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Properties of Hamiltonian in free final multitime problems

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Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th anniversary.

Abstract. In single-time autonomous optimal control problems, the Hamiltonian is constant on optimal evolution. In addition, if the final time is free, the optimal Hamiltonian vanishes on the hole interval of evolution. The purpose of this paper is to extend some of these results to the case of multitime optimal control. The original results include: anti-trace problem, weak and strong multitime maximum principles, multitime-invariant systems and change rate of Hamiltonian, the variational derivative of volume integral, necessary conditions for a free final multitime expressed with the Hamiltonian tensor that replaces the energy-momentum tensor, change of variables in multitime optimal control, conversion of free final multitime problems to problems over fixed interval.

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1. Introduction

The scientific sources for this paper are: necessary conditions for multiple integral problem in the calculus of variations [1], lower semicontinuity of integral functionals [2], time-optimal control of the Bi-Steerable Robot [4], Pontryagin functions for multiple integral control problems [6], multitime maximum principle and multitime dynamic programming [7]- [3].

We give a positive answer to an important question: does the Hamiltonian attached to multitime control problems have properties similar to the Hamiltonian attached to single-time control problems?

Section 2 underlines properties of Hamiltonian in single-time optimal control. Section 3 studies the Hamiltonian in multitime optimal control. Section 4 analyses the strong multitime maximum principle. Section 5 is dedicated to multitime-invariant dynamical systems and change rate of Hamiltonian. The variational derivative of volume integral is analysed in Section 6. The necessary conditions for a free final multitime are given in Section 7. The change of variables in multitime optimal control is described in Section 8. The conversion of free end multitime problems to problems over fixed interval is realized in Section 9. Section 10 contains conclusions.

We tested the theory in relevant applications: multitime control strategies for skilled movements [3], minirobots moving at different partial speeds [16], optimal control of electromagnetic energy [5], multitime optimal control for quantum systems [10] etc.

The basic results are consequences of some properties that deserve to be emphasized: (1) the controlled PDEs used in the paper are completely integrable and this means symmetry conditions, (2) in Section 4 are used the Goursat-Darboux system and Goursat (hyperbolic) PDE, which are totally symmetric, and (3) the dynamical systems analysed in Section 5 are multitime-invariant.

2. Hamiltonian in single-time optimal control

2.1. Maximum principle with algebraic constraints

Single-time optimal control problem. Find

$$\max_u J(u) = \phi(x(t_0), x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))\, dt,$$

subject to

$$(i)\ \dot{x}(t) = X(x(t), u(t)), \quad t \in (t_0, t_f),$$

$$(ii)\ u(t) \in U\,, \quad t \in (t_0, t_f),$$

$$(iii)\ \Phi(x(t_0), x(t_f)) \in K\,.$$

We have $x : R \to R^n$, $u : R \to R^q$, $L : R^n \times R^q \to R$, $\phi : R^n \times R^n \to R$, $X : R^n \times R^q \to R^n$, $\Phi : R^n \times R^n \to R^k$, $U \subseteq R^m$, $K \subseteq R^k$. Usually U is bounded and K is compact and convex.

Consider the Hamiltonian

$$H : R^n \times R^n \times R^q \to R,\ H(x, p, u) = L(x, u) + \langle p, X\rangle$$

and the *endpoints Lagrangian*

$$\Psi : R^n \times R^n \times R^k \to R,$$

$$\Psi((x_0, x_f), \psi) = \phi(x(t_0), x(t_f)) + \langle \psi, \Phi(x(t_0), x(t_f))\rangle.$$

Our problem becomes: find $\max \mathcal{J}(u)$, where

$$\mathcal{J}(u) = \Psi((x_0, x_f), \psi) + \int_{t_0}^{t_f} \left(H(x(t), p(t), u(t)) - \langle p(t), X(x(t), u(t))\rangle\right)\, dt\,.$$

Proposition 2.1. *The optimal solution* (x^*, p^*, u^*) *satisfies the conditions*

$$(a)\quad \dot{x}^*(t) = \frac{\partial H}{\partial p}(x^*(t), p^*(t), u^*(t))\,,$$

$$(b)\quad \dot{p}^*(t) = -\frac{\partial H}{\partial x}(x^*(t), p^*(t), u^*(t))\,,$$

$$(c)\quad H(x^*(t), p^*(t), u^*(t)) = \max_{u\in U} H(x^*(t), p^*(t), u)\,,$$

$$(d)\quad p^*(t_f) = \frac{\partial \Psi}{\partial x_f}(x^*(t_0), x^*(t_f), \psi^*)\,,$$

$$(e)\quad p^*(t_0) = -\frac{\partial \Psi}{\partial x_0}(x^*(t_0), x^*(t_f), \psi^*)\,,$$

and ψ^* *is an element of the normal cone to* K *at the point* $(x^*(t_0), x^*(t_f))$.

If the final time is free, we consider it as a new control which maximizes

$$\mathcal{J} = \Psi(x(t_0), x(t_f), \psi) + \int_{t_0}^{t_f} \big(H(x(t), u(t), p(t)) - p_i(t)\dot{x}^i(t)\big)\, dt\,.$$

The necessary condition for extremum, using (d), is

$$0 = \frac{\partial \mathcal{J}}{\partial t}|_{t=t_f} = \frac{\partial \Psi}{\partial x_f^i}\dot{x}^i|_{t=t_f} + \left(H - p_i\dot{x}^i\right)|_{t=t_f} = H(t_f)\,.$$

Hence, according with the Proposition 2.1, we have

Proposition 2.2. *Let* x^*, p^*, u^* *be the optimal solution for a free final time autonomous problem. Then* $H^*(t) = 0$ *on the hole interval* $t_0 \le t \le t_f$.

2.2. The Hamiltonian as a first integral

For any kind of single-time autonomous problem with bounded control, the following statement is true:

Proposition 2.3. *Let* x^*, p^*, u^* *be the optimal solution and*

$$H^*(t) = H(x^*(t), p^*(t), u^*(t)) \tag{2.1}$$

the pull-back of Hamiltonian on this solution. Then $H^*(t) =$ *constant.*

Proof. According with maximum principle, in any interval of continuity, for each τ and σ, we have

$$H^*(\tau) - H^*(\sigma) \ge H(x^*(\tau), p^*(\tau), u^*(\sigma)) - H^*(\sigma)\,.$$

Then, for $\tau > \sigma$, by the state and costate equations:

$$\lim_{\tau\downarrow\sigma} \frac{H^*(\tau) - H^*(\sigma)}{\tau - \sigma} \ge \lim_{\tau\downarrow\sigma} \frac{H(x^*(\tau), p^*(\tau), u^*(\sigma)) - H^*(\sigma)}{\tau - \sigma}$$

$$= \frac{\partial H}{\partial x}(x^*(\sigma), p^*(\sigma), u^*(\sigma))\,\dot{x}^*(\sigma) + \frac{\partial H}{\partial p}(x^*(\sigma), p^*(\sigma), u^*(\sigma))\,\dot{p}^*(\sigma) = 0\,.$$

Taking $\tau < \sigma$, we obtain in a similar way the opposite inequality. Hence $\dot{H}^*(\sigma) = 0$. The result follows.
With a bit completions at a point of discontinuity we have $H^*(t) \equiv ct$. □

2.3. Conversion to problems over a fixed interval

For an optimal problem with free end time T, consider the change of variable $\tau = t/T$. Then the final time in the new variable is 1.

The functions of time expressed in this new variable become $\widetilde{x}(\tau) = x(\tau T)$, $\widetilde{u}(\tau) = u(\tau T)$, the evolution is $T\,X(x,u)$ and the running cost is $T\,L(x,u)$.

Viewing T as a new state variable for the new problem, with $\dot{T} = 0$ and costate q, the new Hamiltonian will be $\mathcal{H} = T\,H$. The optimality condition gives us

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial T} = H\ ,\ \ q(0) = q(1) = 0\,.$$

Then

$$0 = q(1) - q(0) = \int_0^1 H^*(\tau)\,d\tau$$

and, according with the Proposition 2.2, we have $H^*(\tau) = 0$.

3. Hamiltonian in multitime optimal control

Generally, a multitime optimal control problem [7]-[3] is formulated in the following way: find

$$\max_u\ Q(u(\cdot)) = \int_{\Omega_{0t_0}} L(x(t),u(t))\omega + g(x(t_0))$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t),u(t)),\quad t \in \Omega_{0t_0} \subset R^m_+,$$

a controlled PDEs evolution system which is completely integrable (m-flow), where $\Omega_{0t_0} \subset \mathbb{R}^m_+$ is the parallelepiped determined by the diagonal opposite points 0 and t_0, $x(t) = (x^1(t),...,x^n(t)),\ t = (t^1,...,t^m) \in \Omega_{0t_0}$ is the state vector, $u(t) \in U,\ t \in \Omega_{0t_0}$ is the control vector, the C^1 function $L(x,u)$ is the *running cost*, $\omega = dt^1 \wedge ... \wedge dt^m$ is the volume element, and g is a C^1 function that defines the *terminal cost.*
The multitime maximum principle [7]- [3] involves the Hamiltonian $H = L + p^\alpha_i X^i_\alpha$, the initial and adjoint PDEs

$$\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p^\alpha_i},\ \frac{\partial p^\alpha_i}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i}$$

and the condition $\max_u H$. Since the adjoint PDEs have too many solutions, we attach an anti-trace problem which involves the Hamiltonian tensor field $H^\alpha_\beta = \frac{1}{m}\delta^\alpha_\beta L + p^\alpha_i X^i_\beta$, the initial and adjoint (completely integrable) PDEs

$$\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p^\alpha_i},\ \frac{\partial p^\alpha_i}{\partial t^\beta} = -\frac{\partial H^\alpha_\beta}{\partial x^i}$$

and the condition $\max_u H$.
Anti-trace property: *Any solution of the anti-trace problem is solution of multitime maximum principle.*

Remark 3.1. The complete integrability condition for the adjoint PDEs is essential. Generally, we can write the anti-trace adjoint PDEs in the Pfaff form

$$dp_i^\alpha = -\frac{\partial H_\gamma^\alpha}{\partial x^i} dt^\gamma.$$

Then, let us consider $\omega = dt^1 \wedge ... \wedge dt^m$ and $\omega_\alpha = \frac{\partial}{\partial t^\alpha} \rfloor \omega$. The $(m-1)$-forms $p_i^\alpha \, \omega_\alpha$ have the exterior differentials $d(p_i^\alpha \, \omega_\alpha) = dp_i^\alpha \wedge \omega_\alpha$. This suggests to use

$$dp_i^\alpha \wedge \omega_\alpha = -\frac{\partial H_\gamma^\alpha}{\partial x^i} dt^\gamma \wedge \omega_\alpha$$

and the identities $dt^\beta \wedge \omega_\alpha = \delta_\alpha^\beta \, \omega$. We find the divergence PDEs system

$$\frac{\partial p_i^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i}$$

if and only if $\omega \neq 0$ on the solutions (complete integrability conditions).

For any type of autonomous multitime problem with bounded control, let x^*, p^*, u^* be the optimal solution and we denote $H^*(t) = H(x^*(t), p^*(t), u^*(t))$, where $t = (t^1, ..., t^m)$.

Theorem 3.2. *Suppose we have an autonomous multitime optimal problem (multitime-invariant dynamics and Lagrangian), with bounded control. If the Lagrangian L is independent on $x = (x^i)$ and the optimal solution x^*, p^*, u^* fulfills the anti-trace PDEs, then H^* is constant on the optimal m-sheets.*

Proof. According to the multitime maximum principle for any fixed σ in any m-interval of continuity, $\tau \in \mathbb{R}^m$ and $\varepsilon \in \mathbb{R}$ we have

$$H^*(\sigma + \varepsilon\tau) - H^*(\sigma) \geq H(x^*(\sigma + \varepsilon\tau), p^*(\sigma + \varepsilon\tau), u^*(\sigma)) - H^*(\sigma)\,.$$

Then, for $\varepsilon > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{H^*(\sigma + \varepsilon\tau) - H^*(\sigma)}{\varepsilon} \geq \lim_{\varepsilon \downarrow 0} \frac{H(x^*(\sigma + \varepsilon\tau), p^*(\sigma + \varepsilon\tau), u^*(\sigma)) - H^*(\sigma)}{\varepsilon}$$

$$= \left[\frac{\partial H}{\partial x^i}(x^*(\sigma), p^*(\sigma), u^*(\sigma)) \, \frac{\partial x^{*i}}{\partial t^\gamma}(\sigma) + \frac{\partial H}{\partial p_i^\alpha}(x^*(\sigma), p^*(\sigma), u^*(\sigma)) \, \frac{\partial p_i^{*\alpha}}{\partial t^\gamma}(\sigma)\right] \tau^\gamma\,.$$

By hypotheses (anti-trace property of multitime maximum principle), at $x^*(\sigma)$, $p^*(\sigma)$, $u^*(\sigma)$, we have

$$H_\beta^\alpha = \frac{1}{m}\delta_\beta^\alpha L + p_i^\alpha X_\beta^i, \; H = L + p_i^\alpha X_\alpha^i.$$

$$\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p_i^\alpha}, \; \frac{\partial p_i^\alpha}{\partial t^\gamma} = -\frac{\partial H_\gamma^\alpha}{\partial x^i}.$$

It follows

$$\frac{\partial H}{\partial t^\gamma} = \frac{\partial H}{\partial x^i}\frac{\partial x^i}{\partial t^\gamma} + \frac{\partial H}{\partial p_i^\alpha}\frac{\partial p_i^\alpha}{\partial t^\gamma} = \frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\gamma} + \frac{\partial p_i^\alpha}{\partial t^\gamma} X_\alpha^i + p_i^\alpha \frac{\partial X_\alpha^i}{\partial x^j}\frac{\partial x^j}{\partial t^\gamma}$$

$$= \frac{\partial L}{\partial x^i} X_\gamma^i \left(1 - \frac{1}{m}\right) + p_i^\alpha [X_\gamma, X_\alpha]^i = 0,$$

($[X_\gamma, X_\alpha] = 0$ means the complete integrability condition), i.e., H is a first integral of the anti-trace PDEs.

Hence

$$\lim_{\varepsilon \downarrow 0} \frac{H^*(\sigma + \varepsilon\tau) - H^*(\sigma)}{\varepsilon} \geq 0\,.$$

Taking $\varepsilon < 0$, we obtain in a similar way the opposite inequality; the derivative of H^* at σ in any direction τ vanishes. The result follows.

With some additions for points of discontinuity, it follows $H^*(t) \equiv ct$. □

4. Strong multitime maximum principle

This Section discusses the differences between two kind of evolution systems involved into multitime optimal control problems: (i) a full completely integrable PDEs system and (ii) a hyperbolic (diagonal) PDEs system which is completely integrable via an m-order hyperbolic PDE.

4.1. Full PDEs evolution system, no running cost

This case involves the PDEs evolution system (m-flow)

$$(PDE_f) \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), u(t)), \quad t \in \Omega_{0t_0} \subset \mathbb{R}^m_+$$

with $i = 1, ..., n$, $\alpha = 1, ..., m$, and a terminal cost.

The *Hamiltonian* and the *Hamiltonian tensor* are respectively

$$H(x, p, u) = p^\alpha_i X^i_\alpha(x, u)\,, \;\; H^\alpha_\beta(x, p, u) = p^\alpha_i X^i_\beta(x, u)\,,$$

p^α_i being the costate variables. The (PDE_f) can be written $\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p^\alpha_i}$. According to the *strong multitime maximum principle*, we can built an optimal costate function via the adjoint equations

$$(ADJ\,1,\,2) \qquad \frac{\partial p^\alpha_i}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i}\,, \;\; \frac{\partial p^\alpha_i}{\partial t^\beta} = -\frac{\partial H^\alpha_\beta}{\partial x^i}\,.$$

4.2. Missing equations in PDEs evolution system, no running cost

Let us suppose that a PDEs evolution system does not contain all equations previously indexed by $i = 1, ..., n$, $\alpha = 1, ..., m$. An example could be a diagonal system (hyperbolic system, Goursat-Darboux system)

$$\frac{\partial x^\alpha}{\partial t^\alpha}(t) = X^\alpha_\alpha(x(t), u(t)),\;\; \alpha = 1, ..., m \;\; \text{(no sum)}.$$

To include these kinds of PDEs in the set of all first order normal PDEs, let us use an *indicator (characteristic)* function χ which, generally, is a function defined on a set $\mathcal{A}$ that indicates membership of an element in a subset A of $\mathcal{A}$, having the value 1 for all elements of A and the value 0 for all elements of $\mathcal{A}$ not in A. In our case, $\chi = 1$, if the equation with indices i and α appears in the initial evolution system and $\chi = 0$, if not. So the (PDE_f) can be written

$$(PDE_m) \qquad \chi\frac{\partial x^i}{\partial t^\alpha}(t) = \chi X^i_\alpha(x(t), u(t)), \quad t \in \Omega_{0t_0} \subset \mathbb{R}^m_+$$

with $i = 1, ..., n$, $\alpha = 1, ..., m$.

Then the *Hamiltonian* and the *Hamiltonian tensor* are respectively

$$H(x,p,u) = p_i^\alpha \ \chi X_\alpha^i(x,u)\,, \ \ H_\beta^\alpha(x,p,u) = p_i^\alpha \ \chi X_\beta^i(x,u)\,,$$

p_i^α being the costate variables.

According to the *strong multitime maximum principle*, we can built an optimal costate function via the adjoint equations

$$(ADJ\,1,\,2) \qquad \frac{\partial \chi p_i^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i}\,, \ \ \frac{\partial \chi p_i^\alpha}{\partial t^\beta} = -\frac{\partial H_\beta^\alpha}{\partial x^i}\,.$$

Remark 4.1. In the case of missing equations in PDEs evolution system, we work only with the "active" equations. The formalism of characteristic function is doing this.

4.3. Free endpoint problem with running cost

Let us consider that the cost functional include a running cost, i.e.,

$$(Q) \qquad Q(u(\cdot)) = \int_{\Omega_{0t_0}} X^0(x(t),u(t))\omega + g(x(t_0)),$$

where $x(t) = (x^1(t), ..., x^n(t))$ is the state vector, Ω_{0t_0} is the parallelepiped determined by the diagonal opposite points 0 and t_0, the *running cost* $X^0(x,u)$ is a C^1 function, and g is a C^1 function associated to the *terminal cost.* Suppose the controlled PDEs evolution system

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X_\alpha^i(x(t),u(t)), \quad t \in \Omega_{0t_0} \subset \mathbb{R}_+^m$$

is full.

Adding new variables. Introducing new variables $x^{n+1}, ..., x^{n+m}$, and new costates $p_{n+\alpha}^\alpha(\cdot)$, we convert the theory to the foregoing case. The new state variables are constrained by the diagonal PDEs system (hyperbolic system, Goursat-Darboux system)

$$\frac{\partial x^{n+\alpha}}{\partial t^\alpha} = x_\alpha^{n+\alpha} = x^{n+\alpha+1}, \alpha = \overline{1, m-1}\,, \ \ \frac{\partial x^{n+m}}{\partial t^m} = X^0(x^1, ..., x^n, u),$$

equivalent to the Goursat (hyperbolic) PDE

$$\frac{\partial^m x^{n+1}}{\partial t^1 ... \partial t^m} = X^0(x^1, ..., x^n, u)\,;$$

denote also, for convenience, $x^{n+m+1} = X^0(x^1, ..., x^n, u)$.

We introduce a costate matrix $\bar{p}(\cdot) = (p_i^\alpha(\cdot)) \oplus (p_{n+\alpha}^\alpha(\cdot))$. For the new equations and new costates, $p_{n+\beta}^\alpha(\cdot)$, the values of the indicator χ are summarized by δ_β^α.

The *control Hamiltonian* is

$$H(\bar{x},\bar{p},u) = p_i^\alpha X_\alpha^i(x,u) + \sum_{\alpha=1}^m p_{n+\alpha}^\alpha x^{n+\alpha+1}$$

and the *control Hamiltonian tensor field* H_β^α must have the form

$$H_\beta^\alpha(\bar{x},\bar{p},u) = \begin{cases} p_i^\alpha X_\beta^i(x,u) & \text{if} \quad \alpha \neq \beta \\ p_i^\alpha X_\alpha^i(x,u) + p_{n+\alpha}^\alpha x^{n+\alpha+1} & \text{if} \quad \alpha = \beta \quad \text{(no sum upon } \alpha). \end{cases}$$

To understand that the matrix H^α_β is the anti-trace of H, we need to have in mind the diagonal matrices operations.

According to the *strong multitime maximum principle*, we can built a costate function $\bar{p}^*(\cdot) = ({p^*}^\alpha_i(\cdot)) \oplus ({p^*}^\alpha_{n+\alpha}(\cdot))$ satisfying

$$(EDP1) \qquad x^i_\alpha = \frac{\partial H}{\partial p^\alpha_i} \text{ or } x^i_\alpha = \frac{\partial H^\alpha_\beta}{\partial p^\beta_i}, \text{ (no sum)}, \alpha, \beta = \overline{1,m} \;\; i = \overline{1,n},$$

$$(EDP2) \qquad x^{n+\alpha}_\alpha = \frac{\partial H}{\partial p^\alpha_{n+\alpha}} \text{ or } x^{n+\alpha}_\alpha = \frac{\partial H^\alpha_\alpha}{\partial p^\alpha_{n+\alpha}}, \; \alpha = \overline{1,m} \text{ (no sum)},$$

$$(ADJ1) \qquad \frac{\partial \bar{p}^\alpha_i}{\partial t^\alpha} = -\frac{\partial H}{\partial \bar{x}^i}, \; i = \overline{1,n}; \; \frac{\partial p^\alpha_{n+\alpha}}{\partial t^\alpha} = -\frac{\partial H}{\partial x^{n+\alpha}}, \alpha = \overline{1,m} \text{ (no sum)}$$

$$(ADJ2) \qquad \frac{\partial p^\alpha_i}{\partial t^\beta} = -\frac{\partial H^\alpha_\beta}{\partial x^i}, \; i = \overline{1,n}; \; \frac{\partial p^\alpha_{n+\alpha}}{\partial t^\alpha} = -\frac{\partial H^\alpha_\alpha}{\partial x^{n+\alpha}}, \alpha = \overline{1,m} \text{ (no sum)}.$$

All these PDEs systems are completely integrable.

5. Multitime-invariant dynamical systems and change rate of Hamiltonian

Let us refer to open-end-multitime optimization problem. In the conditions of Section 3, we have $H^* = \textit{constant}$ as an alternative scalar necessary condition for optimality.

Let us consider $\omega = dt^1 \wedge ... \wedge dt^m$ and $\omega_\alpha = \frac{\partial}{\partial t^\alpha} \rfloor \omega$. Since the final multitime t_f is free to vary, we rewrite the functional

$$J = \int_{\partial\Omega_{0t_f}} v^\alpha \omega_\alpha + \int_{\Omega_{0t_f}} \left(H - p^\alpha_i \frac{\partial x^i}{\partial t^\alpha}\right)\omega$$
$$= \int_{\Omega_{0t_f}} Div\, v + \int_{\Omega_{0t_f}} \left(H - p^\alpha_i \frac{\partial x^i}{\partial t^\alpha}\right)\omega,$$

where $v(x(t)) = (v^\alpha(x(t)))$ is the generating vector field, and $\frac{\partial v^\alpha}{\partial x^i}(t_f) = p^\alpha_i(t_f)$. Now t_f is an additional control variable for maximizing J. Consequently the cost sensitivity via the total mixed operator $D_{t^1...t^m}$, to final multitime t_f, should be zero, i.e.,

$$0 = D_{t^1...t^m} J|_{t=t_f} = \frac{\partial v^\alpha}{\partial x^i}\frac{\partial x^i}{\partial t^\alpha}|_{t=t_f} + \left(H - p^\alpha_i \frac{\partial x^i}{\partial t^\alpha}\right)|_{t=t_f} = H(t_f).$$

Consequently, $H^* = 0$ in the closed interval $0 \leq t \leq t_f$, i.e., in the hyperrectangle Ω_{0t_f}.

Lemma 5.1. *Let ϕ be a terminal cost, ψ be an algebraic condition for the terminal point, both of class C^m, and v be a generating C^1 vector field related by the PDE $D_{t^1...t^m}(\phi + \nu\psi) = Div\, v$, where ν is a constant Lagrange multiplier.*

(i) Given v, there exists $\phi + \nu\psi$; (ii) given $\phi + \nu\psi$, there exists v.

Proof. (i) Consequence of the formula $\phi(x(t)) + \nu\psi(x(t)) = \int_{\Omega_{0t}} Div\, v\, \omega.$

(ii) Explicitly $v^1 = \frac{1}{m} D_{t^2 \dots t^m}(\phi + \nu\psi), \dots, v^m = \frac{1}{m} D_{t^1 \dots t^{m-1}}(\phi + \nu\psi).$ □

This Lemma shows that the cost functional can be written also as

$$J = \phi(x(t_f)) + \nu\psi(x(t_f)) + \int_{\Omega_{0t_f}} \left(H - p_i^\alpha \frac{\partial x^i}{\partial t^\alpha} \right) \omega.$$

6. The variational derivative of volume integral

For a domain that evolves with the velocity v from Ω_0 to Ω_ϵ and for the function

$$I(\varepsilon) = \int_{\Omega_\varepsilon} f(t, \varepsilon)\omega,$$

we have

$$\frac{dI}{d\varepsilon}(\varepsilon = 0) = \int_{\Omega_0} \left(\frac{\partial f}{\partial \varepsilon} + div\,(f\, v) \right) \omega\,.$$

If we want that the hyperrectangle $\Omega_0 = [0, T]$ to become $\Omega_\varepsilon = [0, T + \varepsilon\, \delta t]$, it should take the transformation $t \to t + \epsilon v(t)$, where for example, $v = (v^\alpha)$, with $v^\alpha = \frac{t^\alpha}{T^\alpha}\, \delta t^\alpha$ (no summation). Then, for the function

$$\varepsilon \to I(x(\cdot) + \varepsilon h(\cdot); T + \varepsilon \delta t) = \int_{\Omega_\varepsilon} L(x(t) + \varepsilon h(t), x_\alpha(t) + \varepsilon h_\alpha(t))\omega,$$

we find

$$\frac{dI}{d\varepsilon}(\varepsilon = 0) = \int_{\Omega_0} \left[\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial t^\alpha} \left(\frac{\partial L}{\partial x^i_\alpha} \right) \right] h^i\, \omega$$

$$+ \int_{\Omega_0} \left[\frac{\partial}{\partial t^\alpha} \left(\frac{\partial L}{\partial x^i_\alpha}\, h^i \right) + div(L(x(t), x_\gamma(t))\, v) \right] \omega$$

$$= \int_{\Omega_0} \left[\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial t^\alpha} \left(\frac{\partial L}{\partial x^i_\alpha} \right) \right] h^i\, \omega$$

$$+ \int_{\partial\Omega_0} \delta_{\alpha\beta} \left(\frac{\partial L}{\partial x^i_\alpha}\, h^i\ + L(x(t), x_\gamma(t)) \frac{t^\alpha}{T^\alpha}\, \delta t^\alpha \right) n^\beta\, d\sigma\,,$$

with the transversality tensor

$$\mathcal{T}^\alpha = \frac{\partial L}{\partial x^i_\alpha}\, h^i\ + L(x(t), x_\gamma(t)) \frac{t^\alpha}{T^\alpha}\, \delta t^\alpha\,.$$

In this way we have consistency because on the initial faces the integrand is 0, since $h^i = 0$, $t^\alpha = 0$ and canonical normals, and on the final faces $t^\alpha = T^\alpha$. Moreover, using the vector v, we find the connection between h and δt on the faces, from their relationship to the final multitime T.

On the faces, h is related to δt. Indeed

$$x(t + \varepsilon v) + \varepsilon h(t + \varepsilon v) = \phi(t + \varepsilon v),$$

whence, differentiating at $\varepsilon = 0$, we find

$$\frac{\partial x}{\partial t^\gamma}\, v^\gamma + h = \frac{\partial \phi}{\partial t^\gamma}\, v^\gamma \quad \text{or} \quad h^i = \left(\frac{\partial \phi^i}{\partial t^\gamma} - \frac{\partial x^i}{\partial t^\gamma}\right) v^\gamma \,.$$

The transversality vector becomes

$$\mathcal{T}^\alpha = \frac{\partial L}{\partial x^i_\alpha}\left(\frac{\partial \phi^i}{\partial t^\gamma} - \frac{\partial x^i}{\partial t^\gamma}\right) v^\gamma \; + L(x(t), x_\gamma(t))\, v^\gamma$$

$$= \left(\frac{\partial L}{\partial x^i_\alpha}\,\frac{\partial \phi^i}{\partial t^\gamma} - \frac{\partial L}{\partial x^i_\alpha}\,\frac{\partial x^i}{\partial t^\gamma} + \delta^\alpha_\gamma L(x(t), x_\gamma(t))\right) v^\gamma$$

$$= \left(\frac{\partial L}{\partial x^i_\alpha}\,\frac{\partial \phi^i}{\partial t^\gamma} - T^\alpha_\gamma\right) v^\gamma \,,$$

where T^α_γ is the energy-momentum tensor.

Case of boundary integral

We can write

$$0 = \int_{\partial\Omega_0} \delta_{\alpha\beta}\left(\frac{\partial L}{\partial x^i_\alpha}\,\frac{\partial \phi^i}{\partial t^\gamma} - T^\alpha_\gamma\right) v^\gamma\; n^\beta\, d\sigma,$$

where

$$v^\gamma|_{t^\gamma = T^\gamma} = \delta t^\gamma \,, \quad v^\gamma|_{t^\gamma = 0} = 0$$

($2m$ faces, m terms of summation). Hence

$$0 = \delta t^\gamma \; \sum_{\alpha=1}^{m} \int_{F_\alpha} \left(\frac{\partial L}{\partial x^i_\alpha}\,\frac{\partial \phi^i}{\partial t^\gamma} - T^\alpha_\gamma\right) d\sigma.$$

Since δt^γ is arbitrary, it follows

$$0 = \sum_{\alpha=1}^{m} \int_{F_\alpha} \left(\frac{\partial L}{\partial x^i_\alpha}\,\frac{\partial \phi^i}{\partial t^\gamma} - T^\alpha_\gamma\right) d\sigma.$$

Here we have an algebraic system of m equations with m unknowns $T^1, ..., T^m$.

Case of multiple integral

Consequently

$$0 = \int_{\Omega_0} \frac{\partial}{\partial t^\alpha}\left(\left(\frac{\partial L}{\partial x^i_\alpha}\,\frac{\partial \phi^i}{\partial t^\gamma} - T^\alpha_\gamma\right) v^\gamma\right) \omega.$$

7. Necessary conditions for a free final multitime

Let us look for optimization of the functional

$$I(x(\cdot); t_f) = \int_{\Omega_{0t_f}} L(x(t), x_\gamma(t))\ \omega,$$

where the final multitime t_f is free to vary.

Setting the final time free means that we want to use the final time as yet another parameter for optimization. Let us return back to the calculus of variations, having in mind that understanding of the boundary conditions is crucial.

The key idea: derive the necessary conditions with the end point t_f of Ω_{0t_f} on a sheet prescribed by the function $\phi : \Omega_{0t_f} \to \mathbb{R}^n$, $t \to \phi(t)$. This trick is, that the stretching or shrinking of the hyperrectangle Ω_{0t_f} is done by perturbing the stationary value of the final multitime, denoted t_f, with the same ϵ as we use to perturb the functions $x(t)$:

$$I(x(\cdot) + \epsilon h(\cdot); t_f + \epsilon \delta t_f) = \int_{\Omega_{0t_f + \epsilon\delta t_f}} L(x(t) + \epsilon h(t), x_\gamma(t) + \epsilon h_\gamma(t))\omega.$$

Using the differentiation of a multiple integral with a parameter, we impose the necessary condition

$$0 = \frac{d}{d\epsilon} I(\epsilon)|_{\epsilon=0} = \int_{\Omega_{0t_f}} \left(\frac{\partial L}{\partial x^i} - D_\gamma \frac{\partial L}{\partial x^i_\gamma} \right) h^i\ \omega$$

$$+ \int_{\partial\Omega_{0t_f}} \delta_{\alpha\beta} \left(\frac{\partial L}{\partial x^i_\alpha}\, h^i + L\, \delta t^\alpha_f \right) n^\beta d\sigma.$$

Via Euler-Lagrange equations, it remains the only condition

$$0 = \int_{\partial\Omega_{0t_f}} \delta_{\alpha\beta} \left(\frac{\partial L}{\partial x^i_\alpha}\, h^i + L\, \delta t^\alpha_f \right) n^\beta d\sigma,$$

where

$$\delta t^\alpha_f |_{t^\beta = 0} = 0,\ \forall \alpha, \beta = 1, ..., m.$$

The surface integral represents the flux of the vector field

$$\mathcal{T}^\alpha = \frac{\partial L}{\partial x^i_\alpha}\, h^i + L\, \delta t^\alpha_f$$

through the surface $\partial\Omega_{0t_f}$.
Obviously $h(t)$ and δt are related since $x(t)$ is requested to lie in the sheet $\phi(t)$ on the end faces $t^\alpha = t^\alpha_f$, i.e.,

$$x(t + \epsilon\delta t) + \epsilon h(t + \epsilon\delta t) = \phi(t + \epsilon\delta t), \text{ on the end faces of } \Omega_{0t_f}.$$

Differentiating with respect to ϵ and evaluating at $\epsilon = 0$, we find

$$\frac{\partial x}{\partial t^\alpha}\ \delta t^\alpha + h = \frac{\partial \phi}{\partial t^\alpha}\ \delta t^\alpha.$$

Computing $h(t)$, and replacing in $T^\alpha(t)$, we get the *transversality vector*

$$\begin{aligned}\mathcal{T}^\alpha(t) &= \frac{\partial L}{\partial x^i_\alpha}(x(t), x_\gamma(t))\left(\frac{\partial \phi^i}{\partial t^\beta}(t) - \frac{\partial x^i}{\partial t^\beta}(t)\right)\delta t^\beta + L(x(t), x_\gamma(t)\delta^\alpha_\beta \delta t^\beta \\ &= \left(\frac{\partial L}{\partial x^i_\alpha}(x(t), x_\gamma(t))\frac{\partial \phi^i}{\partial t^\beta}(t) - T^\alpha_\beta(t)\right)\delta t^\beta,\end{aligned}$$

where T^α_β is the *energy-momentum tensor*. Since δt^β is arbitrary, and the normal vector field n^α of each face of Ω_{0t_f} belongs to the set of canonical orthonormal versors and their opposites in $\mathbb{R}^m$, the transversality relation can be written as

$$\frac{\partial L}{\partial x^i_\alpha}(x(t), x_\gamma(t))\frac{\partial \phi^i}{\partial t^\beta}(t) - T^\alpha_\beta(t) = 0,\ t \in \text{union of end faces.}$$

The energy-momentum tensor $T^\alpha_\beta = p^\alpha_i x^i_\beta - L\,\delta^\alpha_\beta$ can be changed into Hamiltonian tensor $H^\alpha_\beta = p^\alpha_i x^i_\beta - \frac{1}{m}\,L\,\delta^\alpha_\beta$ by scaling the partial velocities. The trace of the Hamiltonian tensor is $H = p^\alpha_i x^i_\alpha - L$. It follows that for a free-final-multitime and fixed-final-state scenario, in which $\phi(t) = c$, $c \in \mathbb{R}^n$, the transversality condition simplifies to

$$H^\alpha_\beta(t) = 0 \Longrightarrow H(t) = 0,\ t \in \text{union of end faces.}$$

Consequently, $H^* = 0$ in the interval $0 \leq t \leq t_f$, i.e., in the hyperrectangle Ω_{0t_f}.

Remark 7.1. (i) The transition from the multitime calculus of variations to the multitime optimal control, especially when it comes to the definition of Hamiltonian, is somewhat tricky.

(ii) The classical Reynolds' transport theorem is:

$$\frac{d}{d\epsilon}\int_{\Omega(\epsilon)} f(x,\epsilon)dV = \int_{\Omega(\epsilon)} \frac{\partial}{\partial \epsilon} f(x,\epsilon)dV + \int_{\partial\Omega(\epsilon)} (v^b \cdot n) f(x,\epsilon)dA,$$

where $n(x,\epsilon)$ is the outward-pointing unit-normal, x is a point in the region and is the variable of integration, and dV, dA are volume and surface elements at x, and $v^b(x,\epsilon)$ is the velocity of the area element - so not necessarily the flow velocity.

8. Change of variables in multitime optimal control

In order to transform the control conditions to other coordinates and, over all, to converse a free end multitime problem to a fixed end one, we must use the transformation of the independent variables as $t = w(\tau)$, i. e. $t^\alpha = w^\alpha(\tau^1, ..., \tau^m)\,, \alpha = 1, ..., m$. Then a function x will change in $\bar{x}(\tau) = x(w(\tau))$. Consider the Jacobian matrix of the transformation, $J = \left(\frac{\partial w^\alpha}{\partial \tau^\beta}\right)$ and assume that $det(J)$ is not zero at all points of the domain Ω. In the new variables, the domain Ω becomes Ω_τ, the volume element is transformed as

$$dt^1...dt^m = det(J)\,d\tau^1...d\tau^m$$

and the partial derivatives in variables t^α become in the new variables

$$\frac{\partial x^i}{\partial t} = \left(\frac{\partial \bar{x}^i}{\partial \tau^\beta}\frac{\partial \tau^\beta}{\partial t^\alpha}\right) = \frac{\partial \bar{x}^i}{\partial \tau}\,J^{-1}\,.$$

Let us consider a non-autonomous multitime control problem given by a controlled functional

$$I(u) = \int_\Omega L(t, x(t), u(t)) dt^1 ... dt^m$$

and a non-autonomous PDE system

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t), u(t))\,.$$

We may transform this non-autonomous problem by a change of the multitime in two ways.

8.1. Change the problem

The multitime controlled functional I becomes

$$I_\tau = \int_{\Omega_\tau} L(w(\tau), \bar{x}(\tau), \bar{u}(\tau))\, det(J)\, d\tau^1 ... d\tau^m$$

and the constraints change in

$$\frac{\partial \bar{x}^i}{\partial \tau^\alpha} = \frac{\partial x^i}{\partial t^\beta}\frac{\partial w^\beta}{\partial \tau^\alpha} = X^i_\beta(w(\tau), \bar{x}(\tau), \bar{u}(\tau))\,\frac{\partial w^\beta}{\partial \tau^\alpha}\,,$$

or

$$\frac{\partial \bar{x}^i}{\partial \tau} = X^i(w(\tau), \bar{x}(\tau), \bar{u}(\tau))\, J\,.$$

Then we obtain the Lagrange functional, with adjoint vectors $q_i = (q_i^\alpha)$,

$$\mathcal{I} = \int_{\Omega_\tau} \left[L(w(\tau), \bar{x}(\tau), \bar{u}(\tau)) + Tr\, q_i(\tau) \left(X^i(w(\tau), \bar{x}(\tau), \bar{u}(\tau))\, J - \frac{\partial \bar{x}^i}{\partial \tau} \right) \right]$$
$$\times \det(J)\, d\tau^1 ... d\tau^m\,.$$

The new Hamiltonian is

$$\mathcal{H}_1 = \big(L(w(\tau), \bar{x}(\tau), \bar{u}(\tau)) + Tr\, q_i(\tau) X^i(w(\tau), \bar{x}(\tau), \bar{u}(\tau))\, J \big) \det(J)$$
$$= \mathcal{H}\, \det(J)$$

and the corresponding variational equations are:

$$\frac{\partial}{\partial \tau^\alpha}(\det(J)\, q_i^\alpha) = -\, \det(J)\,\frac{\partial \mathcal{H}}{\partial \bar{x}^i},\ \frac{\partial \bar{x}^i}{\partial \tau^\alpha} = \frac{\partial \mathcal{H}}{\partial q_i^\alpha},\ \frac{\partial \mathcal{H}}{\partial u} = 0\,.$$

8.2. Change the variables in the Lagrange functional

The Lagrange functional, with adjoint vectors $p_i = (p_i^\alpha)$, is

$$J = \int_\Omega \left[L(t, x(t), u(t)) + p_i^\alpha(t) \left(X^i_\alpha(t, x(t), u(t)) - \frac{\partial x^i}{\partial t^\alpha} \right) \right] dt^1 ... dt^m\,.$$

Changing the multitime by $t = w(\tau)$, the Lagrange functional becomes

$$\mathcal{J} = \int_{\Omega_\tau} \left[L(w(\tau), \bar{x}(\tau), \bar{u}(\tau)) + Tr\, \bar{p}_i(\tau) \left(X^i(w(\tau), \bar{x}(\tau), \bar{u}(\tau))\, -\, \frac{\partial \bar{x}^i}{\partial \tau}\, J^{-1} \right) \right]$$
$$\times \det(J)\, d\tau^1 ... d\tau^m$$

$$= \int_{\Omega_\tau} \left[L(w(\tau), \bar{x}(\tau), \bar{u}(\tau)) + Tr\, J^{-1}\, \bar{p}_i(\tau) \left(X^i(w(\tau), \bar{x}(\tau), \bar{u}(\tau))\, J - \frac{\partial \bar{x}^i}{\partial \tau} \right) \right] \times \det(J)\, d\tau^1 ... d\tau^m \,.$$

The new Hamiltonian is

$$\mathcal{K}_1 = \big(L(w(\tau), \bar{x}(\tau), \bar{u}(\tau)) + Tr\, \bar{p}_i(\tau) X^i(w(\tau), \bar{x}(\tau), \bar{u}(\tau)) \big) \det(J) = \mathcal{K} \det(J)\,.$$

For the two ways commute, the costates p_i and q_i must be related in a change of variable following the rule

$$\bar{p}_i(\tau) = J\, q_i(\tau)\,.$$

8.3. Conversion to problems over a fixed interval

By the multitime transformation $s^\alpha = \dfrac{1}{T^\alpha}\, t^\alpha$, where $T^\alpha = t^\alpha_f$, for constants $t^\alpha_f > 0$, a free-end multitime problem is converted to problem over the fixed interval $\Omega_{01} = [0,1]^m$. The unknown end multitime T is represented by an additionally state variable $T = (T^\alpha)$, for which $\dfrac{\partial T^\alpha}{\partial s^\beta} = 0$ and $T(0) = t_f$ is assumed. The evolution PDEs will be

$$\frac{\partial \bar{x}}{\partial t^\alpha} = \delta_{\alpha\beta} T^\beta X_\alpha\,, \ \ \frac{\partial T^\alpha}{\partial s^\beta} = 0\,, \ T(0) = t_f.$$

Using the Jacobian $\Delta = T^1 \cdots T^m$, it follows

$$J = \int_{\Omega_{0t_f}} L(x(t), x_\gamma(t)) dt^1 ... dt^m = \int_{\Omega_{01}} L(x(T^\alpha\, s^\alpha), u(T^\alpha\, s^\alpha))\, \Delta\, ds^1 ... ds^m$$

$$= \int_{\Omega_{01}} T^1 \cdots T^m\, L(\bar{x}(s), \bar{u}(s))\, ds^1 ... ds^m\,.$$

Denoting q^β_α the costates associated with the variables T^α we have the following new extended Lagrangian

$$\begin{aligned} \mathcal{L} &= T^1 \cdots T^m \left(L(\bar{x}, \bar{u}) + p^\alpha_i X^i_\alpha T^\alpha - p^\alpha_i x^i_\alpha - q^\beta_\alpha T^\alpha_\beta \right) \\ &= T^1 \cdots T^m \left(\mathcal{H} - p^\alpha_i x^i_\alpha - q^\beta_\alpha T^\alpha_\beta \right), \end{aligned}$$

where $\mathcal{H} = L(\bar{x}, \bar{u}) + p^\alpha_i X^i_\alpha T^\alpha$ is the new Hamiltonian.

The variational Euler equations with respect to $\bar{x}, p, \bar{u}, T$ and q, respectively give us

$$\frac{\partial \mathcal{H}}{\partial \bar{x}^i} + \frac{\partial p^\alpha_i}{\partial s^\alpha} = 0\,, \ \ \frac{\partial \mathcal{H}}{\partial p^\alpha_i} - \bar{x}^i_\alpha = 0\,, \ \ \frac{\partial \mathcal{H}}{\partial \bar{u}} = 0\,,$$

$$\frac{\partial}{\partial T^\alpha} (T^1 \cdots T^m\ \mathcal{H}) + \frac{\partial q^\beta_\alpha}{\partial s^\beta} = 0\,, \ T^\alpha_\beta = 0\,.$$

Let us consider that there exist functions Q_α such that

$$\frac{\partial q^\beta_\alpha}{\partial s^\beta} = \frac{\partial^m Q_\alpha}{\partial s^1 ... \partial s^m}\,.$$

Then we have, by an integral on Ω_{01},

$$\int_{\Omega_{01}} \frac{\partial}{\partial T^\alpha} (T^1 \cdots T^m\ \mathcal{H})\, ds^1 ... ds^m = - \int_{\Omega_{01}} \frac{\partial^m Q_\alpha}{\partial s^1 ... \partial s^m}\ ds^1 ... ds^m = Q_\alpha(1) = 0\,.$$

8.4. Generating costates

In a multitime optimal control problem there exist generating costates p_i such that

$$p_i^\alpha = \frac{\partial^{m-1} p_i}{\partial t^1 ... \widehat{\partial t^\alpha} ... \partial t^m}$$

(analogously for q). So we have

$$\frac{1}{m}\frac{\partial \mathcal{H}}{\partial \bar{x}^i} + \frac{\partial^m \bar{p}_i}{\partial s^1 ... \partial s^m} = 0 \text{ and } \frac{1}{m}\frac{\partial \mathcal{H}}{\partial T^\alpha} + \frac{\partial^m q_\alpha}{\partial s^1 ... \partial s^m} = 0\,.$$

By an integral on Ω_{01} we obtain

$$T^1 ... \widehat{T^\alpha} ... T^m \int_{\Omega_{01}} H\, ds^1 ... ds^m \; = -m \int_{\Omega_{01}} \frac{\partial^m q_\alpha}{\partial s^1 ... \partial s^m} ds^1 ... ds^m \; = q_\alpha(1) = 0\,.$$

But $H^*(t) \equiv ct$ and hence $H^*(t) \equiv 0$.

Let us consider the duality relation $\frac{\partial^m p_i}{\partial t^1 ... \partial t^m} = \frac{\partial p_i^\alpha}{\partial t^\alpha}$ (divergence form, complete integrability condition).

1) If $p_i^\alpha(t)$ are given, then

$$p_i(t) = \int_{\partial\Omega_{0t}} p_i^\alpha\, \omega_\alpha = \int_{\Omega_{0t}} \frac{\partial p_i^\alpha}{\partial t^\alpha}(t)\, dt^1 ... dt^m\,, \quad \text{with } p_i|_{t^\beta = 0} = 0,$$

where $\omega_\alpha = (-1)^{\alpha-1} dt^1 ... \widehat{dt^\alpha} ... dt^m$.
Generally: If

$$\omega = dt^1 \wedge \dots \wedge dt^m, \;\; \omega_\alpha = \frac{\partial}{\partial t^\alpha} \rfloor \omega,$$

then

$$\int_{\partial\Omega_{0t}} p_i^\alpha\, \omega_\alpha = \int_{\Omega_{0t}} d(p_i^\alpha\, \omega_\alpha)$$

$$= \int_{\Omega_{0t}} \frac{\partial p_i^\alpha}{\partial t^\beta} dt^\beta \wedge \omega_\alpha = \int_{\Omega_{0t}} \frac{\partial p_i^\alpha}{\partial t^\alpha}\, \omega = \int_{\Omega_{0t}} \frac{\partial^m p_i}{\partial t^1 ... \partial t^m}\, \omega$$

$$= p_i(t) - \Sigma_\alpha p_i(t)|_{t^\alpha=0} + \Sigma_{\alpha\neq\beta} p_i(t)|_{t^\alpha=0, t^\beta=0} - ... + (-1)^m p_i(0).$$

2) If $p_i(t)$ is given, then we can take

$$p_i^\alpha(t) = \frac{1}{m}\frac{\partial^{m-1} p_i}{\partial t^1 ... \partial \hat{t}^\alpha ... \partial t^m}(t).$$

9. Conversion of free end multitime problems to problems over fixed interval

The control problems considered so far are free end multitime problems, as the end multitime t_f of the interval $\Omega_{0t_f} = [0, t_f]$ is unspecified. By the multitime transformation $s^\alpha = \frac{1}{t_f^\alpha}\, t^\alpha$ (no sum) for constants $t_f^\alpha > 0$, such problems are converted to problems over the fixed interval $\Omega_{01} = [0,1]$. The transformed problems are called fixed end multitime problems. The unknown end multitime t_f is represented by an addition state variable $y = (y_\alpha)$, for which $\frac{\partial y_\alpha}{\partial s^\beta} = 0$ and $y(0) = t_f$ is assumed.

Definition 9.1. [Transformed multitime-optimal control problem with fixed end multitime] A multitime optimal control problem is considered. Let $z = (\bar{x}, y)$ be the extended state, $M \times \mathbb{R}^m_+$ the extended state space, and

$$d\bar{x}^i = y_\alpha X_\alpha ds^\alpha,\ dy = 0$$

the extended control system. The problem to find an initial condition $y(0) = t_f$ and an input map $\bar{u}(\cdot)$ such that a solution $z(\cdot)$ results which satisfies $z(0) = (x_0, t_f)$ and $z(1) = (x_f, t_f)$ and gives the minimal value of the cost function $I(z(\cdot), \bar{u}(\cdot)) = t_f$ is called transformed multitime-optimal control problem with fixed end multitime. Any solution $(z(\cdot), \bar{u}(\cdot))$ to this problem is called transformed multitime-optimal solution.

Let us consider a free end multitime functional

$$J = \int_{\Omega_{0t_f}} L(x(t), x_\gamma(t))\ dt^1...dt^m.$$

We introduce the changing of variables $t^\alpha = t^\alpha_f\, s^\alpha$, that moves Ω_{0t_f} to Ω_{01} and

$$\frac{\partial x}{\partial t^\gamma} = \frac{\partial x}{\partial s^\alpha}\frac{\partial s^\alpha}{\partial t^\gamma} = \frac{1}{t^\gamma_f}\frac{\partial x}{\partial s^\gamma}.$$

Using the Jacobian $\Delta = t^1_f...t^m_f$, it follows

$$J = \int_{\Omega_{0t_f}} L(x(t), x_\gamma(t))\ dt^1...dt^m = \Delta \int_{\Omega_{01}} L(x(t^\alpha_f\, s^\alpha), \frac{1}{t^\gamma_f}\, x_\gamma(t^\alpha_f\, s^\alpha))\, ds^1...ds^m.$$

In this way, the free end multitime variational problem is changed into a fixed end multitime variational problem.
Let us consider a free end controlled multitime functional

$$I(u) = \int_{\Omega_{0t_f}} L(x(t), u(t))\ dt^1...dt^m.$$

We introduce the changing of variables $t^\alpha = t^\alpha_f\, s^\alpha$, that moves Ω_{0t_f} to Ω_{01}. Using the Jacobian $\Delta = t^1_f...t^m_f$, it follows

$$I = \int_{\Omega_{0t_f}} L(x(t), u(t))\ dt^1...dt^m = \Delta \int_{\Omega_{01}} L(x(t^\alpha_f\, s^\alpha), u(t^\alpha_f\, s^\alpha))\, ds^1...ds^m.$$

In this way, the free end controlled multitime problem is changed into a fixed end multitime problem.

Remark 9.2. The evolution PDEs are

$$\frac{\partial \bar{x}}{\partial t^\alpha} = y_\alpha X_\alpha,\ \frac{\partial y_\alpha}{\partial s^\beta} = 0,\ y(0) = t_f.$$

10. Conclusions

We start with a single-time optimal control problem. The Hamiltonian is a function used to solve such a problem for a dynamical system. It was introduced by Lev Pontryagin for single-time optimal control problems as part of his maximum principle. The idea is that a necessary condition for solving an optimal control problem is that the control should be chosen so as to optimize the Hamiltonian. From Pontryagin's maximum principle, special conditions for the Hamiltonian can be derived. When the final time t_f is fixed and the Hamiltonian does not depend explicitly on time (is autonomous), we have $H(x^*(t), u^*(t), p^*(t)) \equiv \text{ct}$, or if the terminal time is free, then $H(x^*(t), u^*(t), p^*(t)) \equiv 0$. Further, if the terminal time tends to infinity, a transversality condition on the Hamiltonian applies and $\lim_{t\to\infty} H(t) = 0$.

The main question: do some of these properties from uni-temporal problems survive for multi-temporal problems? Our goal was to provide positive answers where possible, which we did in this paper.

In order to give positive answers, we had to go through the following steps of original research: any solution of the anti-trace problem is solution of multitime maximum principle, weak and strong multitime maximum principle, multitime-invariant dynamical systems and change rate of Hamiltonian, Hamiltonian tensor, change of variables in multitime optimal control, generated costates. All these combine ideas from differential geometry, multitemporal variational calculus and optimal multitemporal control, topics to which we have made an essential contribution in recent years [5], [7]-[3].

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References

[1] Barbu, V., *Necessary conditions for multiple integral problem in the calculus of variations*, Math. Ann., **260**(1982), 175-189.

[2] Berkovitz, L., *Lower semicontinuity of integral functionals*, Trans. Amer. Math. Soc., **192**(1974), 51-57.

[3] Iliuta, M., Udriste, C., Tevy, I., *Multitime control strategies for skilled movements*, Balkan J. Geom. Appl., **18**(2013), no. 2, 31-46.

[4] Mauder, M., *Time-Optimal Control of the Bi-Steerable Robot, A Case Study in Optimal Control of Nonholonomic Systems*, PhD Thesis, Julius-Maximilians-University at Würzburg, 2012.

[5] Pirvan, M., Udriste, C., *Optimal control of electromagnetic energy*, Balkan J. Geom. Appl., **15**(2010), no. 1, 131-141.

[6] Rund, H., *Pontryagin functions for multiple integral control problems*, J. Optim. Theory Appl., **18**(1976), no. 4, 511-520.

[7] Udriste, C., *Multitime controllability, observability and bang-bang principle*, J. Optim. Theory Appl., **139**(2008), no. 1, 141-157.

[8] Udriste, C., *Simplified multitime maximum principle*, Balkan J. Geom. Appl., **14**(2009), no. 1, 102-119.

[9] Udriste, C., *Multitime maximum principle for curvilinear integral cost*, Balkan J. Geom. Appl., **16**(2011), no. 1, 128-149.

[10] Udriste, C., *Multitime optimal control for quantum systems*, Proceedings of Third International Conference on Lie-Admissible Treatments of Irreversible Processes (ICLATIP-3), Kathmandu University, Dhulikhel, Nepal, January 3-7, 2011, 203-218.

[11] Udriste, C., *Multitime maximum principle for curvilinear integral cost*, Balkan J. Geom. Appl., **16**(2011), no. 1, 128-149.

[12] Udriste, C., Bejenaru, A., *Multitime optimal control with area integral costs on boundary*, Balkan J. Geom. Appl., **16**(2011), no. 2, 138-154.

[13] Udriste, C., Tevy, I., *Multitime dynamic programming for curvilinear integral actions*, J. Optim. Theory Appl., **146**(2010), Online First.

[14] Udriste, C., Tevy, I., *Multitime dynamic programming for multiple integral actions*, J. Glob. Optim., **51**(2011), no. 2, 345-360.

[15] Udriste, C., Tevy, I., *Sturm-Liouville operator controlled by sectional curvature on Riemannian manifolds*, Balkan J. Geom. Appl., **17**(2012), no. 2, 129-140.

[16] Udriste, C., Tevy, I., *Minirobots moving at different partial speeds*, Mathematics, MDPI, **1036(8)**(2020), 1-17.

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