Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 2, 151–162
DOI: 10.24193/subbmath.2017.2.02

The first Zolotarev case in the Erdös-Szegö solution to a Markov-type extremal problem of Schur

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Abstract. Schur's [14] Markov-type extremal problem asks to find the maximum (1) $\sup_{-1\leq\xi\leq 1}\ \sup_{P_n\in\mathbf{B}_{n,\xi,2}} |P_n^{(1)}(\xi)|$, where $\mathbf{B}_{n,\xi,2} = \{P_n \in \mathbf{B}_n : P_n^{(2)}(\xi) = 0\} \subset \mathbf{B}_n = \{P_n : |P_n(x)| \leq 1 \text{ for } |x| \leq 1\}$ and P_n is an algebraic polynomial of degree $\leq n$. Erdös and Szegö [3] found that for $n \geq 4$ this maximum is attained if $\xi = \pm 1$ and $P_n \in \mathbf{B}_{n,\xi,2}$ is a (unspecified) member of the 1-parameter family of hard-core Zolotarev polynomials $Z_{n,t}$. Our first result centers around the proof in [3] for the initial case $n = 4$ and is three-fold: (i) the numerical value for (1) in ([3], (7.9)) is not correct, but sufficiently precise; (ii) from preliminary work in [3] can in fact be deduced a closed analytic expression for (1) if $n = 4$, allowing numerical evaluation to any precision; (iii) even the explicit power form representation of an extremal $Z_{4,t} = Z_{4,t^*}$ can be deduced from [3], thus providing an exemplification of Schur's problem that seems to be novel. Additionally, we cross-check its validity twice: firstly by deriving Z_{4,t^*} conversely from a general formula for $Z_{4,t}$ that we have given in [12], and secondly by applying Theorem 5.10 in [11]. We then turn to a generalized solution of Schur's problem, to k -th derivatives, by Shadrin [16]. Again we provide in explicit form the corresponding maximum as well as an extremizer polynomial for the first non-trivial degree $n = 4$. Finally, we contribute to the fuller description of $Z_{4,t}$ by providing its critical points in explicit form.

Mathematics Subject Classification (2010): 26C05, 26D10, 41A10, 41A17, 41A29, 41A44, 41A50.

Keywords: Chebyshev, derivative, Erdös, extremal problem, inequality, Markov, polynomial, quartic, Schur, Shadrin, Szegö, Zolotarev.

1. Introduction

The famous A. A. Markov inequality of 1889 [8] asserts an estimate on the size of the first derivative of an algebraic polynomial P_n of degree $\leq n$ and can be restated as follows:

$$\sup_{\xi\in\mathbf{I}} \sup_{P_n\in\mathbf{B}_n} |P_n^{(1)}(\xi)| = n^2 = T_n^{(1)}(1), \tag{1.1}$$

where $\mathbf{I} = [-1, 1]$ and $\mathbf{B}_n = \{P_n : |P_n(x)| \leq 1 \text{ for } x \in \mathbf{I}\}$. As indicated, this maximum will be attained if, up to the sign, $P_n = T_n \in \mathbf{B}_n$ is the n-th Chebyshev polynomial of the first kind on $\mathbf{I}$ (defined by $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ with $T_1(x) = x$, $T_0(x) = 1$) and if $\xi = \pm 1$, see e.g. ([10], p. 529), ([13], p. 123).

In 1919 I. Schur ([14], §2), inspired by (1.1), was led to the problem of finding the maximum of $|P_n^{(1)}(\xi)|$ under the additional restriction $P_n^{(2)}(\xi) = 0$: Determine $P_n = P_n^*$ which attains, for $n \geq 3$,

$$\sup_{\xi\in\mathbf{I}} \sup_{P_n\in\mathbf{B}_{n,\xi,2}} |P_n^{(1)}(\xi)| = n^2 M_n, \tag{1.2}$$

where $\mathbf{B}_{n,\xi,2} = \{P_n \in \mathbf{B}_n : P_n^{(2)}(\xi) = 0\}$ and M_n is a constant (depending on n). Schur ([14], (9)) proved that there holds

$$\sup_{\xi\in\mathbf{I}} \sup_{P_n\in\mathbf{B}_{n,\xi,2}} |P_n^{(1)}(\xi)| < \frac{1}{2}n^2, \text{so that } M_n < \frac{1}{2}. \tag{1.3}$$

In 1942 P. Erdös and G. Szegö addressed this problem of Schur and they provided the following solution ([3], Theorem 2):

*The maximum (*1.2*) will be attained, for $n \geq 4$, only if $\xi = 1$ and $P_n = P_n^*$ is a (unspecified) member of the* 1*-parameter family (with parameter t) of hard-core Zolotarev polynomials $\pm Z_{n,t}$; or if $\xi = -1$ and $P_n = P_n^*$ is a (unspecified) member of the family $\pm Z_{n,t}^-$, where $Z_{n,t}^-(x) = Z_{n,t}(-x)$.*

We leave aside the simple case $n = 3$ (with solution $\xi = 0$ and $P_3 = P_3^* = \pm T_3$ ([3], p. 466)). Henceforth we will confine ourselves to specify only one extremal polynomial P_n^* for a given problem on $\mathbf{I}$, but will keep in mind that $-P_n^*$ as well as $\pm Q_n^*$, where $\pm Q_n^*(x) = \pm P_n^*(-x)$, may likewise be extremal. The solutions to (1.1) and (1.2) have in common that the maximum is attained at the endpoints $\xi = \pm 1$ of the unit interval $\mathbf{I}$. But, on the other hand, the solutions differ greatly when it comes to exhibit an explicit extremal polynomial from $\mathbf{B}_n$ resp. $\mathbf{B}_{n,\xi,2}$: Whereas in (1.1) an extremizer is, for all $n \geq 1$, the well-known n -th Chebyshev polynomial T_n [13], the explicit power form of the intricate extremizers $Z_{n,t}$ in (1.2) remained arcane for all $n \geq 4$. This is due to the fact that for a general degree n the explicit power form of a hard-core Zolotarev polynomial $Z_{n,t}$ is not known ([16], p. 1185). Rather, $Z_{n,t}$ can be expressed with the aid of elliptic functions (see ([1], pp. 280), ([10], p. 407), [18]) which amounts to *an extremely complicated concoction of elliptic quantities* ([17], p. 52).

It is a purpose of this note to provide, nearly one hundred years after the origin of

Schur's problem, the explicit power form of a particular hard-core Zolotarev polynomial $Z_{n,t} = Z_{n,t^*}$ which is extremal for (1.2), at least for the first nontrivial case $n = 4$. Such a solution was coined *Schur polynomial* in ([11], Section 5d), where a numerical method (solution of a system of nonlinear equations) is advised in order to determine it.

We will first tackle the explicit analytic expression for (1.2) if $n = 4$. Once it has been established, to calculate its numerical value to arbitrary precision becomes immediate. Incidentally, we notice that the numerical value for $16M_4$ as given in ([3], (7.9)) is not correct from the third decimal place on. We then deduce, in three alternative fashions, an extremal hard-core Zolotarev polynomial $P_4^* = Z_{4,t^*}$ with optimal value t^* of the parameter t. This Schur polynomial P_4^* may well serve as illustrative example of the result in ([3], Theorem 2). Finally, we will consider a recent generalization of Schur's problem (1.2), due to A. Shadrin [16], to higher derivatives of P_n, and again we will exemplify the quartic case $n = 4$. In a closing remark we reveal the critical points of $Z_{4,t}$ to get a fuller picture of the quartic hard-core Zolotarev polynomial.

2. Analytical and numerical value of the maximum in the quartic case

To determine the value in (1.2) for $n = 4$ we rely on preliminary work in ([3], Section 7) and will therefore retain, for the reader's convenience, the notation used there. A sought-for extremal hard-core Zolotarev polynomial P_4^* which solves (1.2) can be assumed to be from class $\mathbf{B}_{4,1,2}$ and be represented as, see ([3], (7.3)),

$$P_4^*(x) = 1 - \lambda(1-x)(B_4 - x)(y_1 - x)^2, \tag{2.1}$$

where λ, B_4, y_1 are parameters which reflect properties of P_4^*, such as:

$$P_4^*(-1) = -1,\ P_4^*(y_1) = 1,\ P_4^{*(1)}(y_1) = 0,\ P_4^*(1) = P_4^*(B_4) = 1.$$

The first and second derivative of P_4^* at $x = 1$ read:

$$P_4^{*(1)}(1) = \lambda(B_4 - 1)(1 - y_1)^2 \text{ and } P_4^{*(2)}(1) = 2\lambda(y_1 - 1)(2(1 - B_4) - (y_1 - 1)), \tag{2.2}$$

so that the condition $P_4^{*(2)}(1) = 0$ yields $y_1 = 3 - 2B_4$ which, when inserted into $P_4^{*(1)}(1)$, eliminates there the parameter y_1. From $P_4^*(-1) = -1$ one deduces, upon inserting the said value of y_1, that

$$\lambda = \frac{1}{(B_4 + 1)(4 - 2B_4)^2},$$

see (2.1). This implies

$$P_4^{*(1)}(1) = \frac{(B_4 - 1)^3}{(B_4 - 2)^2(B_4 + 1)}.$$

The identity

$$\frac{2}{B_4 - 1} = \frac{11 - \sqrt{33} + 2\sqrt{5(5 + \sqrt{33})}}{8},$$

which is given in an equivalent form in ([3], (7.8)), allows to evaluate B_4 (see (3.2) below). Inserting this value of B_4 into the preceding expression for $P_4^{*(1)}(1)$ eventually yields the analytical expression for the maximum, which can be evaluated numerically to any desired precision:

$$\begin{aligned} P_4^{*(1)}(1) &= \sup_{\xi\in\mathbf{I}} \sup_{P_4\in\mathbf{B}_{4,\xi,2}} \left|P_4^{(1)}(\xi)\right| = 16M_4 \\ &= \frac{-561+161\sqrt{33}+\sqrt{30(15215+3329\sqrt{33})}}{288} \\ &= 4.7876468942..., \end{aligned} \tag{2.3}$$

being a root of $P_4(x) = -65536 - 39424x - 1915x^2 + 1683x^3 + 216x^4$.

By contrast, Formula (7.9) in [3] states that

$$P_4^{*(1)}(1) = 4.7881... \tag{2.4}$$

holds, a value which is now seen to be biased in the third and fourth decimal place. But that bias does not harm the argument in [3] for $n = 4$ since the first two valid decimal places are sufficiently conclusive for P_4^* to be the extremal element (as a comparison is drawn with competitor polynomial T_4 and value $\left|T_4^{(1)}\left(\frac{1}{\sqrt{6}}\right)\right| = 4.3546...$, see ([3], (7.2))).

The constant M_4 itself can thus be represented as

$$\begin{aligned} M_4 &= \frac{P_4^{*(1)}(1)}{16} = \sup_{\xi\in\mathbf{I}} \sup_{P_4\in\mathbf{B}_{4,\xi,2}} \frac{\left|P_4^{(1)}(\xi)\right|}{4^2} \\ &= \frac{-561+161\sqrt{33}+\sqrt{30(15215+3329\sqrt{33})}}{4608} \\ &= 0.2992279308... . \end{aligned} \tag{2.5}$$

We note that according to ([3], (1.3), (1.4)) there holds $\lim_{n\to\infty} M_n = 0.3124...$. Schur ([14], p. 277) had obtained the weaker result $0.217... \leq \limsup_{n\to\infty} M_n \leq 0.465...$.

3. Explicit power form representation of an extremal hard-core Zolotarev polynomial in the quartic case

Having expressed the parameters $\lambda = \lambda(B_4)$ and $y_1 = y_1(B_4)$ as functions of B_4 alone and knowing the value of the constant B_4, it is possible to even retrieve the explicit power form of an extremal P_4^*. In fact, according to the preceding Section we have

$$\begin{aligned} P_4^*(x) &= 1 - \lambda(1-x)(B_4-x)(y_1-x)^2 \\ &= 1 - \frac{(1-x)(B_4-x)(3-2B_4-x)^2}{(B_4+1)(4-2B_4)^2} \end{aligned} \tag{3.1}$$

Inserting now

$$B_4 = \frac{177 - 17\sqrt{33} + \sqrt{30(527 + 97\sqrt{33})}}{144} = 1.8034303689... \tag{3.2}$$

and expanding (3.1) leads us, after some algebraic manipulations, to the explicit power form representation of an extremal quartic hard-core Zolotarev polynomial P_4^* with

$$P_4^*(x) = \sum_{i=0}^{4} a_i^* x^i$$

and with coefficients

$$\begin{aligned}
a_0^* &= \frac{21297 - 2081\sqrt{33} - \sqrt{30(3160847 + 628577\sqrt{33})}}{9216} = -0.5328330303...\\
a_1^* &= \frac{291 - 1139\sqrt{33} - \sqrt{30(-1236313 + 427337\sqrt{33})}}{4608} = -2.6688925571...\\
a_2^* &= \frac{-849 + 161\sqrt{33} + \sqrt{30(15215 + 3329\sqrt{33})}}{384} = 2.8407351706...\\
a_3^* &= \frac{4317 + 1139\sqrt{33} + \sqrt{30(-1236313 + 427337\sqrt{33})}}{4608} = 3.6688925571...\\
a_4^* &= \frac{-921 - 1783\sqrt{33} - \sqrt{330(-59555 + 64243\sqrt{33})}}{9216} = -2.3079021403... .
\end{aligned} \tag{3.3}$$

These optimal coefficients a_i^* are roots of the following respective quartic polynomials $P_{4,i}$ with integer coefficients:

$$\begin{aligned}
P_{4,0}(x) &= -7951932 - 7463259x + 11697424x^2 - 4089024x^3 + 442368x^4\\
P_{4,1}(x) &= 12221 + 273251x - 7120x^2 - 3492x^3 + 13824x^4\\
P_{4,2}(x) &= -236196 - 112023x + 17720x^2 + 13584x^3 + 1536x^4\\
P_{4,3}(x) &= 288684 - 303831x + 65348x^2 - 51804x^3 + 13824x^4\\
P_{4,4}(x) &= 314928 + 2644083x - 861584x^2 + 176832x^3 + 442368x^4.
\end{aligned} \tag{3.4}$$

This result constitutes, to the best of our knowledge, the first explicit example of an extremal P_n^* which solves Schur's problem according to Erdös-Szegö ([3], Theorem 2) (here for the first Zolotarev case $n = 4$). It is therefore worth summarizing the properties of that Schur polynomial $P_4^* \in \mathbf{B_4}$:

(i) The equiripple property on **I**, i.e., 4 alternation points, including the endpoints ± 1:

$$\begin{aligned} P_4^*(-1) &= -1, \\ P_4^*(y_1) &= 1 \text{ and } P_4^{*(1)}(y_1) = 0, \text{where} \\ y_1 &= \tfrac{1}{72}(39 + 17\sqrt{33} - \sqrt{30(527 + 97\sqrt{33})}) = -0.6068607378..., \\ P_4^*(y_2) &= -1 \text{ and } P_4^{*(1)}(y_2) = 0, \text{where} \\ y_2 &= \tfrac{1}{72}(105 - \sqrt{33} - \sqrt{30(95 + 17\sqrt{33})}) = 0.3226516930..., \\ P_4^*(1) &= 1. \end{aligned} \tag{3.5}$$

(ii) The Zolotarev property at three points $A_4 < B_4 < C_4$ to the right of **I** (of which B_4 and C_4 are two additional alternation points)

$$\begin{aligned} P_4^{*(1)}(A_4) &= 0, \text{ where} \\ A_4 &= \frac{279 + 25\sqrt{33} + \sqrt{30(2879 + 561\sqrt{33})}}{576} = 1.4764907146..., \\ P_4^*(B_4) &= 1, \text{ where } B_4 \text{ is given in (3.2)}, \\ P_4^*(C_4) &= -1, \text{ where} \\ C_4 &= \frac{201 + 55\sqrt{33} - \sqrt{330(61 + 19\sqrt{33})}}{144} = 1.9444055070... . \end{aligned} \tag{3.6}$$

Additionally, by construction, P_4^* satisfies

$$\begin{aligned} P_4^{*(2)}(1) &= 2(a_2^* + 3a_3^* + 6a_4^*) = 0, \text{i.e.}, P_4^* \in \mathbf{B}_{4,1,2} \\ P_4^{*(1)}(1) &= a_1^* + 2a_2^* + 3a_3^* + 4a_4^* = 16M_4, \text{ see (2.3)}, \end{aligned} \tag{3.7}$$

and we add, by inspection, that

$$a_3^* = 1 - a_1^* \text{ and } a_4^* = -a_0^* - a_2^*. \tag{3.8}$$

That particular hard-core Zolotarev polynomial P_4^* may well serve as elucidating example to provide for explanation purposes in lectures or expository writings on Schur's problem, respectively on its solution by Erdös-Szegö, see e.g. [4].

4. Alternative deductions of an explicit extremal hard-core Zolotarev polynomial in the quartic case

In ([12], p. 357) we have provided explicit expressions for the parameterized coefficients of an arbitrary fourth-degree hard-core Zolotarev polynomial on **I**. But since the assumption was made there that it attains the value 1 at $x = -1$, we prefer to consider here the negative form of that polynomial in order to be compliant with [3]. We hence set

$$Z_{4,t}(x) = \sum_{i=0}^{4} -a_i(t)x^i, \text{ with } 1 < t < 1 + \sqrt{2} \tag{4.1}$$

where the coefficients $a_i(t)$ read as follows:

$$\begin{array}{rcl} a_0(t) & = & \left(-a^5 - b^3 + a^4(-2+3b) + a^3(-1+6b-3b^2) + \right. \\ & & \left. +a(3b^2-2b^3) + a^2(3b+2b^2+b^3)\right)\kappa, \\ a_1(t) & = & (a^2(-16b+8b^2) + a(-12b+8b^2-4b^3))\kappa, \\ a_2(t) & = & (a^2(8-16b) + 6b - 4b^2 + 2b^3 + a(6-4b+2b^2))\kappa, \\ a_3(t) & = & (-4+8a^2+8b+8ab-4b^2)\kappa, \\ a_4(t) & = & (-4-6a+2b)\kappa \end{array} \tag{4.2}$$

with

$$\begin{array}{rcl} \kappa & = & \dfrac{1}{(1+a)^2(-a+b)^3} \\ a & = & \dfrac{1-3t-t^2-t^3}{(1+t)^3} \\ b & = & \dfrac{1+t+3t^2-t^3}{(1+t)^3}. \end{array} \tag{4.3}$$

Here a and b with $a < b$ are the alternation points of $Z_{4,t}$ in the interior of $\mathbf{I}$. We now proceed to determine the optimal parameter $t = t^*$ and the corresponding explicit coefficients $-a_i(t^*)$ of an extremal polynomial Z_{4,t^*} with $Z_{4,t^*}(x) = \sum_{i=0}^{4} -a_i(t^*)x^i$ which, according to the general result in ([3], Theorem 2), solves Schur's problem (1.2) for $n = 4$.

The assumption $Z_{4,t} \in \mathbf{B}_{4,1,2}$, i.e., $Z_{4,t}^{(2)}(1) = 0$, implies

$$a_2(t) + 3a_3(t) + 6a_4(t) = 0. \tag{4.4}$$

Employing the definition of $a_i(t)$ in (4.2),(4.3) this amounts to the following equation, after some algebraic manipulations:

$$\frac{(1+t)^3(3+t(2+t))(-2+t(-7+t(1+3(-1+t)t)))}{4(t+t^3)^2} = 0. \tag{4.5}$$

The numerator vanishes, for $1 < t < 1+\sqrt{2}$, only if we choose

$$t = t^* = \frac{3+\sqrt{33}+\sqrt{30(-1+\sqrt{33})}}{12} = 1.7229220588..., \tag{4.6}$$

which is a root of the polynomial $P_4(x) = -2 - 7x + x^2 - 3x^3 + 3x^4$. Inserting the optimal parameter (4.6) into the coefficients $-a_i(t)$ of $Z_{4,t}$, see (4.2), (4.3), shows that $-a_i(t^*)$ indeed coincides for $i = 0,1,2,3,4$ with a_i^* as given in (3.3). We check only the coefficient $-a_4(t)$ and leave it to the reader to check the remaining coefficients:

$$-a_4(t) = \frac{4+6a-2b}{(1+a)^2(-a+b)^3} = \frac{(1-t)(1+t)^9}{32t^3(1+t^2)^2}, \tag{4.7}$$

and inserting now $t = t^*$ according to (4.6) indeed yields $-a_4(t^*) = a_4^*$ as given in (3.3). After all, we so obtain an alternative and independent deduction of the extremal hard-core Zolotarev polynomial $P_4^* = Z_{4,t^*}$ which we had already found in Section 3, based on preliminary work in [3].

A third argument can be brought forward to prove that $P_4^* = Z_{4,t^*}$ is a sought-for extremizer in (1.2) for $n = 4$, see ([11], Theorem 5.10): It suffices to verify that the following five equations hold true

$$-1 + 2(-y_1 + y_2) - (1 + B_4 - C_4) = 0 \tag{4.8}$$

$$1 + 2(-y_1^2 + y_2^2) - (1 + B_4^2 - C_4^2) = 0 \tag{4.9}$$

$$-1 + 2(-y_1^3 + y_2^3) - (1 + B_4^3 - C_4^3) = 0 \tag{4.10}$$

$$\frac{16(A_4 - 1)^2}{(B_4 - 1)(C_4 - 1)} = 1 + 2\left(\frac{2}{A_4 - 1} - \frac{1}{B_4 - 1} - \frac{1}{C_4 - 1}\right) \tag{4.11}$$

$$A_4 = \frac{3}{8}(B_4 + C_4) - \frac{1}{4}(y_1 + y_2), \tag{4.12}$$

where y_1 and y_2 are defined in (3.5), A_4 and C_4 are defined in (3.6), and B_4 is defined in (3.2). We leave it to the reader to check the validity of equations (4.8) - (4.12). Summarizing, we have thus established

Proposition 4.1. *Polynomial P_4^* with $P_4^*(x) = \sum_{i=0}^{4} a_i^* x^i$ and explicit coefficients a_i^* $(i = 0, 1, 2, 3, 4)$ according to (3.3) is a sought-for extremal hard-core Zolotarev polynomial of degree four which solves, according to Erdös-Szegö ([3], Theorem 2), Schur's problem (1.2) for $n = 4$. The corresponding maximum*

$$\sup_{\xi \in \mathbf{I}} \sup_{P_4 \in \mathbf{B}_{4,\xi,2}} \left| P_4^{(1)}(\xi) \right| = 16 M_4$$

is explicitly given in (2.3), so that M_4 equals the constant given in (2.5).

5. A generalized Schur problem and its solution for the quartic case

A. A. Markov's inequality (1.1) for the first derivative of P_n was generalized in 1892 by his half-brother V. A. Markov ([9], p. 93) to the k -th derivative and can be restated as follows, see also ([10], p. 545), ([13], Theorem 2.24):

$$\sup_{\xi \in \mathbf{I}} \sup_{P_n \in \mathbf{B}_n} \left| P_n^{(k)}(\xi) \right| = \prod_{j=0}^{k-1} \frac{n^2 - j^2}{2j + 1} = T_n^{(k)}(1), (1 \le k \le n), \tag{5.1}$$

indicating that the maximum is attained if $P_n = T_n$ and $\xi = 1$. Shadrin [16] has analogously generalized Schur's problem (1.2) to the k -th derivative. This generalized problem can be stated as follows:
Determine, for $1 \le k \le n - 2$ and $n \ge 4$, an algebraic polynomial $P_n = P_n^*$ of degree $\le n$ which attains the maximum

$$\sup_{\xi \in \mathbf{I}} \sup_{P_n \in \mathbf{B}_{n,\xi,k+1}} \left| P_n^{(k)}(\xi) \right| = \prod_{j=0}^{k-1} \frac{n^2 - j^2}{2j + 1} M_{n,k} = T_n^{(k)}(1) M_{n,k}, \tag{5.2}$$

where $\mathbf{B}_{n,\xi,k+1} = \{P_n \in \mathbf{B}_n : P_n^{(k+1)}(\xi) = 0\}$ and $M_{n,k}$ is a constant (depending on n and k). Shadrin ([16], Proposition 4.4) found that, for $k \ge 2$, this maximum is attained if $\xi = 1$ and $P_n = P_n^* \in \mathbf{B}_{n,1,k+1}$ is a Zolotarev polynomial Z_n (not necessarily a hard-core one), or if $\xi = \omega_{k,n}$, the rightmost zero of $T_n^{(k+1)}$, and $P_n = P_n^* = T_n$, so that altogether there holds:

$$\sup_{\xi \in \mathbf{I}} \sup_{P_n \in \mathbf{B}_{n,\xi,k+1}} \left| P_n^{(k)}(\xi) \right| = \max\{|Z_n^{(k)}(1)|, |T_n^{(k)}(\omega_{k,n})|\}. \tag{5.3}$$

We are now going to determine that maximum as well as an extremizer polynomial for the quartic case $n = 4$ and for the second derivative, i.e., $k = 2 = n - 2$ (the case $k = 1$ is settled in Proposition 4.1). It is well known that Zolotarev polynomials Z_n of degree $n \geq 4$ on $\mathbf{I}$ satisfy $||Z_n||_\infty = 1$ (maximum-norm) and exhibit at least n equiripple points on $\mathbf{I}$ where the values ± 1 are attained alternately, see ([16], p. 1190). Apart from sign and reflection, the Zolotarev polynomial Z_4 takes on the role (see also ([1], pp. 280), ([10], p. 406)):

(i) $Z_4 = T_3$, with $T_3(x) = -3x + 4x^3$,

(ii) $Z_4 = T_4$, with $T_4(x) = 1 - 8x^2 + 8x^4$,

(iii) $Z_4 = T_{4,\beta}$, with $T_{4,\beta}(x) = T_4\left(\frac{2x - \beta + 1}{1 + \beta}\right)$
where $1 < \beta \leq 1 + 2\tan^2\left(\frac{\pi}{8}\right) = 7 - 4\sqrt{2} = 1.3431457505...$,

(iv) $Z_4 = Z_{4,t}$, the hard-core Zolotarev polynomial, as given in (4.1).

We first calculate $|Z_4^{(2)}(1)|$, subject to the constraint $Z_4^{(3)}(1) = 0$, and observe that polynomials (i), (ii), (iii) cannot be extremal due to $T_3^{(3)}(1) = 24 \neq 0$, resp. $T_4^{(3)}(1) = 192 \neq 0$, resp. $T_{4,\beta}^{(3)}(1) = \dfrac{1536(3-\beta)}{(1+\beta)^4} \neq 0$ if $1 < \beta \leq 7 - 4\sqrt{2}$. For polynomial (iv) we get, after some algebraic manipulations,

$$|Z_{4,t}^{(3)}(1)| = \left| \frac{3(1+t)^6(-1 + t(-8 + 2t + 3t^3))}{8t^3(1+t^2)^2} \right|. \tag{5.4}$$

The numerator vanishes for $1 < t < 1 + \sqrt{2}$ only if

$$t = t^{**} = \frac{1 + \sqrt{2(-1+\sqrt{3})}}{\sqrt{3}} = 1.2759444802... . \tag{5.5}$$

Inserting this parameter t^{**} into $|Z_{4,t}^{(2)}(1)|$ yields, again after some manipulations,

$$|Z_{4,t^{**}}^{(2)}(1)| = \left| -12 - \frac{22}{\sqrt{3}} + 4\sqrt{\frac{10}{3} + 2\sqrt{3}} \right| = 14.2729495641... . \tag{5.6}$$

In view of (5.3), we have next to compare (5.6) to $|T_4^{(2)}(\omega_{2,4})|$. Since the only, and hence the rightmost, zero of $T_4^{(3)}$ is $\omega_{2,4} = 0$, we get

$$|T_4^{(2)}(0)| = |-16| = 16 > |Z_{4,t^{**}}^{(2)}(1)|.$$

So eventually we arrive at the identity

$$\begin{aligned} \sup_{\xi\in\mathbf{I}} \sup_{P_4\in\mathbf{B}_{4,\xi,3}} |P_4^{(2)}(\xi)| &= \max\{|Z_4^{(2)}(1)|, |T_4^{(2)}(0)|\} = 16 \\ &= \prod_{j=0}^{1} \frac{4^2-j^2}{2j+1} M_{4,2} = 80M_{4,2}, \end{aligned} \tag{5.7}$$

yielding $M_{4,2} = \dfrac{1}{5} = 0.2$. Summarizing, we have thus established

Proposition 5.1. *Polynomial $P_4^* = T_4$ with $T_4(x) = 1 - 8x^2 + 8x^4$ is a sought-for extremal polynomial of degree four which solves, according to Shadrin (*[16]*, Proposition 4.4), the generalized Schur problem (*5.2*) for $n = 4$ and $k = 2$. The corresponding maximum $\sup_{\xi\in\mathbf{I}} \sup_{P_4\in\mathbf{B}_{4,\xi,3}} |P_4^{(2)}(\xi)| = 80M_{4,2}$ is* 16*, so that $M_{4,2}$ equals the constant $\frac{1}{5}$.*

Shadrin ([16], Theorem 7.1) has added to (5.3) the following estimate which can be viewed as an extension, to the k-th derivative, of Schur's estimate (1.3):

$$\sup_{\xi\in\mathbf{I}} \sup_{P_n\in\mathbf{B}_{n,\xi,k+1}} |P_n^{(k)}(\xi)| \le \prod_{j=0}^{k-1} \frac{n^2-j^2}{2j+1} \lambda_{n,k} = T_n^{(k)}(1)\lambda_{n,k} \quad (1 \le k \le n-2), \tag{5.8}$$

where $\lambda_{n,k} = \dfrac{1}{k+1} \cdot \dfrac{n-1}{n-1+k}$.

Thus for $k = 2$ and $n = 4$ we get $\lambda_{4,2} = \dfrac{1}{3} \cdot \dfrac{3}{5} = \dfrac{1}{5} = 0.2 = M_{4,2}$, see (5.7). However, for $k = 1$ and $n = 4$ we get $\lambda_{4,1} = \dfrac{1}{2} \cdot \dfrac{3}{4} = \dfrac{3}{8} = 0.375 > M_4 = 0.299...$, see (2.5) and ([16], Remark 5.5).

6. Concluding remarks

1. In deducing (2.3) we have been guided by a computer algebra system which the authors of [3], who have paved the way, certainly did not have at their disposal.

2. Our explicit power form representation ([12], p. 357) for the fourth hard-core Zolotarev polynomial $Z_{4,t}$ remained unnoticed, and several related formulas have been published afterwards, e.g. ([2], p. 184), ([15], p. 242), ([18], p. 721). Shadrin [15] attributes his formula (with a different range of the parameter t) to V. A. Markov [9] and remarks: *But already for $n = 4$ it seems that nobody really believed that an explicit form can be found. As a matter of fact it was, by V. Markov in 1892.* In a private communication Professor Shadrin kindly called our attention to p. 73 in [9] from which his formula can be recovered. However, one has first to exploit the relation $4z = t^3 + t$ (see p. 71 in [9]), then fix the parameter α and finally rearrange the Taylor form of the given fourth-degree polynomial, centered at $x_0 = 2z$, to the usual power form centered at $x_0 = 0$. It is under these side conditions that priority for the power form representation of $Z_{4,t}$ belongs indeed to V. A. Markov [9].

3. In Section 4 we have alternatively deduced the Schur polynomial P_4^* from the explicit power form $Z_{4,t}(x) = ...$ as given, up to the sign, in ([12], p. 357). P_4^* can

likewise be deduced from the explicit power form $Z_4(x,t) = ...$ as given in ([15], p. 242), however instead of $Z_{4,t}^{(2)}(1) = 0$ (see (4.4)) one has then to set $Z_4^{(2)}(-1,t) = 0$.

4. For the quartic Schur polynomial $P_4^* = Z_{4,t^*}$ we have determined its five critical points $y_1, y_2 \in \mathbf{I}$ and A_4, B_4, C_4 with $1 < A_4 < B_4 < C_4$. As Z_{4,t^*} is a special case of the general quartic hard-core Zolotarev polynomial $Z_{4,t}$ it is desirable to know the corresponding five (general) critical points of $Z_{4,t}$ as well. These are, as can be verified by insertion: $y_1(t) = a(t) = a$ and $y_2(t) = b(t) = b$ as given in (4.3), and furthermore

$$A_4(t) = \frac{1+4t+2t^2+4t^3+t^4}{2(-1+t)(1+t)^3} \tag{6.1}$$

$$B_4(t) = \frac{1+2t+6t^3-t^4}{(-1+t)(1+t)^3} \tag{6.2}$$

$$C_4(t) = \frac{-1+6t+2t^3+t^4}{(-1+t)(1+t)^3}. \tag{6.3}$$

Choosing $t = t^*$ according to (4.6) takes us back to the five critical points of $P_4^* = Z_{4,t^*}$.

5. The optimal parameter $t = t^*$ according to (4.6) which selects the quartic Schur polynomial Z_{4,t^*} among all $Z_{4,t}$ with $1 < t < 1+\sqrt{2}$ can alternatively be determined as follows: In (4.11) replace A_4 by $A_4(t)$, B_4 by $B_4(t)$ and C_4 by $C_4(t)$ according to (6.1) - (6.3). Then solve this generalized equation (4.11) for the unknown number t. The solution will turn out as $t = t^*$.

6. As some progress has been achieved in the computation of $Z_{n,t}$ for the next higher polynomial degrees $n \geq 5$ (see [5], [6], [7], [11]), we hope that we will be able to extend our results to some $n \geq 5$. Meanwhile, we have succeeded so for the case $n = 5$. (The second Zolotarev case in the Erdös-Szegö solution to a Markov-type extremal problem of Schur, J. Numer. Anal. Approx. Theory 46(2017), to appear).

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