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Existence of solutions for fractional q-difference equations

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Abstract. In this paper, we obtain some existence results for the integral boundary value problems of nonlinear fractional q-difference equations. The differential operator is taken in the Riemann-Liouville sense.

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1. Introduction

In this paper we will study the existence and uniqueness of solutions for the following singular boundary value problem of fractional q-difference equations

$$\left(D_q^{\alpha} u\right)(t) + \varphi(t) f(t, u(t)) = 0 , \qquad 0 < t < 1, \tag{1.1}$$

$$u(0) = 0, \quad u(1) = a \int_0^1 h(s)\, u(s)\, d_q A(s) + b, \tag{1.2}$$

where D_q^{α} is a fractional q-derivative of Riemann-Liouville type with $1 < \alpha \leq 2$, $\int_0^1 x(t)\, d_q A(t)$, is the Riemann-Stieltjes q-integral of x with respect to $A(t)$ such that $d_q A(t) = D_q A(t) d_q t$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $h : [0,1] \to \mathbb{R}$ is a continuous function, φ is defined on the interval $(0,1)$ and φ may be singular at 0 or 1.

In the last few years, fractional differential equations have been studied extensively, because of their demonstrated applications in various fields of science and

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engineering; see [5, 16, 19, 27, 36, 39]. Recently, many researchers study the existence of solutions of fractional differential equations such as the Riemann-Liouville fractional derivative problem [3, 12, 17, 31, 32, 34, 35, 37, 38, 40, 41, 42] the Caputo fractional boundary value problem [3, 33], the Hadamard fractional boundary value problem [28, 30], conformable fractional boundary value problem [20, 24, 25] etc.

Quantum calculus is ordinary calculus without limits. There are several types of quantum calculus: h-calculus, q-calculus and Hahn's calculus. In this paper we are concerned with the q-calculus. The q-derivative and the q-integral were first defined by Jackson [14, 15]. For some recent existence results on q-difference equations see [2, 6, 10, 13, 22, 26] and the references there in.

There has also been a growing interest on the subject of discrete fractional equations. Fractional q-difference equations have recently attracted the attention of several researchers for the applications of fields such as physics, chemistry, biology, economics, control theory, signal and image processing, electricity etc. Some recent work on the existence theory of fractional q-difference equations can be found in [4, 7, 8, 9, 23]. Motivated by all the works above, in this paper we discuss the problem (1.1)-(1.2) and we will give the existence results for this problem.

The paper is organized as follows. In Section 2, we give some preliminary results that will be used in the proof of our main results. In Section 3, we establish the existence of a solution for the nonlinear fractional q-difference boundary value problems (1.1)-(1.2).

2. Preliminaries

In this section, we list some useful definitions and preliminaries, which will be used in the proofs of the main results.
Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q}, a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^k$, $k \in N_0 = \{0, 1, 2, \dots\}$ is

$$(a-b)^0 = 1 \ , \qquad (a-b)^{(k)} = \prod_{i=0}^{k-1} \left(a - bq^i\right), \quad k \in N, \ a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$.
The q-gama function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \qquad x \in \mathbb{R} \backslash \{0, -1, -2, \dots\}$$

then

$$\Gamma_q(x+1) = [x] \, \Gamma_q(x).$$

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)^x}, (D_q f)(0) = \lim_{x\to 0} (D_q f)(x) \quad \text{for} \quad x \neq 0$$

and q-derivatives of higher order by

$$\left(D_q^0 f\right)(x) = f(x) \quad \text{and} \quad \left(D_q^n f\right)(x) = D_q\left(D_q^{n-1} f\right)(x), \quad n \in N.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \qquad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, i.e.,

$$\left(I_q^0 f\right)(x) = f(x) \quad \text{and} \quad \left(I_q^n f\right)(x) = I_q\left(I_q^{n-1} f\right)(x) \ , \ n \in N.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x)$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

We now point out two formulas that will be used later (${}_tD_q$ denotes the derivative with respect to variable t)

$$\begin{aligned} {}_tD_q (t-s)^{(\alpha)} &= [\alpha]_q (t-s)^{(\alpha-1)}, \\ \left({}_xD_q \int_0^x f(x,t)\, d_q t\right)(x) &= \int_0^x {}_xD_q f(x,t) d_q t + f(qx, x). \end{aligned}$$

Remark 2.1. If $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

Definition 2.2. [1] *Let $\alpha \geq 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is*

$$(I_q^0 f)(x) = f(x)$$

and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t \ , \qquad x \in [0,1].$$

Definition 2.3. [29] *The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by*

$$(D_q^\alpha f)(x) = f(x)$$

and

$$(D_q^\alpha f)(x) = \left(D_q^p I_q^{p-\alpha} f\right)(x), \qquad \alpha > 0,$$

where p is the smallest integer greater than or equal to α.

Next, we list some properties about q-derivative and q-integral that are already known in the literature, which are helpful in proofs of our main results.

Lemma 2.4. [21]

(1) *If f and g are q-integral on the interval $[a,b]$, $\alpha \in \mathbb{R}$, $c \in [a,b]$, then*

1. $\int_a^b (f(t) + g(t))\, d_qt = \int_a^b f(t)\, d_qt + \int_a^b g(t) d_qt$
2. $\int_a^b \alpha f(t)\, d_qt = \alpha \int_a^b f(t) d_qt$
3. $\int_a^b f(t) d_qt = \int_a^c f(t) d_qt + \int_c^b f(t) d_qt$
4. $\int x^\alpha d_qs = \frac{x^{\alpha+1}}{[\alpha+1]}$, $(\alpha \neq -1)$;

(2) *If $|f|$ is q-integral on the interval $[0,x]$, then*

$$\left| \int_0^x f(t) d_qt \right| \leq \int_0^x |f(t)|\, d_qt;$$

(3) *If f and g are q-integral on the interval $[0,x]$, $f(t) \leq g(t)$, $\forall t \in [0,x]$, then*

$$\int_0^x f(t) d_qt \leq \int_0^x g(t) d_qt.$$

Lemma 2.5. [9] *Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:*

$$\left(I_q^\alpha D_q^p f\right)(x) = \left(D_q^p I_q^\alpha f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

Now, we will give the existence theorems used in our main results.

Theorem 2.6. [11] *Let $T : X \to X$ be a map on a complete non-empty metric space. If some iterate T^n of T is a contraction, then T has a unique fixed point.*

Theorem 2.7. [18] *Let X be a Banach space and $P \subseteq X$ be a cone. Suppose that Ω_1 and Ω_2 are bounded open sets contained in X such that $0 \subseteq \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ is a completely continuous operator. If either*

1. $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *or*
2. $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *then*

T has at least one fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Theorem 2.8. [4] *(Nonlinear alternative for single valued maps) Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

1. *F has a fixed point in $\overline{U}$, or*
2. *There is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda\, F(u)$.*

The next result is important in the sequel.

Lemma 2.9. *Let $g(t) : [0,1] \to [0,\infty)$ be a given continuous function. Then the boundary value problem*

$$\left(D_q^\alpha u\right)(t) + g(t) = 0,\ 0 < t < 1, \tag{2.1}$$

$$u\left(0\right)=0,\quad u\left(1\right)=a\int_0^1 h\left(s\right)u\left(s\right)d_qA\left(s\right)+b \tag{2.2}$$

has a unique solution

$$u\left(t\right)=\int_0^1 H\left(t,qs\right)g\left(s\right)d_qs+\frac{b}{k}\ t^{\alpha-1}$$

where

$$H\left(t,s\right)=G\left(t,s\right)+\frac{a\ t^{\alpha-1}}{k}G_A(s)$$

such that

$$G\left(t,s\right)=\frac{1}{\Gamma_q(\alpha)}\begin{cases}\left(t\left(1-s\right)\right)^{(\alpha-1)}-\left(t-s\right)^{(\alpha-1)}, & s\leq t\\ \left(t\left(1-s\right)\right)^{(\alpha-1)}, & s\geq t,\end{cases}$$

$$G_A\left(s\right)=\int_{t=0}^1 h\left(t\right)G\left(t,s\right)d_qA(t)$$

and

$$k=1-a\ \int_0^1 h\left(s\right)s^{\alpha-1}\ d_qA(s)\neq 0.$$

Proof. From *Lemma 2.5* and *Definition 2.2*, we have

$$u(t)=-\frac{1}{\Gamma_q\left(\alpha\right)}\int_0^t\left(t-qs\right)^{(\alpha-1)}g\left(s\right)d_qs+c_1t^{\alpha-1}+c_2t^{\alpha-2}.$$

Since $u\left(0\right)=0$ we get $c_2=0$. Thus, we have

$$u(t)=-\frac{1}{\Gamma_q\left(\alpha\right)}\int_0^t\left(t-qs\right)^{(\alpha-1)}g\left(s\right)d_qs+c_1t^{\alpha-1}.$$

Using the second boundary condition we get

$$\begin{aligned}&-\frac{1}{\Gamma_q\left(\alpha\right)}\int_0^1\left(1-qs\right)^{(\alpha-1)}g\left(s\right)d_qs+c_1\\&\qquad=a\int_0^1 h\left(s\right)\left[-\frac{1}{\Gamma_q\left(\alpha\right)}\int_0^s\left(s-qw\right)^{(\alpha-1)}g\left(w\right)d_qw+c_1s^{\alpha-1}\right]d_qA\left(s\right)+b\end{aligned}$$

Thus, we have

$$\begin{aligned}c_1\left[1-a\int_0^1 h\left(s\right)s^{\alpha-1}d_qA\left(s\right)\right]&=\frac{1}{\Gamma_q\left(\alpha\right)}\int_0^1\left(1-qs\right)^{(\alpha-1)}g\left(s\right)d_qs\\&-\frac{a}{\Gamma_q\left(\alpha\right)}\int_0^1\left[\int_{wq}^1 h\left(s\right)\left(s-qw\right)^{(\alpha-1)}d_qA\left(s\right)\right]g\left(w\right)d_qw+b\end{aligned}$$

and

$$\begin{aligned}c_1=\frac{1}{k}\Bigg\{&\frac{1}{\Gamma_q\left(\alpha\right)}\int_0^1\left(1-qs\right)^{(\alpha-1)}g\left(s\right)d_qs\\&-\frac{a}{\Gamma_q\left(\alpha\right)}\int_0^1\left[\int_{wq}^1 h\left(s\right)\left(s-qw\right)^{(\alpha-1)}d_qA\left(s\right)\right]g\left(w\right)d_qw\Bigg\}+\frac{b}{k}\end{aligned}$$

so

$$
\begin{aligned}
c_1 = &\frac{1}{k\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s \\
&- \frac{a}{k\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 h(t)(t-qs)^{(\alpha-1)} d_q A(t) \right] g(s)\, d_q s \ + \frac{b}{k}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} g(s)\, d_q s + \frac{t^{\alpha-1}}{k} \left\{ \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s \right. \\
&\quad \left. - \frac{a}{\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 h(t)(t-qs)^{(\alpha-1)} d_q A(t) \right] g(s)\, d_q s \right\} + \frac{b}{k} t^{\alpha-1} \\
&= -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} g(s)\, d_q s + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s \\
&\quad - \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s + \frac{t^{\alpha-1}}{k} \left\{ \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s \right. \\
&\quad \left. - \frac{a}{\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 h(t)(t-qs)^{(\alpha-1)} d_q A(t) \right] g(s)\, d_q s \ \right\} + \frac{b}{k} t^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
u(t) = &\int_0^1 G(t,qs)\, g(s)\, d_q s \\
&+ \frac{at^{\alpha-1}}{k\Gamma_q(\alpha)} \left\{ \int_{s=0}^1 \left[\int_{t=0}^1 h(t)\, t^{\alpha-1} d_q A(t) \right] (1-qs)^{(\alpha-1)} g(s)\, d_q s \right. \\
&\qquad \left. - \int_{s=0}^1 \left[\int_{t=sq}^1 h(t)(t-qs)^{(\alpha-1)} d_q A(t) \right] g(s)\, d_q s \right\} + \frac{b}{k} t^{\alpha-1}.
\end{aligned}
$$

Thus

$$
u(t) = \int_0^1 G(t,qs)\, g(s)\, d_q s + \frac{at^{\alpha-1}}{k} \int_0^1 G_A(s)\, g(s)\, d_q s + \frac{b}{k} t^{\alpha-1},
$$

where

$$
G(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (t(1-s))^{(\alpha-1)} - (t-s)^{(\alpha-1)}, & s \le t \\ (t(1-s))^{(\alpha-1)}, & s \ge t \end{cases}
$$

$$
G_A(s) = \int_{t=0}^1 h(t)\, G(t,s)\, d_q A(t),
$$

and

$$
k = 1 - a \int_0^1 h(s)\, s^{\alpha-1} \ d_q A(s) \neq 0.
$$

Consequently, we can write

$$
u(t) = \int_0^1 H(t,qs)\, g(s)\, d_q s + \frac{b}{k} t^{\alpha-1}
$$

where

$$H(t,s) = G(t,s) + \frac{a\ t^{\alpha-1}}{k} G_A(s). \qquad \square$$

Lemma 2.10. *Assume that* $0 < k < 1$ *and* $G_A(s) \geq 0$ *for* $s \in [0,1]$*, then* $H(t,s)$ *satisfies followings:*

1. $H(t,s) \geq 0, \quad \forall t, s \in [0,1]$
2. *There exist a constant*

$$L = \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{a.H}{k}\right)$$

such that

$$\frac{at^{\alpha-1}}{k} G_A(s) \leq H(t,s) \leq L\ (1-s)^{(\alpha-1)} t^{\alpha-1},$$

where

$$H = \int_0^1 h(t)\, t^{\alpha-1}\ d_qA(t).$$

Proof. 1. (i) For $s \leq t$, we know that

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \left[(t(1-s))^{(\alpha-1)} - (t-s)^{(\alpha-1)} \right].$$

Since

$$t < 1 \Rightarrow \frac{1}{t} > 1 \Rightarrow -\frac{s}{t} < -s \Rightarrow \left(1 - \frac{s}{t}\right)^{(\alpha-1)} < (1-s)^{(\alpha-1)},$$

we get

$$\begin{aligned} &\frac{1}{\Gamma_q(\alpha)} \left[t^{\alpha-1}(1-s)^{(\alpha-1)} - t^{\alpha-1}\left(1 - \frac{s}{t}\right)^{(\alpha-1)} \right] \\ &> \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1} \left[(1-s)^{(\alpha-1)} - \left(1 - \frac{s}{t}\right)^{(\alpha-1)} \right] > 0, \end{aligned}$$

so $G(t,s) \geq 0$.

(ii) For $s \geq t$, it is clear that $G(t,s) > 0$.

Thus we get $G(t,s) \geq 0, \quad \forall t, s \in [0,1]$.

Since $G_A(s) = \int_{t=0}^1 h(t)\, G(t,s)\, d_qA(t) > 0$, then $H(t,s) \geq 0$, for $t, s \in [0,1]$.

2. Since

$$G(t,s) \leq \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}(1-s)^{(\alpha-1)} < \frac{1}{\Gamma_q(\alpha)}(1-s)^{(\alpha-1)},$$

we have

$$\begin{aligned} G_A(s) &= \int_{t=0}^1 h(t)\, G(t,s)\, d_qA(t) < \int_{t=0}^1 h(t)\, \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}(1-s)^{(\alpha-1)} d_qA(t) \\ &= \frac{(1-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} \int_{t=0}^1 h(t)\ t^{\alpha-1} d_qA(t) = \frac{(1-s)^{(\alpha-1)}}{\Gamma_q(\alpha)}\ H. \end{aligned}$$

Also, we know

$$H(t,s) = G(t,s) + \frac{a\ t^{\alpha-1}}{k} G_A(s)$$

that

$$\begin{aligned} H(t,s) &\leq \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}(1-s)^{(\alpha-1)} + \frac{a}{k\Gamma_q(\alpha)} t^{\alpha-1}(1-s)^{(\alpha-1)} H \\ &= \frac{1}{\Gamma_q(\alpha)} \left(1+\frac{aH}{k}\right) t^{\alpha-1}(1-s)^{(\alpha-1)} \\ &\leq L\,(1-s)^{(\alpha-1)} t^{\alpha-1}. \end{aligned}$$

In conclusion, we have

$$\frac{at^{\alpha-1}}{k} G_A(s) \leq H(t,s) \leq \frac{1}{\Gamma_q(\alpha)} \left(1+\frac{aH}{k}\right)(1-s)^{(\alpha-1)} t^{\alpha-1}. \qquad \square$$

3. Main results

We are now in a position to state and prove our main results in this paper. Transform the problem $(1.1)-(1.2)$ into a fixed point problem. We define the operator $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$(Tu)(t) = \int_0^1 H(t,qs)\,\varphi(s)\,f(s,u(s))\,d_qs + \frac{b}{k} t^{\alpha-1}. \tag{3.1}$$

It's easy to show that, from *Lemma 2.9*, the fixed points of operator T coincide with the solutions of boundary value problems $(1.1)-(1.2)$.

Suppose that the following conditions are satisfied.

(H_1) $\varphi(t)$ is nonnegative on $(0,1)$ and

$$\int_0^1 (1-qs)^{(\alpha-1)} \varphi(s)\,d_qs < \infty$$

(H_2) $|f(t,u) - f(t,v)| \leq K.\,|u-v|$, for all $t \in [0,1]$, $u,v \in \mathbb{R}$

(H_3) $f \in C([0,1]\times\mathbb{R},\ [0,\infty))$, $C \subset B$, $C = \{u \in C[0,1] : u(t) \geq 0\}$

(H_4) $f \in C([0,1]\times[0,\infty),[0,\infty))$, $f(t,u_1) \leq f(t,u_2)$ for $0 \leq u_1 < u_2$ and any $t \in [0,1]$.

Let $B = C([0,1],R)$ is the Banach space with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ and $C = \{u \in B : u(t) \geq 0\}$. Then C is a normal cone on B. Also we denote $u_1 \preccurlyeq u_2$ if and only if $u_{2-}u_1 \in C$ for $u_1, u_2 \in B$.

Lemma 3.1 *If there holds (H_1) and f meets (H_3). Then the operator $T : C \to B$*

$$(Tu)(t) = \int_0^1 H(t,qs)\,\varphi(s)\,f(s,u(s))\,d_qs + \frac{b}{k} t^{\alpha-1}$$

satisfies $T(C) \subset C$ and T is completely continuous.

Proof. It follows from (H_1) and the nonnegativeness and continuity of $H(t,qs)$ and $f(t,u(t))$ that T has definition and satisfies $T(C) \subset C$. The next proof will be given in several steps.

Step 1. T is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$. Then for each $t \in [0,1]$, according to Lebesgue control convergence theorem and *Lemma 2.10*, we have

$$\|Tu_n - Tu\| = \sup_{t\in[0,1]} |(Tu_n)(t) - (Tu)(t)|$$

$$\begin{aligned}
&= \sup_{t\in[0,1]} \left| \int_0^1 H(t,qs)\varphi(s) f(s,u_n(s))\, d_qs - \int_0^1 H(t,qs)\varphi(s) f(s,u(s))\, d_qs \right| \\
&\le \sup_{t\in[0,1]} \int_0^1 H(t,qs)\ \varphi(s)\ |f(s,u_n(s)) - f(s,u(s))|\ d_qs \\
&\le \frac{1}{\Gamma_q(\alpha)}\left(1+\frac{aH}{k}\right) t^{\alpha-1} \int_0^1\ (1-qs)^{(\alpha-1)}\varphi(s)\ |f(s,u_n(s)) - f(s,u(s))|\ d_qs \\
&\to\ 0\,,\quad n\to\infty
\end{aligned}$$

Therefore, T is continuous.
Step 2. T maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.
Indeed, it is enough to show that for any $\mu > 0$, there exists a positive constant

$$r = M\int_0^1 \frac{1}{\Gamma_q(\alpha)}\left(1+\frac{aH}{k}\right)(1-qs)^{(\alpha-1)}\varphi(s)\ d_qs + \frac{b}{k}.$$

Such that for each $u \in B_\mu = \{u \in C([0,1],\mathbb{R}) : \|u\| \le \mu\}$, we have $\|T_u\| \le r$.
Denote $M = max_{t\in[0,1],\|u\|\le\mu}\ \{f(t,u(t)) + 1\}$. We have for each $t \in [0,1]$,

$$\begin{aligned}
|Tu(t)| &= \int_0^1 H(t,qs)\varphi(s) f(s,u(s))\, d_qs + \frac{b}{k} t^{\alpha-1} \\
&\le \int_0^1 H(t,qs)\varphi(s) f(s,u(s))\, d_qs + \frac{b}{k} \\
&\le M\int_0^1 H(t,qs)\varphi(s)\, d_qs + \frac{b}{k} \\
&\le M\int_0^1 \frac{1}{\Gamma_q(\alpha)}\left(1+\frac{aH}{k}\right)(1-qs)^{(\alpha-1)}\varphi(s)\ d_qs + \frac{b}{k} = r.
\end{aligned}$$

Thus we get $\|Tu\| \le r$.
Step 3. T maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$.
Let $t_1, t_2 \in [0,1],\ t_1 < t_2,\ B_\mu$ be bounded set of $C([0,1],\mathbb{R})$ as in Step 2 and let $u \in B_\mu$. Then

$$\begin{aligned}
|(Tu)(t_2) - (Tu)(t_1)| = \Bigg| &\int_0^1 (H(t_2,qs) - H(t_1,qs))\varphi(s) f(s,u(s))\, d_qs \\
&+ \frac{b}{k}\left({t_2}^{\alpha-1} - {t_1}^{\alpha-1}\right)\Bigg|
\end{aligned}$$

$$
\begin{aligned}
\leq & \left| - \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s) f(s, u(s)) \, d_q s \right. \\
& + \frac{{t_2}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \varphi(s) \; f(s, u(s)) \, d_q s \\
& - \frac{a{t_2}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 (t_2 - qs)^{(\alpha-1)} h(t_2) \, d_q A(t_2) \right] \varphi(s) \; f(s, u(s)) \, d_q s + \frac{b}{k} {t_2}^{\alpha-1} \\
& + \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s) f(s, u(s)) \, d_q s \\
& - \frac{{t_1}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \varphi(s) \; f(s, u(s)) \, d_q s \\
& \left. + \frac{a{t_1}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 (t_1 - qs)^{(\alpha-1)} h(t_1) \, d_q A(t_1) \right] \varphi(s) \; f(s, u(s)) \, d_q s - \frac{b}{k} {t_1}^{\alpha-1} \right|
\end{aligned}
$$

Furthermore, we deduce that

$$
\begin{aligned}
\left| (Tu)(t_2) - (Tu)(t_1) \right| \leq & \left| \int_0^{t_1} \frac{(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s) f(s, u(s)) \, d_q s \right. \\
& \left. + \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s) f(s, u(s)) \, d_q s \right| \\
& + \left| \frac{{t_2}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \varphi(s) \; f(s, u(s)) \, d_q s \right. \\
& \left. - \frac{{t_1}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \varphi(s) \; f(s, u(s)) \, d_q s \right| \\
& + \left| \frac{a{t_2}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 (t_2 - qs)^{(\alpha-1)} h(t_2) \, d_q A(t_2) \right] \varphi(s) \; f(s, u(s)) \, d_q s \right. \\
& \left. - \frac{a{t_1}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 (t_1 - qs)^{(\alpha-1)} h(t_1) \, d_q A(t_1) \right] \varphi(s) \; f(s, u(s)) \, d_q s \right| \\
& + \frac{b}{k} \left| {t_2}^{\alpha-1} - {t_1}^{\alpha-1} \right| \\
& \leq M \int_0^{t_1} \frac{(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s) \, d_q s
\end{aligned}
$$

$$+\frac{M}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}(1-qs)^{(\alpha-1)}\varphi(s)\,d_qs+\frac{M\left|t_2{}^{\alpha-1}-t_1{}^{\alpha-1}\right|}{k}\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varphi(s)\ d_qs$$

$$+\frac{aM}{k\Gamma_q(\alpha)}\int_0^1\left\{t_2{}^{\alpha-1}\int_{sq}^1(t_2-qs)^{(\alpha-1)}h(t_2)\,d_qA(t_2)\right.$$

$$\left.-t_1{}^{\alpha-1}\int_{sq}^1(t_1-qs)^{(\alpha-1)}h(t_1)\,d_qA(t_1)\right\}$$

$$+\frac{b}{k}\left|t_2{}^{\alpha-1}-t_1{}^{\alpha-1}\right|. \tag{3.2}$$

Obviously,

$$\int_0^{t_1}\left((t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right)\ \varphi(s)\,d_qs$$
$$\le\int_0^1\left(\frac{(t_2-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}-\frac{(t_1-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}\right)(1-qs)^{(\alpha-1)}\ \varphi(s)\,d_qs.$$

The function $\frac{(t-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}$ is continuous with respect to t and s on $[0,1]\times[0,1]$ and so it is uniformly continuous on $[0,1]\times[0,1]$.

Therefore, for any $t_1,t_2\in[0,1]$, $t_1<t_2$, $s\in[0,1]$, as $t_1\to t_2$, we can conclude that

$$\frac{(t_2-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}-\frac{(t_1-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}\to 0.$$

So we can see

$$\int_0^{t_1}\left((t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right)\ \varphi(s)\,d_qs$$
$$\le\int_0^1\left(\frac{(t_2-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}-\frac{(t_1-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}\right)(1-qs)^{(\alpha-1)}\ \varphi(s)\,d_qs$$
$$\to 0\ ,\qquad t_1\to t_2.$$

For

$$\int_{t_1}^{t_2}(1-qs)^{(\alpha-1)}\varphi(s)\,d_qs,$$

according to Cauchy criterion for convergence of an improper integral, as $t_2\to t_1$,

$$\int_{t_1}^{t_2}(1-qs)^{(\alpha-1)}\varphi(s)\,d_qs\to 0.$$

In conclusion, as $t_2\to t_1$, the right-hand side of the above inequality (3.2) tends to zero. As a consequence of Step 1 to 3 together with the Arzela-Ascoli theorem. Hence T is completely continuous. The proof is complete. □

Our first result is based on the generalization of Banach contraction principle.

Theorem 3.2. *Assume* (H_1) *and* (H_2) *hold. Let*

$$M = \int_0^1 s^{\alpha-1}\left(1-qs\right)^{(\alpha-1)}\varphi\left(s\right)d_q s$$

and

$$MK\frac{1}{\Gamma_q\left(\alpha\right)}\left(1+\frac{aH}{k}\right) < 1$$

Then the boundary value problems $(1.1)-(1.2)$ *have a unique solution.*

Proof. We shall prove that under the conditions (H_1) and (H_2), the operator T^n is a contraction map in the space $C\left[0,1\right]$ for sufficiently large n.

$$T: C\left[0,1\right] \to C\left[0,1\right]$$

By *Lemma 2.10* we have

$$\begin{aligned}\left|\left(Tu\right)\left(t\right)-\left(Tv\right)\left(t\right)\right| = &\left|\int_0^1 H\left(t,qs\right)\varphi\left(s\right)f\left(s,u\left(s\right)\right)d_q s+\frac{b}{k}t^{\alpha-1}\right.\\ &\left.-\int_0^1 H\left(t,qs\right)\varphi\left(s\right)f\left(s,v\left(s\right)\right)d_q s-\frac{b}{k}t^{\alpha-1}\right|\end{aligned}$$

$$\begin{aligned}&\le \int_0^1 \left|H\left(t,qs\right)\right|\left|\varphi\left(s\right)\right|\ \left|f\left(s,u\left(s\right)\right)-f(s,v\left(s\right))\right|d_q s\\ &\le \int_0^1 \frac{1}{\Gamma_q\left(\alpha\right)}\left(1+\frac{aH}{k}\right)\left(1-qs\right)^{(\alpha-1)}t^{\alpha-1}\varphi\left(s\right)\ K\left|u\left(s\right)-v(s)\right|d_q s\\ &\le \frac{1}{\Gamma_q\left(\alpha\right)}\left(1+\frac{aH}{k}\right)\ K\left\|u-v\right\|\ t^{\alpha-1}\underbrace{\int_0^1 \left(1-qs\right)^{(\alpha-1)}\varphi\left(s\right)d_q s}_{l}\\ &\le \frac{1}{\Gamma_q\left(\alpha\right)}\left(1+\frac{aH}{k}\right)\ K\left\|u-v\right\|\ t^{\alpha-1}\ l\end{aligned}$$

$$\left|\left(T^2u\right)\left(t\right)-\left(T^2v\right)\left(t\right)\right| \le \int_0^1 \left|H\left(t,qs\right)\right|\left|\varphi\left(s\right)\right|\ \left|f\left(s,\left(Tu\right)\left(s\right)\right)-f(s,\left(Tv\right)\left(s\right))\right|d_q s$$

$$\begin{aligned}&\le \int_0^1 \frac{1}{\Gamma_q\left(\alpha\right)}\left(1+\frac{aH}{k}\right)\left(1-qs\right)^{(\alpha-1)}t^{\alpha-1}\varphi\left(s\right)\ K\left|Tu-Tv\right|d_q s\\ &\le \int_0^1 \frac{1}{{\Gamma_q}^2\left(\alpha\right)}\left(1+\frac{aH}{k}\right)^2\left(1-qs\right)^{(\alpha-1)}t^{\alpha-1}\varphi\left(s\right)\ K^2\left\|u-v\right\|\ s^{\alpha-1}\ l\ d_q s\\ &< \frac{1}{{\Gamma_q}^2\left(\alpha\right)}\left(1+\frac{aH}{k}\right)^2K^2\ \left\|u-v\right\|\ t^{\alpha-1}\ l\ \underbrace{\int_0^1 s^{\alpha-1}\left(1-qs\right)^{(\alpha-1)}\varphi\left(s\right)d_q s}_{M}\end{aligned}$$

By the induction method, we have

$$\begin{aligned} |(T^n u)(t) - (T^n v)(t)| &\le \frac{\|u-v\|}{{\Gamma_q}^n(\alpha)}\left(1+\frac{aH}{k}\right)^n K^n\ l\ M^{n-1}\ t^{\alpha-1} \\ &\le \frac{K^n\ l\ M^{n-1}}{{\Gamma_q}^n(\alpha)}\left(1+\frac{aH}{k}\right)^n \|u-v\|, \end{aligned}$$

we can choose enough large n, such that

$$\frac{K^n\ l\ M^{n-1}}{{\Gamma_q}^n(\alpha)}\left(1+\frac{aH}{k}\right)^n < \frac{1}{2},$$

then it follows that

$$|(T^n u)(t) - (T^n v)(t)| \le \frac{1}{2}\|u-v\|.$$

By means of *Theorem 2.6*, we claim that the operator T has a unique fixed point. □

Theorem 3.3. *If there holds* (H_1)*, define two constants*

$$W = \max_{(t,s)\in[0,1]\times[0,1]} H(t,qs) \quad \text{and} \quad Q = \int_0^1 W\ \varphi(s)d_qs.$$

If there exist two positive constants $r_2 > r_1$ *such that*

$$\frac{b}{k} + Q \max_{(t,u)\in[0,1]\times[0,r_2]} f(t,u) \le r_2$$

and

$$\frac{b}{k} + Q \min_{(t,u)\in[0,1]\times[0,r_1]} f(t,u) \ge r_1$$

then the boundary value problems $(1.1)-(1.2)$ *have at least one solution satisfying* $r_1 \le \|u\| \le r_2$.

Proof. It follows from continuity of $H(t,qs)$ and $f(t,u)$ that $H(t,qs)$, $f(t,u)$ has a maximum on any closed field.

Let $\Omega_1 = \{u \in C:\ \|u\| < r_1\}$. For $u \in C \cap \partial\Omega_1$, we have $0 \le u(t) \le r_1$ on $[0,1]$,

$$\begin{aligned} \|Tu\| &= \sup_{t\in[0,1]}\left(\int_0^1 H(t,qs)\varphi(s)f(s,u(s))\,d_qs + \frac{b}{k}t^{\alpha-1}\right) \\ &= \int_0^1 \max_{t\in[0,1]} H(t,qs)\varphi(s)f(s,u(s))d_qs + \frac{b}{k} \\ &= \int_0^1 W\ \varphi(s)f(s,u(s))\,d_qs + \frac{b}{k} \\ &\ge \min_{(t,u)\in[0,1]\times[0,r_1]} f(t,u)\int_0^1 W\ \varphi(s)\,d_qs + \frac{b}{k} \\ &\ge r_1 = \|u\|. \end{aligned}$$

Let $\Omega_2 = \{u \in C:\ \|u\| < r_2\}$. For $u \in C \cap \partial\Omega_2$, we have $0 \le u(t) \le r_2$ on $[0,1]$,

$$\begin{aligned}\|Tu\| &= \underset{t\in[0,1]}{sup}\left(\int_0^1 H(t,qs)\varphi(s) f(s,u(s))\,d_qs+\frac{b}{k}t^{\alpha-1}\right)\\ &\le \max_{(t,u)\in[0,1]\times[0,r_2]} f(t,u)\int_0^1 W\ \varphi(s)\,d_qs+\frac{b}{k}\\ &\le r_2=\|u\|\,.\end{aligned}$$

By *Theorem 2.7* and *Lemma 3.1*, we can conclude that the operator equation $Tu = u$ has a solution satisfying $r_1 \le \|u\| \le r_2$. The proof is complete. □

Theorem 3.4. *Assume that $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Let (H_1) and (H_3) be satisfied. If there exists a constant R such that*

$$\frac{R}{r} > 1. \tag{3.3}$$

Then the boundary value problems $(1.1)-(1.2)$ have at least one solution, where r is given in Lemma 3.1.

Proof. Let u be a solution. Then for $t\in[0,1]$, using the computations in proving that T is bounded, we have $|u(t)| = |\lambda Tu(t)| \le r$ and thus we have

$$\frac{\|u\|}{r} \le 1.$$

In view of (3.3) there exists R such that $\|u\| \ne R$. Let us set

$$U = \{u\in C([0,1],\mathbb{R}):\ \|u\| < R+1\}$$

Note that the operator $T : \overline{U} \to C([0,1],\mathbb{R})$ is completely continuous. From the choice of U, there is no $u\in\partial U$ such that $u = \lambda Tu(t)$ for some $\lambda\in(0,1)$.

Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that T has a fixed point $u\in\overline{U}$ which is a solution of $(1.1)-(1.2)$. □

Now we will give the upper and lower solutions result.

Definition 3.5. *Let $x\in C^2[0,1]$, we say that x is a lower solution of the boundary value problems $(1.1)-(1.2)$, if*

$$\left(D_q^\alpha x\right)(t)+\varphi(t)f(t,x(t))\ge 0,\qquad t\in(0,1)$$

$$x(0)=0,\quad x(1)\le a\int_0^1 h(s)\,x(s)\,d_qA(s)+b$$

Let $y\in C^2[0,1]$, we say that y is a upper solution of the boundary value problems $(1.1)-(1.2)$, if

$$\left(D_q^\alpha y\right)(t)+\varphi(t)f(t,y(t))\le 0,\qquad t\in(0,1)$$

$$y(0)=0,\quad y(1)\ge a\int_0^1 h(s)\,y(s)\,d_qA(s)+b$$

Theorem 3.6. *Assume that (H_4) holds, boundary value problems $(1.1)-(1.2)$ has a lower solution $u_0\in C$ and an upper solution $v_0\in C$ such that $u_0\preccurlyeq v_0$. The boundary*

value problems $(1.1)-(1.2)$ *has the maximal lower solution* u^* *and the minimal upper solution* v^* *on* $[u_0,\ v_0] \subset C$, *both* u^* *and* v^* *are positive solutions of boundary value problems* $(1.1)-(1.2)$.
Furthermore,

$$0 \le u_0 \le u^* \le v^* \le v_0.$$

Proof. The proof will be given with three steps.
Step 1. We will obtain the lower solution sequence $\{u_k\}$ and the upper solution sequence $\{v_k\}$. According to *Lemma 2.9* for given $u_0 \in C$,

$$D_q^{\alpha} u_1(t) + \varphi(t) f(t, u_0(t)) = 0, \qquad t \in (0,1)$$

$$u_1(0) = 0, \quad u_1(1) = a \int_0^1 h(s) u_0(s) d_q A(s) + b$$

has a unique solution u_1.
Since u_0 is a lower solution of boundary value problems $(1.1)-(1.2)$ then

$$D_q^{\alpha} u_0(t) + \varphi(t) f(t, u_0(t)) \ge 0, \qquad t \in (0,1)$$

$$u_0(0) = 0, \quad u_0(1) = a \int_0^1 h(s) u_0(s) d_q A(s) + b.$$

Thus we can get that

$$D_q^{\alpha}(u_1(t) - u_0(t)) \le 0$$

and

$$(u_1 - u_0)(0) = 0, \quad (u_1 - u_0)(1) \ge a \int_0^1 h(s)(u_0 - u_0)(s) d_q A(s) \ge 0.$$

If we define $u_1(t) - u_0(t) = k(t)$, we get

$$D_q^{\alpha} k(t) = g(t)$$

$$k(0) = 0, \quad k(1) = \gamma$$

so we know that

$$k(t) = -\int_0^1 G(t, qs) g(s) d_q s + \gamma t^{\alpha-1},$$

since $g(t) \le 0$ and $\gamma \ge 0$ we say that $k(t) \ge 0$ and so $u_1(t) \ge u_0(t)$.
So we can get that if $u_0 \preccurlyeq u_1$ than $f(t, u_1) \ge f(t, u_0)$, from the condition (H_4).
Using this, we get

$$D_q^{\alpha} u_1(t) = -\varphi(t) f(t, u_0(t)) \ge -\varphi(t) f(t, u_1(t))$$

$$u_1(0) = 0, u_1(1) = a \int_0^1 h(s) u_0(s) d_q A(s) + b \le a \int_0^1 h(s) u_1(s) d_q A(s) + b.$$

Since

$$D_q^{\alpha} u_1(t) + \varphi(t) f(t, u_1(t)) \ge 0, \qquad t \in (0,1)$$

$$u_1(0) = 0, \quad u_1(1) \le a \int_0^1 h(s)\ u_1(s) d_q A(s) + b,$$

then $u = u_1(t)$ is a lower solution of boundary value problems $(1.1)-(1.2)$.

Starting from the initial function u_0 by the following iterative scheme

$$D_q^\alpha u_k\left(t\right)+\varphi\left(t\right)f\left(t,u_{k-1}\left(t\right)\right)=0,\qquad t\in\left(0,1\right),\quad k=1,2,\ldots$$

$$u_k\left(0\right)=0,\quad u_k\left(1\right)=a\int_0^1 h\left(s\right)u_{k-1}\left(s\right)d_qA\left(s\right)+b, \tag{3.4}$$

we can obtain the sequence $\{u_k\}$, where $u=u_k(t)$ are lower solutions of boundary value problems (1.1) – (1.2) and $u_{k-1}\preccurlyeq u_k$, so that $\{u_k\}$ is monotonically increasing.

Starting from the initial function v_0 by the following iterative scheme

$$D_q^\alpha v_k\left(t\right)+\varphi\left(t\right)f\left(t,v_{k-1}\left(t\right)\right)=0,\qquad t\in\left(0,1\right),\quad k=1,2,\ldots$$

$$v_k\left(0\right)=0,\quad v_k\left(1\right)=a\int_0^1 h\left(s\right)v_{k-1}\left(s\right)d_qA\left(s\right)+b, \tag{3.5}$$

we can get the sequence $\{v_k\}$, where $v=v_k(t)$ are upper solutions of boundary value problems (1.1) – (1.2) and $\{v_k\}$ is monotonically decreasing.

Step 2. We prove that $u_k\preccurlyeq v_k$ if $u_{k-1}\preccurlyeq v_{k-1},\ \ k=1,2,\ldots$
Since $u_{k-1}\preccurlyeq v_{k-1}$, then $u_{k-1}(t)\leq v_{k-1}\left(t\right)$ and $D_q^\alpha u_{k-1}\left(t\right)\geq D_q^\alpha v_{k-1}\left(t\right)$ and from (H_4), we have

$$f\left(t,u_{k-1}\left(t\right)\right)\leq f\left(t,v_{k-1}\left(t\right)\right).$$

Thus, by (3.4) and (3.5), we get

$$D_q^\alpha\left(v_k\left(t\right)-u_k(t)\right)=-\,\varphi\left(t\right)\left(f\left(t,v_{k-1}\left(t\right)\right)-f\left(t,u_{k-1}\left(t\right)\right)\right)\leq 0,$$

$$v_k\left(0\right)-u_k\left(0\right)=0,$$

$$v_k\left(1\right)-u_k\left(1\right)=a\int_0^1 h\left(s\right)v_{k-1}\left(s\right)d_qA\left(s\right)-a\int_0^1 h\left(s\right)u_{k-1}\left(s\right)d_qA\left(s\right)\geq 0.$$

Similarly we can show that $u_k\preccurlyeq v_k$ in the same way as the above.
Therefore,

$$u_0\preccurlyeq u_1\preccurlyeq\cdots\preccurlyeq u_k\preccurlyeq\cdots\preccurlyeq\cdots\preccurlyeq v_k\preccurlyeq\cdots\preccurlyeq v_1\preccurlyeq v_0.$$

Since C is a normal cone on B, the $\{u_k\}$ is uniformly bounded. Because H,G,φ and f are continuous, we can easily get that $\{u_k\}$ is equicontinuous. Hence the $\{u_k\}$ is relatively compact. Then there exist u^* and v^* such that

$$\lim_{k\to\infty}u_k=u^*\ ,\qquad \lim_{k\to\infty}D_q^\alpha u_k=D_q^\alpha u^* \tag{3.6}$$

$$\lim_{k\to\infty}v_k=v^*\ ,\qquad \lim_{k\to\infty}D_q^\alpha v_k=D_q^\alpha v^* \tag{3.7}$$

which imply that u^* is the maximal lower solution, v^* is the minimal upper solution of boundary value problems (1.1) – (1.2) in $[u_0,v_0]\subset C$ and $u^*\preccurlyeq v^*$.

Step 3. We prove that u^* and v^* are the solution of boundary value problems (1.1) – (1.2).
According to *Lemma 2.9* and (3.4), we can get that

$$u_k\left(t\right)=\int_0^1 H\left(t,qs\right)\varphi\left(s\right)f\left(s,u_{k-1}\left(s\right)\right)d_qs+\frac{b}{k}t^{\alpha-1}.$$

From (3.6) and by the continuity of H, f and Lebesgue dominated convergence theorem, we have

$$u^*(t) = \int_0^1 H(t, qs)\varphi(s) f(s, u^*(s))\, d_q s + \frac{b}{k} t^{\alpha-1},$$

which implies that u^* is a solution of boundary value problems $(1.1) - (1.2)$. In the same way, we can show that v^* is a solution of boundary value problems $(1.1) - (1.2)$, too.

Furthermore,

$$0 \leq u_0(t) \leq u^*(t) \leq v^*(t) \leq v_0(t). \qquad \square$$

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