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Linear delay-differential operator of a meromorphic function sharing two sets or small function together with values with its c-shift or q-shift

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Abstract. The paper is devoted to study the uniqueness problem of linear delay-differential operator of a meromorphic function sharing two sets or small function together with values with its c-shift and q-shift operator. Results of this paper drastically improve two recent results of Meng-Liu [J. Appl. Math. Inform. 37(1-2)(2019), 133-148] and Qi-Li-Yang [Comput. Methods Funct. Theory, 18(2018), 567-582]. In addition to this, one of our results improves and extends that of Qi-Yang [Comput. Methods Funct. Theory, 20(2020), 159-178].

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1. Introduction, Definitions and Results

Throughout the paper we use standard notations of Nevanlinna theory as stated in [7] and by any meromorphic function f we always mean that it is defined on $\mathbb{C}$. Let f and g be such two non-constant meromorphic functions. For $a \in \mathbb{C} \cup \{\infty\}$, the following two quantities

$$\delta(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \longrightarrow \infty} \frac{m(r, a; f)}{T(r, f)}$$

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and

$$\Theta(a;f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}$$

are respectively known as Nevanlinna deficiency of the value a and ramification index.

In the beginning of the nineteenth century R. Nevanlinna inaugurated the value distribution theory with his famous Five value and Four value theorems which can be considered as the backbone of the modern uniqueness theory. Illuminated by these two basic results initially the research were performed on the value sharing of meromorphic functions. After five decades, uniqueness theory moved to a new direction led by F. Gross [4], who transformed the traditional value sharing problem to a more general set up namely shared set problems. Now we recall the definition of set sharing.

Definition 1.1. For some $a \in \mathbb{C}$, we denote by $E_f(a)$, the collection of the zeros of $f - a$, where a zero is counted according to its multiplicity. In addition to this, when $a = \infty$, the above notation implies that we are considering the poles. In the same manner, by $\overline{E}_f(a)$, we denote the collection of the distinct zeros or poles of $f - a$ according as $a \in \mathbb{C}$ or $a = \infty$ respectively.

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$. For a non-constant meromorphic function f, let $E_f(S) = \bigcup_{a\in S} E_f(a)$ $\big(\overline{E}_f(S) = \bigcup_{a\in S} \overline{E}_f(a)\big)$. Then we say f, g share the set S CM(IM) if $E_f(S) = E_g(S)$ $\big(\overline{E}_f(S) = \overline{E}_g(S)\big)$.

Evidently, if S is a singleton, then it coincides with the traditional definition of CM(IM) sharing of values, which are known to the readers.

In 2001, due to a revolutionary approach by Lahiri [8, 9], the notion of weighted sharing of values or sets appeared in the literature and expedite the research work there in. Though now-a-days the definition is widely circulated, we invoke the definition.

Definition 1.2. [8, 9] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k and denote it by (a,k). The IM and CM sharing corresponds to $(a,0)$ and (a,∞) respectively.

Definition 1.3. [8] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_f(S,k)$ the set $\cup_{a\in S} E_k(a;f)$. Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.

If $E_f(S,k) = E_g(S,k)$, then we say that f, g share the set S with weight k and write it as f, g share (S,k).

By $N(r,a;f \mid< m)$ we mean the counting function of those a-points of f whose multiplicities are less than m where each a-point is counted according to its multiplicity and by $\overline{N}(r,a;f \mid\geq m)$ we mean the counting function of those a-points of f whose multiplicities are not less than m where each a-point is counted ignoring multiplicity. We also denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2)$.

Usually, $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside of a possible exceptional set of finite linear measure. Also $S_1(r,f)$ denotes any quantity satisfying $S_1(r,f) = o(T(r,f))$ for all r on a set of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap F} \frac{dt}{t}.$$

Throughout the paper for a positive integer n, S_1, S_1^* and S_2 represents respectively the sets $\{1, \omega, \ldots, \omega^{n-1}\}$, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\infty\}$, where $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and α_i, $i = 1, 2, \ldots, n$ are non-zero constants.

Let $a_{t-1}(\neq 0)$, a_{t-2}, $\ldots$, a_0 and $C(\neq 0)$ be complex numbers. We define

$$P(z) = CzQ(z) = Cz(a_{t-1}z^{t-1} + a_{t-2}z^{t-2} + \ldots + a_1 z + a_0). \tag{1.1}$$

For the polynomial $P(z)$ as given in (1.1) let us define the following two functions:

$$\chi_0^{t-1} = \begin{cases} 1, & if\ a_0 \neq 0 \\ 0, & if\ a_0 = 0 \end{cases}$$

and

$$\mu_0^{t-1} = \begin{cases} 1, & if\ a_0 = 0, a_1 \neq 0 \\ 0, & otherwise. \end{cases}$$

In view of (1.1), corresponding to the set S_1^*, let us consider the polynomial $P_*(z)$ as follows:

$$P_*(z) = CzQ_*(z), \text{ where } C = \frac{1}{(-1)^{n+1}\alpha_1\alpha_2\ldots\alpha_n} \text{ and} \tag{1.2}$$

$$Q_*(z) = \sum_{r=0}^{n-1}(-1)^r \sum \alpha_1\alpha_2\ldots\alpha_r\ z^{n-r-1},$$

$\sum \alpha_1\alpha_2\ldots\alpha_r =$ sum of the products of the values $\alpha_1, \alpha_2, \ldots, \alpha_n$ taken r into account. We also denote by m_1 and m_2 as the number of simple and multiple zeros of $Q_*(z)$ respectively.

Next we define linear shift operator, delay operator and differential operator respectively as follows:

$$L_1(f(z)) = a_k f(z+c_k) + a_{k-1}f(z+c_{k-1}) + \ldots + a_1 f(z+c_1) + a_0 f(z),$$
$$L_2(f(z)) = b_s f^{(s)}(z+c_s) + b_{s-1}f^{(s-1)}(z+c_{s-1}) + \ldots + b_1 f'(z+c_1),$$
$$L_3(f(z)) = d_t f^{(t)}(z) + d_{t-1}f^{(t-1)}(z) + \ldots + d_1 f'(z),$$

where a_k, b_s and d_t are non-zero and k, s, t are natural numbers and all $c_i's$ are non-zero. For the sake of convenience we shall call $L_2(f(z)) + L_3(f(z))$ as delay-differential operator which is denoted by $\tilde{L}(f(z))$.

As far as the knowledge of the authors are concerned, Qi-Li-Yang [13] were the first authors who initiated two shared set problems for the derivative of a meromorphic function $f(z)$ with its shift $f(z+c)$ as follows:

Theorem A. [13] *Let $f(z)$ be a non-constant meromorphic function of finite order, $n \geq 9$ be an integer and a be a non-zero complex constant. If $[f'(z)]^n$ and $f^n(z+c)$ share (a,∞) and (∞,∞), then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

Recently employing the notion of weighted sharing, Meng-Liu [12] further investigated *Theorem A* to obtain the following result.

Theorem B. [12] *Let $f(z)$ be a non-constant meromorphic function of finite order, $n \geq 10$ be an integer. If $[f'(z)]^n$ and $f^n(z+c)$ share $(1,2)$ and $(\infty,0)$, then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

Considering $f(z) = e^z$ and $\omega = e^{-c}$ satisfying $\omega^n = 1$, it is easy to see that f' and $f(z+c)$ share the sets (S_1, ∞), (∞, ∞) and $f'(z) = \omega f(z+c)$ for each n. So it is natural to conjecture that in *Theorem A* and *Theorem B* the cardinality of n could further be reduced. To this end, we have performed our investigations and have been able to reduce the cardinality of n in *Theorem B* up to 6. In fact, we have proved our theorem for a more general setting S_1^* rather than to consider only the set S_1.

Theorem 1.1. *Let $f(z)$ be a non-constant meromorphic function of finite order such that $\tilde{L}(f(z))$ and $f(z+c)$ share $(S_1^*, 2)$ and $(S_2, 0)$. If*

$$n > 2(\chi_0^{n-1} + \mu_0^{n-1} + m_1 + 2m_2) + \frac{15}{(2n-3)}(\chi_0^{n-1} + m_1 + m_2), \quad \textit{then}$$

$$\prod_{i=1}^{n}(\tilde{L}(f(z)) - \alpha_i) \equiv \prod_{i=1}^{n}(f(z+c) - \alpha_i).$$

Remark 1.1. From the definitions, we easily can calculate the value of χ_0^{n-1}, μ_0^{n-1}, m_1 and m_2 for a particular set S_1^*. Clearly for the set S_1, $\chi_0^{n-1} = 0$; $\mu_0^{n-1} = 0$; $m_1 = 0$ and $m_2 = 1$. Therefore in above theorem for the set S_1 if $n > 4 + \frac{15}{(2n-3)}$ i.e., if $n \geq 6$ then $\tilde{L}(f(z)) = tf(z+c)$, for a constant t that satisfies $t^n = 1$. For a particular choices of coefficients of $\tilde{L}(f(z))$ we can easily make $\tilde{L}(f(z)) = f'$.

Corresponding to q-shift Meng-Liu [12] also investigated the same result like *Theorem B* as follows :

Theorem C. [12] *Let $f(z)$ be a non-constant meromorphic function of zero order, $n \geq 10$ be an integer. If $[f'(z)]^n$ and $f^n(qz)$ share $(1,2)$ and $(\infty,0)$, then $f'(z) = tf(qz)$, for a constant t that satisfies $t^n = 1$.*

In connection to *Theorem C* below we present our result which improves the same.

Theorem 1.2. *Let $f(z)$ be a non-constant meromorphic function of zero order such that $\tilde{L}(f(z))$ and $f(qz)$ share $(S_1^*, 2)$ and $(S_2, 0)$. If*

$$n > 2(\chi_0^{n-1} + \mu_0^{n-1} + m_1 + 2m_2) + \frac{15}{(2n-3)}(\chi_0^{n-1} + m_1 + m_2) \quad \textit{then}$$

$$\prod_{i=1}^{n}(\tilde{L}(f(z)) - \alpha_i) \equiv \prod_{i=1}^{n}(f(qz) - \alpha_i).$$

In the next theorem we shall show that the lower bound of n can further be reduced at the expense of allowing both the range sets S_1^*, S_2 to be shared CM.

Theorem 1.3. *Let $f(z)$ be a non-constant meromorphic function of finite order such that $\tilde{L}(f(z))$ and $f(z+c)$ share (S_1^*, ∞) and (S_2, ∞) with*

$$T(r,f) = N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f)$$

then for $n > 2(\chi_0^{n-1} + m_1 + m_2) + 1$,

$$\prod_{i=1}^{n}(\tilde{L}(f(z)) - \alpha_i) \equiv \prod_{i=1}^{n}(f(z+c) - \alpha_i).$$

Remark 1.2. In connection of *Remark 1.1*, for the set S_1 in *Theorem 1.3* the result holds for $n \geq 4$.

Our next theorem is analogous theorem of *Theorem 1.3* corresponding to q-shift.

Theorem 1.4. *Let $f(z)$ be a non-constant meromorphic function of zero order such that $\tilde{L}(f(z))$ and $f(qz)$ share (S_1^*, ∞) and (S_2, ∞). If $n > 2(\chi_0^{n-1} + m_1 + m_2) + 1$ then*

$$\prod_{i=1}^{n}(\tilde{L}(f(z)) - \alpha_i) \equiv \prod_{i=1}^{n}(f(qz) - \alpha_i).$$

Recently, corresponding to *Theorem A*, Qi-Yang [14] obtained the value sharing problem for entire function as follows:

Theorem D. [14] *Let $f(z)$ be a transcendental entire function of finite order and let $(a \neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share $(0,\infty)$ and $(a,0)$, then $f'(z) \equiv f(z+c)$.*

In view of *Theorem 1.1*, [14] we know that $f(z)$ actually becomes a transcendental entire function. Since we are dealing with $\tilde{L}(f(z))$ instead of f', it will be reasonable to consider the above theorem for meromorphic function under small function sharing category. In this respect we prove the following theorem.

Theorem 1.5. *Let $f(z)$ be a transcendental meromorphic function of finite order and let $a(z)(\not\equiv 0) \in S(f)$ be an entire function. If $\tilde{L}(f(z))$ and $f(z+c)$ share $(0,\infty)$, (∞,∞) and $(a(z),0)$ with $\Theta(0;f) + \Theta(\infty;f) > 0$, then $\tilde{L}(f(z)) \equiv f(z+c)$.*

From *Theorem 1.5* we can immediately deduce the following corollary.

Corollary 1.1. *Let $f(z)$ be a transcendental entire function of finite order and let $a(z)(\not\equiv 0) \in S(f)$. If $\tilde{L}(f(z))$ and $f(z+c)$ share $(0,\infty)$ and $(a(z),0)$, then $\tilde{L}(f(z)) \equiv f(z+c)$.*

Following example shows that in *Theorem 1.5* the CM pole sharing can not be replaced by IM.

Example 1.1. Let $f(z) = \frac{2e^{2\sqrt{2}iz} - 8e^{\sqrt{2}iz} + 2}{(e^{\sqrt{2}iz}+1)^2}$ and $c = \sqrt{2}\pi$. Choose the coefficients of $\tilde{L}(f(z))$ in such a way that $\tilde{L}(f(z)) = f''$. Then

$$\tilde{L}(f(z)) \left(= \frac{24e^{\sqrt{2}iz}[e^{2\sqrt{2}iz} - 4e^{\sqrt{2}iz} + 1]}{(e^{\sqrt{2}iz}+1)^4} \right)$$

and $f(z+c)$ share $(0,\infty)$, $(1,0)$ and $(\infty,0)$ and $\Theta(0;f)+\Theta(\infty;f)=\frac{1}{2}>0$ but $\tilde{L}(f(z))\not\equiv f(z+c)$.

From the next example we can show that in Theorem 1.5 sharing of 0 can not be replaced by sharing of a non-zero value.

Example 1.2. Let $f(z)=(e^{\lambda z}-1)^2+1$. Choose $e^{\lambda c}=1$,

$$\sum_{i=1}^{s} b_i(2\lambda)^i e^{2\lambda c_i}+\sum_{i=1}^{t} d_i(2\lambda)^i=0$$

and

$$\sum_{i=1}^{s} b_i(\lambda)^i e^{\lambda c_i}+\sum_{i=1}^{t} d_i(\lambda)^i=-\frac{1}{2}.$$

Then $f(z+c)=(e^{\lambda z}-1)^2+1$ and $\tilde{L}(f(z))=e^{\lambda z}$. Clearly $f(z+c)$ and $\tilde{L}(f(z))$ share $(2,\infty)$, (∞,∞) and $(1,0)$ with $\Theta(0;f)+\Theta(\infty;f)>0$. But $\tilde{L}(f(z))\neq f(z+c)$.

In *Theorem 1.5*, sharing of the value 0 can be removed at the cost of slightly manipulating the deficiency condition. In this respect, we state the following theorem for transcendental meromorphic function.

Theorem 1.6. *Let $f(z)$ be a transcendental meromorphic function of finite order and let $a(z)(\not\equiv 0)\in S(f)$ be an entire function. If $\tilde{L}(f(z))$ and $f(z+c)$ share $(a(z),\infty)$ and (∞,∞) with $\delta(0;f)>0$, then $\tilde{L}(f(z))\equiv f(z+c)$.*

By an example we now show that $a(z)$ CM sharing can not be replaced by IM in *Theorem 1.6*.

Example 1.3. Let $f(z)=\frac{-2e^z-1}{e^{2z}}$ and $c=\pi i$. Choose $\tilde{L}(f(z))=L_3(f(z))$ with

$$2\sum_{i=1}^{t}(-1)^{i+1}d_i=1 \text{ and } \sum_{i=1}^{t}(-2)^i d_i=0.$$

Then $\tilde{L}(f(z))=\frac{1}{e^z}$ and $f(z+c)=\frac{2e^z-1}{e^{2z}}$ share $(1,0)$, (∞,∞) and $\delta(0;f)=\frac{1}{2}>0$. Clearly $\tilde{L}(f(z))\neq f(z+c)$.

Our next example shows that $a(z)\not\equiv 0$ in *Theorem 1.6* can not be dropped as well as $(a(z),0)$ sharing in *Theorem 1.5* can not be removed.

Example 1.4. Let $f(z)=e^{\frac{\pi iz}{c}}$. Choose $\tilde{L}(f(z))=f'$. Then clearly $f(z+c)$ and $\tilde{L}(f(z))$ share $(0,\infty)$, (∞,∞) and $\delta(0;f)>0$. But $\tilde{L}(f(z))\neq f(z+c)$.

Following two examples show that $\delta(0;f)>0$ in *Theorem 1.6* can not be removed.

Example 1.5. In *Example 1.2* though $f(z+c)$ and $\tilde{L}(f(z))$ share $(2,\infty)$, (∞,∞) but $\delta(0;f)=0$. Here $\tilde{L}(f(z))\neq f(z+c)$.

Example 1.6. Let $f(z)=\frac{e^z+z}{2}$ and $a(z)=z$. Choose $\tilde{L}(f(z))=L_3(f(z))$ with $d_1=2c$ and

$$\sum_{j=2}^{t} d_j = 2(e^c-c).$$

Then

$$f(z+c)\left(=\frac{e^c e^z+z+c}{2}\right) \text{ and } \tilde{L}(f(z))\ (=e^c e^z+c)$$

share $(a(z),\infty)$ and (∞,∞) but $\delta(0;f)=0$. Clearly $\tilde{L}(f(z))\neq f(z+c)$.

2. Lemmas

In this section some lemmas will be presented which will be needed in the sequel.

Lemma 2.1. [3] *Let $f(z)$ be a meromorphic function of finite order ρ and let $c\in\mathbb{C}\setminus\{0\}$ be fixed. Then, for each $\varepsilon>0$, we have*

$$T(r,f(z+c))=T(r,f(z))+O(r^{\rho-1+\varepsilon})+O(\log r).$$

Lemma 2.2. [5] *Let $f(z)$ be a meromorphic function of finite order and $c\in\mathbb{C}\setminus\{0\}$. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=S(r,f).$$

Lemma 2.3. [6] *Let f be a non-constant meromorphic function of finite order and $c\in\mathbb{C}$. Then*

$$N\left(r,\frac{1}{f(z+c)}\right)\leq N\left(r,\frac{1}{f(z)}\right)+S(r,f),$$

$$N\left(r,f(z+c)\right)\leq N\left(r,f(z)\right)+S(r,f),$$

$$\overline{N}\left(r,\frac{1}{f(z+c)}\right)\leq \overline{N}\left(r,\frac{1}{f(z)}\right)+S(r,f)$$

and

$$\overline{N}\left(r,f(z+c)\right)\leq \overline{N}\left(r,f(z)\right)+S(r,f).$$

Lemma 2.4. [2] *Let $f(z)$ be a meromorphic function of zero order and $q\in\mathbb{C}\setminus\{0\}$. Then*

$$m\left(r,\frac{f(qz)}{f(z)}\right)=S_1(r,f).$$

Lemma 2.5. [16] *Let $f(z)$ be a non-constant zero order meromorphic function and $q\in\mathbb{C}\setminus\{0\}$, then*

$$T(r,f(qz))=(1+o(1))T(r,f(z))$$

and

$$N(r,f(qz))=(1+o(1))N(r,f(z))$$

on a set of lower logarithmic measure 1.

Using *Lemma 2.4* and *Lemma 2.5* and by the help of simple transformation one can easily prove the next lemma.

Lemma 2.6. *Let $f(z)$ be a meromorphic function of zero order and $q \in \mathbb{C}\setminus\{0\}$. Then*

$$m\left(r, \frac{f(z)}{f(qz)}\right) = S_1(r, f).$$

Lemma 2.7. [15] *Let $f(z)$ be a non-constant meromorphic function in the complex plane, and let $R(f) = \frac{P(f)}{Q(f)}$, where*

$$P(f) = \sum_{k=0}^{p} a_k(z) f^k \quad and \quad Q(f) = \sum_{j=0}^{q} b_j(z) f^j$$

are two mutually prime polynomials in f. If the coefficients $a_k(z)$ for $k = 0, 1, \ldots, p$ and $b_j(z)$ for $j = 0, 1, \ldots, q$ are small functions of f with $a_p(z) \not\equiv 0$ and $b_q(z) \not\equiv 0$, then

$$T(r, P(f)) = \max\{p, q\} T(r, f) + S(r, f).$$

Lemma 2.8. [11] *Suppose that h is a non-constant meromorphic function satisfying*

$$N(r, h) + N\left(r, \frac{1}{h}\right) = S(r, h).$$

Let $f = a_0 h^p + a_1 h^{p-1} + \ldots + a_p$, and $g = b_0 h^q + b_1 h^{q-1} + \ldots + b_q$ be polynomials in h with coefficients $a_0, a_1, \ldots, a_p$; $b_0, b_1, \ldots, b_q$ being small functions of h and $a_0 b_0 a_p \not\equiv 0$. If $q \le p$, then $m\left(r, \frac{g}{f}\right) = S(r, h)$.

Lemma 2.9. [10] *If $N(r, 0; f^{(k)} \mid f \ne 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r, 0; f^{(k)} \mid f \ne 0) \le k\overline{N}(r, \infty; f) + N(r, 0; f \mid< k) + k\overline{N}(r, 0; f \mid\ge k) + S(r, f).$$

Lemma 2.10. *Let F be a meromorphic function. Then*

$$\overline{N}(r, 1; F \mid\ge k+1) \le \frac{1}{k}\{\overline{N}(r, 0; F) + \overline{N}(r, \infty; F)\} + S(r, F).$$

Since the proof is straight forward, it is omitted.

Lemma 2.11. [1] *Let F, G be two meromorphic functions sharing $(1, 2)$ and (∞, k), where $0 \le k \le \infty$. Then one of the following cases holds*

$$\begin{aligned} (i)\ T(r, F) + T(r, G) \le\ & 2\{N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) \\ & + \overline{N}_*(r, \infty; F, G)\} + S(r, F) + S(r, G), \end{aligned}$$

where $\overline{N}_*(r, \infty; f, g)$ is the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G,
(ii) $F \equiv G$,
(iii) $FG \equiv 1$.

Lemma 2.12. *Let $P_*(f)$ and $P_*(g)$ be defined in (1.2), for two non-constant meromorphic functions f and g. Then*

$$\overline{N}(r,0;P_*(f)) \leq (\chi_0^{n-1}+m_1+m_2)T(r,f);$$
$$N_2(r,0;P_*(f)) \leq (\chi_0^{n-1}+\mu_0^{n-1}+m_1+2m_2)T(r,f).$$

Similar results occur for $P_*(g)$.

Proof. Rewrite $P_*(f)$ and $P_*(g)$ as

$$P_*(f) = Cf(f-\beta_1)\ldots(f-\beta_{m_1})(f-\beta_{m_1+1})^{n_{m_1+1}}\ldots(f-\beta_{m_1+m_2})^{n_{m_1+m_2}} \tag{2.1}$$

and

$$P_*(g) \;=\; Cg(g-\beta_1)\ldots(g-\beta_{m_1})(g-\beta_{m_1+1})^{n_{m_1+1}}\ldots(g-\beta_{m_1+m_2})^{n_{m_1+m_2}},$$

where $\beta_i's$ $(i=1,2,\ldots,m_1+m_2)$ are distinct complex constants and n_i is the multiplicity of the factor $(z-\beta_i)$ in $P_*(z)$ for $i=1,2,\ldots,m_1+m_2$ with $n_1=n_2=\ldots=n_{m_1}=1$ and $n_{m_1+1},\ldots,n_{m_1+m_2}\geq 2$.

Here we have to consider two cases:

Case 1. Suppose none of $\beta_i's$ $(i=1,2,\ldots,m_1+m_2)$ be zero. Then

$$\overline{N}(r,0;P_*(f)) \;\leq\; \overline{N}(r,0;f)+\sum_{i=1}^{m_1+m_2}\overline{N}(r,\beta_i;f) \;\leq\; (1+m_1+m_2)T(r,f);$$

$$N_2(r,0;P_*(f)) \leq N(r,0;f)+\sum_{i=1}^{m_1}N(r,\beta_i;f)+2\sum_{i=m_1+1}^{m_1+m_2}\overline{N}(r,\beta_i;f) \leq (1+m_1+2m_2)T(r,f).$$

Case 2. Next let one of $\beta_i's$ $(i=1,2,\ldots,m_1+m_2)$ be zero.

Subcase 1: Suppose one among $\beta_i's$ $(i=1,2,\ldots,m_1)$ be zero. Without loss of generality let us assume that $\beta_1=0$. Then

$$\overline{N}(r,0;P_*(f)) \;\leq\; \overline{N}(r,0;f)+\sum_{i=2}^{m_1+m_2}\overline{N}(r,\beta_i;f) \;\leq\; (m_1+m_2)T(r,f)\;;$$

$$\begin{aligned}N_2(r,0;P_*(f)) &\leq 2\overline{N}(r,0;f)+\sum_{i=2}^{m_1}N(r,\beta_i;f)+2\sum_{i=m_1+1}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq (1+m_1+2m_2)T(r,f).\end{aligned}$$

Subcase 2: Next suppose one among $\beta_i's$ $(i=m_1+1,m_1+2,\ldots,m_1+m_2)$ be zero. Without loss of generality let us assume that $\beta_{m_1+1}=0$. Then

$$\begin{aligned}\overline{N}(r,0;P_*(f)) &\leq \overline{N}(r,0;f)+\sum_{i=1}^{m_1}\overline{N}(r,\beta_i;f)+\sum_{i=m_1+2}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq (m_1+m_2)T(r,f);\end{aligned}$$

$$\begin{aligned}N_2(r,0;P_*(f)) &\leq 2\overline{N}(r,0;f)+\sum_{i=1}^{m_1}N(r,\beta_i;f)+2\sum_{i=m_1+2}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq (m_1+2m_2)T(r,f).\end{aligned}$$

Combining all cases we can write

$$\overline{N}(r,0;P_*(f)) \leq (\chi_0^{n-1}+m_1+m_2)T(r,f);$$
$$N_2(r,0;P_*(f)) \leq (\chi_0^{n-1}+\mu_0^{n-1}+m_1+2m_2)T(r,f).$$

Similarly we can obtain the same conclusions for the function g. □

Lemma 2.13. *Let $P_*(f)$ and $P_*(g)$ for two non-constant meromorphic functions f and g (as defined in (1.2)) share $(1,2)$ and $(\infty,0)$. If*

$$n > 2(\chi_0^{n-1}+\mu_0^{n-1}+m_1+2m_2)+\frac{15}{(2n-3)}(\chi_0^{n-1}+m_1+m_2),$$

then either $P_*(f)(z) \equiv P_*(g)(z)$ or $P_*(f)(z).P_*(g)(z) \equiv 1$.

Proof. Set

$$\Phi = \frac{P_*(f)(P_*(g)-1)}{P_*(g)(P_*(f)-1)}.$$

Clearly $S(r,\Phi)$ can be replaced by $S(r,f)+S(r,g)$. It is obvious that $\Phi \not\equiv 0$. If $\Phi \equiv 0$ then either $P_*(f)=0$ or $P_*(g)=1$, which gives f and g are constants, a contradiction. First suppose that $\Phi \not\equiv 1$. So $P_*(f) \not\equiv P_*(g)$.
Therefore, using *Lemma 2.10* we get

$$\begin{aligned}
& \overline{N}(r,0;\Phi)+\overline{N}(r,\infty;\Phi) \\
\leq\ & \overline{N}(r,1;P_*(f)\ \ |\geq 3)+\overline{N}(r,0;P_*(f))+\overline{N}(r,0;P_*(g)) \\
\leq\ & \frac{1}{2}\left(\overline{N}(r,0;P_*(f))+\overline{N}(r,\infty;P_*(f))\right)+\overline{N}(r,0;P_*(f)) \\
& +\overline{N}(r,0;P_*(g))+S(r,P_*(f)) \\
\leq\ & \frac{3}{2}\overline{N}(r,0;P_*(f))+\frac{1}{2}\overline{N}(r,\infty;f)+\overline{N}(r,0;P_*(g))+S(r,f).
\end{aligned}$$

Now,

$$\Phi-1=\frac{P_*(g)-P_*(f)}{P_*(g)(P_*(f)-1)} \ \text{ and } \ \Phi' = \left[\frac{P_*(g)'}{P_*(g)(P_*(g)-1)}-\frac{P_*(f)'}{P_*(f)(P_*(f)-1)}\right]\Phi.$$

If $\Phi' \equiv 0$ then

$$\left[\frac{P_*(g)'}{P_*(g)(P_*(g)-1)}-\frac{P_*(f)'}{P_*(f)(P_*(f)-1)}\right] \equiv 0.$$

Integrating we have

$$\frac{P_*(f)-1}{P_*(f)} \equiv A\,\frac{P_*(g)-1}{P_*(g)},$$

where A is non-zero constant. i.e.,

$$1-\frac{1}{P_*(f)} \equiv A-\frac{A}{P_*(g)}.$$

Since $P_*(f)$ and $P_*(g)$ share $(\infty,0)$ so $A=1$. Then $P_*(f) \equiv P_*(g)$ which gives $\Phi \equiv 1$, a contradiction. Therefore $\Phi' \not\equiv 0$. Clearly all poles of $P_*(f)$ and $P_*(g)$ are multiple poles which are multiple zeros of $\Phi-1$ and so zeros of Φ' with multiplicity at least $(n-1)$ but not zeros of Φ. Therefore by *Lemma 2.9*,

$$\begin{aligned}(n-1)\overline{N}(r,\infty;f) &= (n-1)\overline{N}(r,\infty;P_*(f))=(n-1)\overline{N}(r,\infty;P_*(f)\mid\geq n)\\ &\leq N(r,0;\Phi^{'}\mid\Phi\neq 0)\leq\overline{N}(r,0;\Phi)+\overline{N}(r,\infty;\Phi)+S(r,\Phi).\end{aligned}$$

So,

$$(2n-3)\overline{N}(r,\infty;f)\leq 3\overline{N}(r,0;P_*(f))+2\overline{N}(r,0;P_*(g))+S(r,f).$$

Applying *Lemma 2.12* we obtain

$$\begin{aligned}\overline{N}(r,\infty;f)\leq{}&\frac{3(\chi_0^{n-1}+m_1+m_2)}{2n-3}T(r,f)\\ &+\frac{2(\chi_0^{n-1}+m_1+m_2)}{2n-3}T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Similarly

$$\begin{aligned}\overline{N}(r,\infty;g)\leq{}&\frac{3(\chi_0^{n-1}+m_1+m_2)}{2n-3}T(r,g)\\ &+\frac{2(\chi_0^{n-1}+m_1+m_2)}{2n-3}T(r,f)+S(r,f)+S(r,g).\end{aligned}$$

That is

$$\begin{aligned}\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g) &\leq \frac{5(\chi_0^{n-1}+m_1+m_2)}{2n-3}\left(T(r,f)+T(r,g)\right)\\ &\quad +S(r,f)+S(r,g).\end{aligned}\tag{2.2}$$

If possible, we suppose that (i) of *Lemma 2.11* holds. Therefore

$$\begin{aligned}&T(r,P_*(f))+T(r,P_*(g))\\ \leq\ &2\{N_2(r,0;P_*(f))+N_2(r,0;P_*(g))+\overline{N}(r,\infty;P_*(f))+\overline{N}(r,\infty;P_*(g))\\ &+\overline{N}_*(r,\infty;P_*(f),P_*(g))\}+S(r,P_*(f))+S(r,P_*(g)).\end{aligned}$$

Then using *Lemma 2.7*, *Lemma 2.12* and (2.2) we have

$$\begin{aligned}&n\left(T(r,f)+T(r,g)\right)\\ \leq\ &\left(2(\chi_0^{n-1}+\mu_0^{n-1}+m_1+2m_2)+\frac{15(\chi_0^{n-1}+m_1+m_2)}{2n-3}\right)\left(T(r,f)+T(r,g)\right)\\ &+S(r,f)+S(r,g),\end{aligned}$$

which contradicts our assumption. So by *Lemma 2.11* we have

$$P_*(f)(z).P_*(g)(z)\equiv 1.$$

If $\Phi\equiv 1$, then $P_*(f)(z)\equiv P_*(g)(z)$.
Hence the lemma is proved. □

Lemma 2.14. *Let f and g be two non-constant meromorphic functions of finite order. Let $n \geq 2$, and let $\{a_1(z), a_2(z), \ldots, a_n(z)\} \in S(f)$ be distinct meromorphic periodic functions with period c. If $m\left(r, \frac{g}{f-a_k}\right) = S(r,f)$, for $k = 1, 2, \ldots, n$, then*

$$\sum_{k=1}^{n} m\left(r, \frac{1}{f-a_k}\right) \leq m\left(r, \frac{1}{g}\right) + S(r,f),$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Proof. Set

$$P(f) = \prod_{k=1}^{n}(f - a_k).$$

Rewriting we have

$$\frac{1}{P(f)} = \sum_{k=1}^{n} \frac{\alpha_k}{f-a_k},$$

where $\alpha_k \in S(f)$ are certain periodic function with period c. Now,

$$m\left(r, \frac{g}{P(f)}\right) \leq \sum_{k=1}^{n} m\left(r, \frac{g}{f-a_k}\right) + S(r,f) = S(r,f),$$

and so

$$m\left(r, \frac{1}{P(f)}\right) = m\left(r, \frac{g}{P(f)}\right) + m\left(r, \frac{1}{g}\right) \leq m\left(r, \frac{1}{g}\right) + S(r,f).$$

By the first fundamental theorem and using the above inequation we get

$$m\left(r, \frac{1}{g}\right) \geq m\left(r, \frac{1}{P(f)}\right) + S(r,f) = T(r, P(f)) - N\left(r, \frac{1}{P(f)}\right) + S(r,f)$$

$$\geq nT(r,f) - \sum_{k=1}^{n} N\left(r, \frac{1}{f-a_k}\right) + S(r,f) = \sum_{k=1}^{n} m\left(r, \frac{1}{f-a_k}\right) + S(r,f). \qquad \square$$

Lemma 2.15. *If f be a meromorphic function of finite order then $\tilde{L}(f(z))$ is of finite order and*

$$m\left(r, \frac{\tilde{L}(f(z))}{f(z+c)}\right) = S(r,f), \; m\left(r, \frac{\tilde{L}(f(z))}{f(z)-\beta_i}\right) = S(r,f)$$

and

$$m\left(r, \frac{\tilde{L}(f(z))}{f(qz)}\right) = S_1(r,f).$$

Proof. Using logarithmic derivative lemma and *Lemma 2.2* we have

$$\begin{aligned} m\left(r, \frac{\tilde{L}(f(z))}{f(z+c)}\right) &= m\left(r, \frac{\sum_{j=1}^{s} b_j f^{(j)}(z+c_j) + \sum_{j=1}^{t} d_j f^{(j)}(z)}{f(z+c)}\right) \\ &\leq \sum_{j=1}^{s} m\left(r, \frac{f^{(j)}(z+c_j)}{f^{(j)}(z)}\right) + \sum_{j=1}^{s} m\left(r, \frac{f^{(j)}(z)}{f(z)}\right) \\ &\quad + \sum_{j=1}^{t} m\left(r, \frac{f^{(j)}(z)}{f(z)}\right) + (s+t)\, m\left(r, \frac{f(z)}{f(z+c)}\right) + O(1) \\ &= S(r,f). \end{aligned} \tag{2.3}$$

Also,

$$\begin{aligned} m\left(r, \frac{\tilde{L}(f(z))}{f(z)-\beta_i}\right) &= m\left(r, \frac{\sum_{j=1}^{s} b_j f^{(j)}(z+c_j) + \sum_{j=1}^{t} d_j f^{(j)}(z)}{f(z)-\beta_i}\right) \\ &\leq \sum_{j=1}^{s} m\left(r, \frac{f^{(j)}(z+c_j)}{f^{(j)}(z)}\right) + \sum_{j=1}^{t} m\left(r, \frac{f^{(j)}(z)}{f(z)-\beta_i}\right) \\ &\quad + \sum_{j=1}^{s} m\left(r, \frac{f^{(j)}(z)}{f(z)-\beta_i}\right) + O(1) = S(r,f). \end{aligned}$$

Using (2.3) and *Lemma 2.1* we have

$$T(r, \tilde{L}(f(z))) \leq \frac{s^2+t^2+3(s+t)+2}{2}\, T(r,f) + S(r,f).$$

As f is of finite order so $\tilde{L}(f(z))$ and $f(z+c)$ is of finite order and $S(r, \tilde{L}(f(z)))$ can be replaced by $S(r,f)$.

Similarly by using *Lemma 2.4*, *Lemma 2.5* and *Lemma 2.6* as and when required we can prove $f(qz)$ and $\tilde{L}(f(z))$ are zero order when f is of zero order and

$$m\left(r, \frac{\tilde{L}(f(z))}{f(qz)}\right) = S_1(r,f). \qquad \square$$

3. Proofs of the theorems

Proof of Theorem 1.1. Since

$$E_{f(z+c)}(S_1^*, 2) = E_{\tilde{L}(f(z))}(S_1^*, 2) \text{ and } E_{f(z+c)}(S_2, 0) = E_{\tilde{L}(f(z))}(S_2, 0),$$

it follows that $P_*(f(z+c))$, $P_*(\tilde{L}(f(z)))$ share $(1,2)$ and $(\infty,0)$. So by *Lemma 2.13* we have either $P_*(f(z+c)) \equiv P_*(\tilde{L}(f(z)))$ or $P_*(f(z+c)).P_*(\tilde{L}(f(z))) \equiv 1$. Suppose that

$$P_*(f(z+c)).P_*(\tilde{L}(f(z))) \equiv 1. \tag{3.1}$$

Noting that $P_*(f(z+c))$, $P_*(\tilde{L}(f(z)))$ share $(\infty,0)$, so we can conclude that $P_*(f(z+c))$, $P_*(\tilde{L}(f(z)))$ both are entire functions.
So

$$N\left(r,\infty;\frac{P_*(\tilde{L}(f(z)))}{P_*(f(z+c))}\right) = N(r,0;P_*(f(z+c))).$$

Therefore using *Lemma 2.12* and *Lemma 2.1*, we get

$$N\left(r,\infty;\frac{P_*(\tilde{L}(f(z)))}{P_*(f(z+c))}\right) \leq (\chi_0^{n-1}+m_1+m_2)T(r,f(z+c)) \leq nT(r,f)+S(r,f).$$

Using *Lemma 2.2* and *Lemma 2.15* we have

$$\begin{aligned}
m\left(r,\ \frac{P_*(\tilde{L}(f(z)))}{P_*(f(z+c))}\right) &= m\left(r,\ \frac{\tilde{L}(f(z))}{f(z+c)}\prod_{i=1}^{m_1+m_2}\left(\frac{\tilde{L}(f(z))-\beta_i}{f(z+c)-\beta_i}\right)^{n_i}\right)\\
&\leq m\left(r,\frac{\tilde{L}(f(z))}{f(z+c)}\right)+m\left(r,\ \prod_{i=1}^{m_1+m_2}\left(\frac{\tilde{L}(f(z))-\beta_i}{f(z+c)-\beta_i}\right)^{n_i}\right)+O(1)\\
&\leq \sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{\tilde{L}(f(z))-\beta_i}{f(z+c)-\beta_i}\right)+S(r,f)\\
&\leq \sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{\tilde{L}(f(z))}{f(z)-\beta_i}\right)+\sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{1}{f(z)-\beta_i}\right)\\
&+\sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{f(z)-\beta_i}{f(z+c)-\beta_i}\right)+S(r,f)\\
&\leq \sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{1}{f(z)-\beta_i}\right)+S(r,f)\\
&\leq (n_1+n_2+\ldots+n_{m_1+m_2})T(r,f)+S(r,f)\\
&\leq (n-1)T(r,f)+S(r,f).
\end{aligned}$$

By *Lemma 2.1*, *Lemma 2.7* and (3.1),

$$\begin{aligned}
2nT(r,f) &= 2nT(r,f(z+c))+S(r,f) = 2T(r,P_*(f(z+c)))+S(r,f)\\
&\leq T\left(r,\ \frac{1}{P_*(f(z+c))^2}\right)+S(r,f) \leq T\left(r,\ \frac{P_*(\tilde{L}(f(z)))}{P_*(f(z+c))}\right)+S(r,f)\\
&\leq (2n-1)T(r,f)+S(r,f),
\end{aligned}$$

which is a contradiction.

Therefore $P_*(\tilde{L}(f(z))) \equiv P_*(f(z+c))$, which yields
$$\prod_{i=1}^{n}(\tilde{L}(f(z))-\alpha_i) \equiv \prod_{i=1}^{n}(f(z+c)-\alpha_i). \qquad \square$$

Proof of Theorem 1.2. By proceeding in a similar way of the proof of *Theorem 1.1* we can prove this theorem using *Lemma 2.4*, *Lemma 2.5* and *Lemma 2.6* as and when required instead of *Lemma 2.1 and Lemma 2.2.* $\square$

Proof of Theorem 1.3. Since the finite order meromorphic functions $f(z+c)$ and $\tilde{L}(f(z))$ share (S_1^*, ∞), (S_2, ∞), it follows that $P_*(f(z+c))$, $P_*(\tilde{L}(f(z)))$ share $(1,\infty)$ and (∞,∞) which yields
$$N(r, \tilde{L}(f(z))) = N(r, f(z+c)) \tag{3.2}$$
and
$$\frac{P_*(\tilde{L}(f(z)))-1}{P_*(f(z+c))-1} = e^{\gamma(z)}, \tag{3.3}$$
where $\gamma(z)$ is a polynomial.
Now,
$$T(r, e^{\gamma(z)}) = m(r, e^{\gamma(z)}) = m\left(r, \frac{P_*(\tilde{L}(f(z)))-1}{P_*(f(z+c))-1}\right).$$
Using the definition of $P_*(z)$ we have
$$\begin{aligned} T(r, e^{\gamma(z)}) &= m\left(r, \frac{(\tilde{L}(f(z)))-\alpha_1)(\tilde{L}(f(z)))-\alpha_2)\ldots(\tilde{L}(f(z)))-\alpha_n)}{(f(z+c)-\alpha_1)(f(z+c)-\alpha_2)\ldots(f(z+c)-\alpha_n)}\right) \\ &\leq \sum_{j=1}^{n} m\left(r, \frac{\tilde{L}(f(z)))-\alpha_j}{f(z+c)-\alpha_j}\right) + O(1) \\ &\leq \sum_{j=1}^{n} m\left(r, \frac{\tilde{L}(f(z))}{f(z)-\alpha_j}\right) + \sum_{j=1}^{n} m\left(r, \frac{1}{f(z)-\alpha_j}\right) + \sum_{j=1}^{n} m\left(r, \frac{f(z)-\alpha_j}{f(z+c)-\alpha_j}\right) \\ &\quad + O(1). \end{aligned}$$
In view of *Lemma 2.2*, *Lemma 2.14*, *Lemma 2.15* and then by the first fundamental theorem and (3.2) we have
$$\begin{aligned} T(r, e^{\gamma(z)}) &= \sum_{j=1}^{n} m\left(r, \frac{1}{f(z)-\alpha_j}\right) + S(r,f) \leq m\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f) \\ &\leq T(r, \tilde{L}(f(z))) - N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f) \\ &\leq m\left(r, \frac{\tilde{L}(f(z))}{f(z+c)}\right) + m(r, f(z+c)) + N(r, \tilde{L}(f(z))) - N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f) \\ &\leq T(r, f(z+c)) - N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f) \end{aligned}$$

$$\leq T(r,f) - N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f).$$

According to the given condition

$$T(r,f) = N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f),$$

so $T(r, e^{\gamma(z)}) = S(r,f)$.
Now from (3.3) we have

$$P_*(\tilde{L}(f(z))) = e^{\gamma(z)}\left(P_*(f(z+c)) - 1 + e^{-\gamma(z)}\right).$$

Set

$$W(z) = \frac{P_*(f(z+c))}{1 - e^{-\gamma(z)}}.$$

If $e^{\gamma(z)} \not\equiv 1$, then by applying Nevanlinna's second fundamental theorem to $W(z)$ and using (3.2) and *Lemma 2.12* we obtain

$$\begin{aligned}
&T(r, P_*(f(z+c))) \leq T(r,W) + S(r,f)\\
&\leq \overline{N}(r,0;W) + \overline{N}(r,\infty;W) + \overline{N}(r,0;W-1) + S(r,f)\\
&\leq \overline{N}(r,0;P_*(f(z+c))) + \overline{N}(r,\infty;P_*(f(z+c))) + \overline{N}(r,0;P_*(\tilde{L}(f(z)))) + S(r,f)\\
&\leq (\chi_0^{n-1} + m_1 + m_2)\left(T(r,f(z+c)) + T(r,\tilde{L}(f(z)))\right) + N(r,\infty;f(z+c)) + S(r,f)\\
&\leq (\chi_0^{n-1} + m_1 + m_2)\left(T(r,f(z+c)) + m(r,f(z+c)) + m\left(r, \frac{\tilde{L}(f(z))}{f(z+c)}\right)\right.\\
&\left. + N(r,\infty;f(z+c))\right) + N(r,\infty;f) + S(r,f).
\end{aligned}$$

Using *Lemma 2.1* and *Lemma 2.15* we get

$$nT(r,f) \leq (2\chi_0^{n-1} + 2m_1 + 2m_2 + 1)T(r,f) + S(r,f),$$

which contradicts $n > 2(\chi_0^{n-1} + m_1 + m_2) + 1$. This gives $e^{\gamma(z)} \equiv 1$, that yields

$$\prod_{i=1}^{n}(\tilde{L}(f(z)) - \alpha_i) \equiv \prod_{i=1}^{n}(f(z+c) - \alpha_i). \qquad \square$$

Proof of Theorem 1.4. Here $\tilde{L}(f(z))$ and $f(qz)$ are of zero order. Since $f(qz)$ and $\tilde{L}(f(z))$ share (S_1^*, ∞) and (S_2, ∞), it follows that $P_*(f(qz))$ and $P_*(\tilde{L}(f(z)))$ share $(1,\infty)$ and (∞,∞). Therefore

$$\frac{P_*(\tilde{L}(f(z))) - 1}{P_*(f(qz)) - 1} = A,$$

where A is a non-zero constant.
This gives

$$P_*(\tilde{L}(f(z))) = A\left(P_*(f(qz)) - 1 + \frac{1}{A}\right).$$

Set $W_1(z) = \frac{P_*(f(qz))}{1-\frac{1}{A}}$. If $A \not\equiv 1$, then applying Nevanlinna's second fundamental theorem to $W_1(z)$ and using *Lemmas 2.4* and *2.5* and *2.15* as and when required we can calculate the rest of the proof similar to *Theorem 1.3*. □

Proof of Theorem 1.5. Here $f(z+c)$ and $\tilde{L}(f(z))$ are of finite order. Since $f(z+c)$ and $\tilde{L}(f(z))$ share $(0,\infty)$ and (∞,∞), so

$$\frac{\tilde{L}(f(z))}{f(z+c)} = e^{\delta(z)}, \tag{3.4}$$

where $\delta(z)$ is a polynomial.
Clearly by *Lemma 2.15* we get

$$T(r, e^{\delta(z)}) = S(r,f).$$

When $e^{\delta(z)} \equiv 1$ then $\tilde{L}(f(z)) \equiv f(z+c)$.
When $e^{\delta(z)} \not\equiv 1$, using the fact that $f(z+c)$ and $\tilde{L}(f(z))$ share $(a(z),0)$ we have

$$\begin{aligned}\overline{N}\left(r, \frac{1}{\tilde{L}(f(z)) - a(z)}\right) = \overline{N}\left(r, \frac{1}{f(z+c) - a(z)}\right) &\le \overline{N}\left(r, \frac{1}{e^{\delta(z)}-1}\right) + \overline{N}\left(r, \frac{1}{a(z)}\right)\\ &\le T(r, e^{\delta(z)}) + S(r,f) = S(r,f).\end{aligned}$$

Rewriting (3.4) we get

$$\tilde{L}(f(z)) - a(z) = e^{\delta(z)}(f(z+c) - a(z)e^{-\delta(z)}).$$

Clearly $a(z)e^{-\delta(z)} \not\equiv a(z)$. So,

$$\overline{N}\left(r, \frac{1}{f(z+c) - a(z)e^{-\delta(z)}}\right) = \overline{N}\left(r, \frac{1}{\tilde{L}(f(z)) - a(z)}\right) = S(r,f).$$

Using *Lemma 2.1*, *2.3* and the second fundamental theorem we obtain

$$\begin{aligned} & 2T(r,f) = 2T(r, f(z+c)) + S(r,f)\\ \le\ & \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}\left(r, \frac{1}{f(z+c) - a(z)}\right)\\ & + \overline{N}\left(r, \frac{1}{f(z+c) - a(z)e^{-\delta(z)}}\right) + S(r,f) \le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f),\end{aligned}$$

which is a contradiction to $\Theta(0;f) + \Theta(\infty;f) > 0$. Hence $\tilde{L}(f(z)) \equiv f(z+c)$. □

Proof of Theorem 1.6. Here $f(z+c)$ and $\tilde{L}(f(z))$ are of finite order. Since $f(z+c)$ and $\tilde{L}(f(z))$ share $(a(z),\infty)$ and (∞,∞), so

$$\frac{\tilde{L}(f(z)) - a(z)}{f(z+c) - a(z)} = e^{\zeta(z)}, \tag{3.5}$$

where $\zeta(z)$ is a polynomial. Using logarithmic derivative lemma, *Lemma 2.1* and *Lemma 2.2* we get

$$
\begin{aligned}
&T(r,e^{\zeta(z)}) = m(r,e^{\zeta(z)}) = m\left(r,\ \frac{\tilde{L}(f(z))-a(z)}{f(z+c)-a(z)}\right)\\
\leq\ & m\left(r,\ \frac{\tilde{L}(f(z))-\tilde{L}(a(z-c))}{f(z+c)-a(z)}\right)+m\left(r,\ \frac{\tilde{L}(a(z-c))-a(z)}{f(z+c)-a(z)}\right)+O(1)\\
\leq\ & m\left(r,\ \frac{\tilde{L}(f(z))-\tilde{L}(a(z-c))}{f(z)-a(z-c)}\right)+m\left(r,\ \frac{f(z)-a(z-c)}{f(z+c)-a(z)}\right)\\
& +m\left(r,\ \frac{1}{f(z+c)-a(z)}\right)+S(r,f)\\
\leq\ & m\left(r,\ \frac{\sum_{j=1}^{s} b_j(f^{(j)}(z+c_j)-a^{(j)}(z-c+c_j))+\sum_{j=1}^{t} d_j(f^{(j)}(z)-a^{(j)}(z-c))}{f(z)-a(z-c)}\right)\\
& +T(r,f(z+c))+S(r,f)\\
\leq\ & \sum_{j=1}^{s} m\left(r,\ \frac{f^{(j)}(z+c_j)-a^{(j)}(z-c+c_j)}{f^{(j)}(z)-a^{(j)}(z-c)}\right)+\sum_{j=1}^{t} m\left(r,\ \frac{f^{(j)}(z)-a^{(j)}(z-c)}{f(z)-a(z-c)}\right)\\
& +\sum_{j=1}^{s} m\left(r,\ \frac{f^{(j)}(z)-a^{(j)}(z-c)}{f(z)-a(z-c)}\right)+T(r,f)+S(r,f)\\
\leq\ & T(r,f)+S(r,f).
\end{aligned}
$$

So $S(r,e^{\zeta(z)})$ can be replaced by $S(r,f)$. When $e^{\zeta(z)}\equiv 1$ then $\tilde{L}(f(z))\equiv f(z+c)$. Suppose $e^{\delta(z)}\not\equiv 1$. Now rewriting (3.5) we can obtain

$$\frac{1}{f(z+c)} = -\frac{\tilde{L}(f(z))}{a(z)f(z+c)(e^{\zeta(z)}-1)}+\frac{e^{\zeta(z)}}{a(z)(e^{\zeta(z)}-1)}.$$

Therefore in view of *Lemma 2.15* we have

$$m\left(r,\frac{1}{f(z+c)}\right)\leq 2\,m\left(r,\frac{1}{e^{\zeta(z)}-1}\right)+S(r,f).$$

If $\zeta(z)$ is constant then automatically $m\left(r,\frac{1}{f(z+c)}\right)=S(r,f)$. If $\zeta(z)$ is non-constant then by *Lemma 2.8* we get

$$m\left(r,\frac{1}{f(z+c)}\right)=S(r,e^{\zeta(z)})=S(r,f).$$

By *Lemma 2.1* and *Lemma 2.3* we have

$$\begin{aligned} T(r,f) &= T(r,f(z+c))+S(r,f)=T\left(r,\frac{1}{f(z+c)}\right)+S(r,f) \\ &\leq N\left(r,\frac{1}{f(z+c)}\right)+S(r,f)\leq N\left(r,\frac{1}{f}\right)+S(r,f)\leq T(r,f)+S(r,f). \end{aligned}$$

Therefore,

$$N\left(r,\frac{1}{f}\right)=T(r,f)+S(r,f),$$

which contradicts the fact that $\delta(0,f)>0$. Hence $\tilde{L}(f(z))\equiv f(z+c)$. □

4. Observation

Take $\tilde{L}(f(z))=L_3$ with all coefficients are 1. Then we see that choosing

$$c=\frac{\log(\alpha+\alpha^2+\ldots+\alpha^t)}{\alpha},$$

where $1+\alpha+\ldots+\alpha^{t-1}\neq 0$, we somehow get a solution $f(z)=e^{\alpha z}$ $(\alpha\neq 0)$ of

$$\tilde{L}(f(z))=f(z+c). \tag{4.1}$$

However choosing $c=\frac{\pi}{2}$, we can present the solution of $f'=f(z+c)$ as the linear combination of two independent solutions. e.g., $f(z)=d_1e^{iz}+d_2e^{-iz}$. So it is a matter of concern that how the solutions of (4.1) looks like. Unfortunately we can not elucidated in this matter.

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References

[1] Banerjee, A., *Meromorphic functions sharing two sets*, Czech. Math. J., **57**(132)(2007), 1199-1214.

[2] Barnett, D.C., Halburd, R.G., Korhonen, R.J., Morgan, W., *Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations*, Proc. Roy. Soc. Edinburgh Sect. A, **137**(2007), 457-474.

[3] Chiang, Y.M., Feng, S.J., *On the Nevanlinna characteristic $f(z+\eta)$ and difference equations in complex plane*, Ramanujan J., **16**(2008), 105-129.

[4] Gross, F., **Univ. Kentucky**, Lexington, Kentucky (1976); Lecture Notes in Math., **599**(1977), 51-69, Springer, Berlin.

[5] Halburd, R.G., Korhonen, R.J., *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl., **314**(2006), 477-487.

[6] Halburd, R., Korhonen, R., Tohge, K., *Holomorphic curves with shift-invariant hyperplane preimages*, Trans. Amer. Math. Soc., **366**(2014), 4267-4298.

[7] Hayman, W.K., *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[8] Lahiri, I., *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161**(2001), 193-206.

[9] Lahiri, I., *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theo. Appl., **46**(2001), 241-253.

[10] Lahiri, I., Dewan, S., *Value distribution of the product of a meromorphic function and its derivative*, Kodai Math. J., **26**(2003), 95-100.

[11] Li, P., Wang, W.J., *Entire functions that share a small function with its derivative*, J. Math. Anal. Appl., **328**(2007), 743-751.

[12] Meng, C., Liu, G., *Uniqueness of meromorphic functions concerning their shifts and derivatives*, J. Appl. Math. Inform., **37**(2019), no. 1-2, 133-148.

[13] Qi, X., Li, N., Yang, L., *Uniqueness of meromorphic functions concerning their differences and solutions of difference Painlevé equations*, Comput. Methods Funct. Theory, **18**(2018), 567-582.

[14] Qi, X., Yang, L., *Uniqueness of meromorphic functions concerning their shifts and derivatives*, Comput. Methods Funct. Theory, **20**(2020), 159-178.

[15] Yang, C.C., Yi, H.X., *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht, 2003.

[16] Zhang, J.L., Korhonen, R.J., *On the Nevanlinna characteristic of $f(qz)$ and its application*, J. Math. Anal. Appl., **369**(2010), 537-544.

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