, for every $n \in N_2$,

$$i_{\overline{K}_n}(F_n, U_n \backslash \overline{V}_n) = -1 \quad (\text{resp. } i_{\overline{K}_n}(F_n, V_n \backslash \overline{U}_n) = -1).$$

So, there exists

$$z_n \in F_n(z_n) \cap U_{\overline{K}_n} \backslash \overline{V}_{\overline{K}_n} \quad (\text{resp. } z_n \in F_n(z_n) \cap V_{\overline{K}_n} \backslash \overline{U}_{\overline{K}_n}).$$

The conclusion follows from Proposition 2.7. □

Remark 5.9. Even in the particular case where E is a Banach space, Theorem 5.8 generalizes Theorem 2.5 in [6]. Again, the cone is not assumed to be normal and solid, and no monotonicity condition is imposed on f.

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Marlène Frigon
Département de mathématiques et de statistique
Université de Montréal
Montréal, H3C 3J7, Canada
e-mail: `frigon@dms.umontreal.ca`