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A nonlocal Cauchy problem for nonlinear generalized fractional integro-differential equations

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Abstract. In this paper, we study the existence of solutions of a nonlocal Cauchy problem for nonlinear fractional integro-differential equations involving generalized Katugampola fractional derivative. By using fixed point theorems, the results are obtained in weighted space of continuous functions. In the last, results are illustrated with suitable examples.

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1. Introduction

The idea of fractional differentiation was introduced by Riemann and Liouville in the nineteenth century. It is the generalization of ordinary differentiation and integration to arbitrary non-integer order, for details, see [1, 2, 4, 5, 6, 15, 16] and the references therein.

The area of fractional differential equations is now considered to be very important due to its various applications in different fields of science and technology such as control theory, rheology, signal processing, modelling, fractals, chaotic dynamics, bioengineering and biomedical and so on, for example see [6, 13, 17] and the references therein. Recently, many researchers studied the fractional differential and integro-differential equations and obtained many interesting existence and uniqueness results, see [3, 7, 12, 18, 20, 19, 21, 22, 23] and the references therein.

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Recently, the authors in [8] discussed the existence and stability of solution of the initial value problem (IVP):

$$({}^{\varrho}D_{a+}^{\alpha,\beta}x)(t) = f(t, x(t)),\ t \in J := (a, T], \tag{1.1}$$

$$({}^{\varrho}I_{a+}^{1-\gamma}x)(a) = c_2, \gamma = \alpha + \beta(1-\alpha), c_2 \in \mathbb{R}, \tag{1.2}$$

for generalized Katugampola fractional differential equation by using Schauder fixed point theorem and the equivalence between IVP (1.1)-(1.2) and the integral equation

$$x(t) = \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} f(s, x(s))\mathrm{d}s. \tag{1.3}$$

In [9], using Krasnoselskii's fixed point theorem, Schauder fixed point theorem and Schaefer fixed point theorem, authors discussed the existence of solution of IVP with nonlocal initial condition:

$$({}^{\varrho}D_{a+}^{\alpha,\beta}x)(t) = f(t, x(t)),\ t \in J := (a, T], \tag{1.4}$$

$$({}^{\varrho}I_{a+}^{1-\gamma}x)(a+) = \sum_{j=1}^{m}\eta_j x(\xi_j), \alpha \le \gamma = \alpha + \beta(1-\alpha),\ \xi_j \in (a, T], \tag{1.5}$$

where ${}^{\varrho}D_{a+}^{\alpha,\beta}$ is the generalized Katugampola fractional derivative of order $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$ and ${}^{\varrho}I_{a+}^{1-\gamma}$ is the generalized Katugampola fractional integral with $\varrho > 0$. Authors also proved the equivalence between (1.4)-(1.5) and the integral equation

$$\begin{aligned} x(t) = &\frac{K}{\Gamma(\alpha)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} f(s, x(s))\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} f(s, x(s))\mathrm{d}s, \end{aligned} \tag{1.6}$$

where

$$K = \left(\Gamma(\gamma) - \sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\right)^{-1}. \tag{1.7}$$

The above results motivate us and therefore, in this paper, we obtain the existence of solution of the following Nonlinear Generalized Fractional Integro–Differential Equation (NGFIDE) of order α $(0 < \alpha < 1)$ and type $\beta \in [0, 1]$:

$$({}^{\varrho}D_{a+}^{\alpha,\beta}x)(t) = f\left(t, x(t), \int_a^t h(t, s)x(s)\mathrm{d}s\right),\ t \in J := (a, T], \tag{1.8}$$

$$({}^{\varrho}I_{a+}^{1-\gamma}x)(a+) = \sum_{j=1}^{m}\eta_j x(\xi_j), \alpha \le \gamma = \alpha + \beta(1-\alpha),\ \xi_j \in (a, T], \tag{1.9}$$

where ${}^{\varrho}D_{a+}^{\alpha,\beta}$ is the generalized Katugampola fractional derivative of order $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$ and ${}^{\varrho}I_{a+}^{1-\gamma}$ is the generalized Katugampola fractional integral with $\varrho > 0$. Function $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a given function, ξ_j are pre–fixed points satisfy $0 < a < \xi_1 \le \ldots \le \xi_m < T$ and η_j, $j = 1, 2, \ldots, m$ are real numbers.

First, we establish an equivalent mixed-type nonlinear Volterra integral equation

$$\begin{aligned} x(t) =& \frac{K}{\Gamma(\alpha)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \\ &\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s \\ &+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} f\left(s,x(s),\int_a^t h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s, \end{aligned} \tag{1.10}$$

where

$$K=\left(\Gamma(\gamma)-\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\right)^{-1}, \tag{1.11}$$

for NGFIDE (1.8)-(1.9) in the weighted space of continuous functions $C_{1-\gamma,\varrho}[a,T]$ presented in the next section. We use the Krasnoselskii's fixed point theorem and Schauder fixed point theorem to prove the existence results for NGFIDE (1.8)-(1.9).

The rest of the paper is organized as follows. In Section 2, some definitions, notations and basic results are given. We prove the equivalent integral equation in Section 2 and the existence results are proved in Section 3. Illustrative examples are given in the last section.

2. Preliminaries

Here we introduce some definitions and present preliminary results needed in our proofs later.
Let the Euler gamma and beta functions be defined, respectively, by

$$\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}\mathrm{d}x,\quad \mathbf{B}(\alpha,\beta)=\int_0^1(1-x)^{\alpha-1}x^{\beta-1}\mathrm{d}x,\ \alpha>0,\ \beta>0.$$

It is well known that $\mathbf{B}(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ for $\alpha>0,\ \beta>0$, see [13]. Throughout the paper, we consider $[a,T],\ 0<a<T<\infty$ being a finite interval on $\mathbb{R}^+$ and $\varrho>0$.

Definition 2.1 ([13])**.** The space $X_c^p(a,T),\ c\in\mathbb{R},\ p\geq 1$ consists of those real valued Lebesgue measurable functions g on (a,T) for which $\|g\|_{X_c^p}<\infty$, where

$$\|g\|_{X_c^p}=\left(\int_a^b|t^c g(t)|^p\frac{\mathrm{d}t}{t}\right)^{1/p},\quad p\geq 1\quad \text{and}\quad \|g\|_{X_c^{\infty}}=\operatorname*{ess\,sup}_{a\leq t\leq T}|t^c g(t)|$$

In particular, when $c=1/p$, we see that $X_{1/p}^c(a,T)=L_p(a,T)$.

Definition 2.2 ([14])**.** We denote by $C[a,T]$ a space of continuous functions g on $(a,T]$ with the norm

$$\|g\|_C=\max_{t\in[a,T]}|g(t)|$$

The weighted space $C_{\gamma,\varrho}[a,T],\ 0\leq\gamma<1$ of functions g on $(a,T]$ is defined as

$$C_{\gamma,\varrho}[a,T]=\left\{g:(a,T]\to\mathbb{R}:\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma}g(t)\in C[a,T]\right\} \tag{2.1}$$

with the norm

$$\|g\|_{C_{\gamma,\varrho}}=\left\|\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma}g(t)\right\|_{C}=\max_{t\in[a,t]}\left|\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma}g(t)\right|,$$

and $C_{0,\varrho}[a,T]=C[a,T]$

Definition 2.3 ([14])**.** Let $\delta_{\varrho}=\left(t^{\varrho-1}\mathrm{d}/\mathrm{d}t\right),\ 0\leq\gamma<1$. Denote $C^{n}_{\delta_{\varrho}\gamma}[a,T]$ the Banach space of functions g which are continuously differentinble, with δ_{ϱ}, on $[a,T]$ upto order $(n-1)$ and have the derivative $\delta^{n}_{\varrho}g$ on $(a,T]$ such that $\delta^{n}_{\varrho}g\in C_{\gamma,\varrho}[a,T]$:

$$C^{n}_{\delta_{\varrho,\gamma}}[a,T]=\left\{\delta^{k}_{\varrho}g\in C[a,T],k=0,1,\ldots,n-1,\delta^{n}_{\varrho}g\in C_{\gamma,\varrho}[a,T]\right\},\quad n\in\mathrm{N}$$

with the norm

$$\|g\|_{C^{n}_{\delta_{\varrho,\gamma}}}=\sum_{k=0}^{n-1}\left\|\delta^{k}_{\varrho}g\right\|_{C}+\left\|\delta^{n}_{\varrho}g\right\|_{C_{\gamma,\varrho}},\quad \|g\|_{C^{n}_{\delta_{\varrho}}}=\sum_{k=0}^{n}\max_{t\in\Omega}\left|\delta^{k}_{\varrho}g(t)\right|.$$

In particular, for $n=0$ we have $C^{0}_{\delta_{\varrho}\gamma}[a,T]=C_{\gamma,\varrho}[a,T]$.

Definition 2.4 ([10])**.** Let $\alpha>0$ and $f\in X^{p}_{c}(a,T)$, where X^{p}_{c} is as in Definition 2.1. The left-sided Katugampola fractional integral ${}^{\varrho}I^{\alpha}_{a+}$ of order α is defined as

$${}^{\varrho}I^{\alpha}_{a+}f(t)=\int_{a}^{t}s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{f(s)}{\Gamma(\alpha)}\mathrm{d}s,\ t>a. \tag{2.2}$$

Definition 2.5 ([11])**.** Let $\alpha\in\mathbb{R}^{+}\backslash N$ and $n=[\alpha]+1$, where $[\alpha]$ is the integer part of α. The left-sided Katugampola fractional derivative ${}^{\varrho}D^{\alpha}_{a+}$ is defined as

$$\begin{aligned}{}^{\varrho}D^{\alpha}_{a+}f(t)&=\delta^{n}_{\varrho}\left({}^{\varrho}I^{n-\alpha}_{a+}f(s)\right)(t)\\&=\left(t^{\varrho-1}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\int_{a}^{t}s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{n-\alpha-1}\frac{f(s)}{\Gamma(n-\alpha)}\mathrm{d}s.\end{aligned} \tag{2.3}$$

Definition 2.6 ([14])**.** The left-sided generalized Katugampola fractional derivative ${}^{\varrho}D^{\alpha,\beta}_{a+}$ of order $0<\alpha<1$ and type $0\leq\beta\leq1$ is defined as

$$\left({}^{\varrho}D^{\alpha,\beta}_{a+}f\right)(t)=\left({}^{\varrho}I^{\beta(1-\alpha)}_{a+}\delta_{\varrho}\ {}^{\varrho}I^{(1-\beta)(1-\alpha)}_{a+}f\right)(t), \tag{2.4}$$

for the functions for which the right-hand side expression exists.

Lemma 2.7 ([9])**.** *Suppose that $\alpha>0,\ \beta>0,\ p\geq1$ and $\varrho,c\in\mathbb{R}$ such that $\varrho\geq c$. Then for $f\in X^{p}_{c}(a,T)$, the semigroup property of Katugampola integral is valid. This is*

$${}^{\varrho}I^{\alpha}_{a+}\,{}^{\varrho}I^{\beta}_{a+}f(t)={}^{\varrho}I^{\alpha+\beta}_{a+}f(t). \tag{2.5}$$

Lemma 2.8 ([11])**.** *Suppose that $\alpha>0,\ 0\leq\gamma<1$ and $f\in C_{\gamma,\varrho}[a,T]$. Then for all $t\in(a,T]$,*

$${}^{\varrho}D^{\alpha}_{a+}\,{}^{\varrho}I^{\alpha}_{a+}f(t)=f(t). \tag{2.6}$$

Lemma 2.9 ([11]). *Suppose that $\alpha > 0$, $0 \le \gamma < 1$, $f \in C_{\gamma,\varrho}[a,T]$ and ${}^{\varrho}I_{a+}^{1-\alpha} f \in C_{\gamma,\varrho}^{1}[a,T]$. Then*

$$ {}^{\varrho}I_{a+}^{\alpha}\, {}^{\varrho}D_{a+}^{\alpha} f(t) = f(t) - \frac{{}^{\varrho}I_{a+}^{1-\alpha} f(a)}{\Gamma(\alpha)} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho} \right)^{\alpha - 1}. \tag{2.7} $$

Lemma 2.10 ([9]). *Suppose ${}^{\varrho}I_{a+}^{\alpha}$ and ${}^{\varrho}D_{a+}^{\alpha}$ are defined as in Definitions 2.4 and 2.5, respectively. Then*

$$ {}^{\varrho}I_{a+}^{\alpha} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho} \right)^{\sigma - 1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma + 1)} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho} \right)^{\alpha + \sigma - 1}, \ \alpha \le 0,\ \sigma > 0,\ t > a, \tag{2.8} $$

$$ {}^{\varrho}D_{a+}^{\alpha} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho} \right)^{\alpha - 1} = 0,\ 0 < \alpha < 1. \tag{2.9} $$

Remark 2.11. For $0 < \alpha < 1$, $0 \le \beta \le 1$, the generalized Katugampola fractional derivative ${}^{\varrho}D_{a+}^{\alpha,\beta}$ can be written in terms of Katugampola fractional derivative as

$$ {}^{\varrho}D_{a+}^{\alpha,\beta} = {}^{\varrho}I_{a+}^{\beta(1-\alpha)} \delta_{\varrho}\, {}^{\varrho}I_{a+}^{1-\gamma} = {}^{\varrho}I_{a+}^{\beta(1-\alpha)}\, {}^{\varrho}D_{a+}^{\gamma}, \quad \gamma = \alpha + \beta(1-\alpha). $$

Lemma 2.12 ([14]). *Let $\alpha > 0$, $0 < \gamma \le 1$ and $f \in C_{1-\gamma,\varrho}[a,b]$. If $\alpha > \gamma$, then*

$$ \left({}^{\varrho}I_{a+}^{\alpha} f\right)(a) = \lim_{x \to a+} \left({}^{\varrho}I_{a+}^{\alpha} f\right)(t) = 0. $$

To discuss the existence of a solution of NGFIDE (1.8)-(1.9), we need the following spaces:

$$ C_{1-\gamma,\varrho}^{\alpha,\beta}[a,T] = \left\{ g \in C_{1-\gamma,\varrho}[a,T] : {}^{\varrho}D_{a+}^{\alpha,\beta} g \in C_{1-\gamma,\varrho}[a,T] \right\},\ 0 < \gamma \le 1 \tag{2.10} $$

and

$$ C_{1-\gamma,\varrho}^{\gamma}[a,T] = \left\{ g \in C_{1-\gamma,\varrho}[a,T] : {}^{\varrho}D_{a+}^{\gamma} g \in C_{1-\gamma,\varrho}[a,T] \right\},\ 0 < \gamma \le 1. \tag{2.11} $$

Since ${}^{\varrho}D_{a+}^{\alpha,\beta} g = {}^{\varrho}I_{a+}^{\beta(1-\alpha)}\, {}^{\varrho}D_{a+}^{\gamma} g$, it is obvious that $C_{1-\gamma,\varrho}^{\gamma}[a,T] \subset C_{1-\gamma,\varrho}^{\alpha,\beta}[a,T]$.

Lemma 2.13 ([9]). *Let $\alpha > 0$, $\beta > 0$ and $\gamma = \alpha + \beta - \alpha\beta$. If $g \in C_{1-\gamma,\varrho}^{\gamma}[a,T]$, then*

$$ {}^{\varrho}I_{a+}^{\gamma}\, {}^{\varrho}D_{a+}^{\gamma} g(t) = {}^{\varrho}I_{a+}^{\alpha}\, {}^{\varrho}D_{a+}^{\alpha,\beta} g(t) = {}^{\varrho}D_{a+}^{\beta(1-\alpha)} g(t). $$

To prove the equivalence between NGFIDE (1.8)-(1.9) with Volterra integral equation (1.10), we note the following lemmas.

Lemma 2.14 ([14]). *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta - \alpha\beta$. If $f : (a,T] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma,\varrho}[a,T]$ for any $x(\cdot) \in C_{1-\gamma,\varrho}[a,T]$, then $x(\cdot) \in C_{1-\gamma,\varrho}^{\gamma}[a,T]$ satisfies IVP* (1.1)-(1.2) *if and only if $x(\cdot)$ satisfies the nonlinear Volterra integral equation.* (1.3)

Lemma 2.15 ([9]). *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta - \alpha\beta$. If $f : (a,T] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma,\varrho}[a,T]$ for any $x(\cdot) \in C_{1-\gamma,\varrho}[a,T]$, then $x \in C_{1-\gamma,\varrho}^{\gamma}[a,T]$ satisfies IVP* (1.4)-(1.5) *if and only if x satisfies the nonlinear Volterra integral equation* (1.6).

Using the aforementioned equivalence, we prove a new equivalent mixed-type integral equation for NGFIDE (1.8)-(1.9).

Lemma 2.16. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. Suppose that $f : (a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot), y(\cdot)) \in C_{1-\gamma,\varrho}[a,T]$ for any $x(\cdot) \in C_{1-\gamma,\varrho}[a,T]$. Function $x(\cdot) \in C^{\gamma}_{1-\gamma,\varrho}[a,T]$ is a solution of NGFIDE* (1.8)–(1.9) *if and only if $x(\cdot)$ is a solution of the mixed-type nonlinear Volterra integral equation.* (1.10)

Proof. First, we start with necesssary part. By appling Lemma 2.14 and Lemma 2.15, a solution of NGFIDE (1.8)-(1.9) can be expressed as

$$x(t) = \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1} + \int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s. \tag{2.12}$$

By putting $t = \xi_j$ in (2.12), we obtain

$$x(\xi_j) = \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)}\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1} + \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s, \tag{2.13}$$

and by multiplying both sides of (2.13) by η_j, we get

$$\eta_j x(\xi_j) = \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1} + \eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s. \tag{2.14}$$

Using the initial condition of NGFIDE (1.8)-(1.9), we have

$$({}^{\varrho}I^{1-\gamma}_{a+}x)(a+) = \sum_{j=1}^{m}\eta_j x(\xi_j) = \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\rho}-a^{\ell}}{\varrho}\right)^{\gamma-1} + \sum_{j-1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s,$$

which gives

$$({}^{\varrho}I^{1-\gamma}_{a+}x)(a+)\left(\Gamma(\gamma) - \sum_{j-1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{j-1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho^{-1}}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s, \tag{2.15}$$

i.e.

$$({}^{\varrho}I_{a+}^{1-\gamma}x)(a+) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j-1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho^{-1}}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s, \tag{2.16}$$

where K is as in (1.11). Substituting (2.16) into (2.12), we obtain the integral equation (1.10).

Secondly, we prove the sufficient part.

Applying ${}^{\varrho}I_{a+}^{1-\gamma}$ on both sides of the integral equation (1.10), we get

$$\begin{aligned}{}^{\varrho}I_{a+}^{1-\gamma}x(t) =& \frac{K}{\Gamma(\alpha)}{}^{\varrho}I_{a+}^{1-\gamma}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\ &\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s\\ &+{}^{\varrho}I_{a+}^{1-\gamma}{}^{\varrho}I_{a+}^{\alpha}f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s,\end{aligned}$$

using Lemmas 2.7 and 2.10 , we have

$$\begin{aligned}{}^{\varrho}I_{a+}^{1-\gamma}x(t) =& \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\ &\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s\\ &+{}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}f\left(t,x(t),\int_a^t h(t,s)x(s)\mathrm{d}s\right).\end{aligned} \tag{2.17}$$

Since $1-\gamma<1-\beta(1-\alpha)$, Lemma 2.12 can be utilized and limit $t\to a+$ gives

$${}^{\varrho}I_{a+}^{1-\gamma}x(a) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s. \tag{2.18}$$

By putting $t=\xi_j$ in (1.10), we have

$$x(\xi_j) = \frac{K}{\Gamma(\alpha)}\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s.$$

Further,

$$\begin{aligned}\sum_{j=1}^m \eta_j x(\xi_j) = & \frac{K}{\Gamma(\alpha)}\sum_{j-1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\ & \times f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s \sum_{j=1}^m \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \\ & + \sum_{j=1}^m \eta_j \frac{1}{\Gamma(\alpha)}\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s \\ = & \sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s \\ & \times \left(1 + K\sum_{j=1}^m \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1}\right) \\ = & \frac{\Gamma(\gamma)}{\Gamma(\alpha)} K \sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s. \end{aligned} \tag{2.19}$$

Equations (2.18) and (2.19), implies that

$$^{\varrho}I_{a+}^{1-\gamma} x(a+) = \sum_{j=1}^m \eta_j x(\xi_j).$$

Applying $^{\varrho}D_{a+}^{\gamma}$ to both sides of (1.10), from Lemmas 2.10 and 2.14 if follows that

$$^{\varrho}D_{a+}^{\gamma} x(t) = {}^{\varrho}D_{a+}^{\beta(1-\alpha)} f\left(t, x(t), \int_a^t h(t,s)x(s)\mathrm{d}s\right), \tag{2.20}$$

since $x \in C_{1-\gamma,\varrho}^{\gamma}[a,T]$, from the definition of $C_{1-\gamma,\varrho}^{\gamma}[a,T]$ we have $^{\varrho}D_{a+}^{\gamma} x \in C_{1-\gamma,\varrho}[a,T]$ then $^{\varrho}D_{a+}^{\beta(1-\alpha)} f = \delta_\varrho {}^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\varrho}[a,T]$.
For $f \in C_{1-\gamma,\varrho}[a,T]$, obviously $^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\varrho}[a,T]$, then $^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\varrho}^{\delta_\varrho}[a,T]$. This means f and $^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f$ satisfy the conditions of Lemma 2.9. Lastly, applying $^{\varrho}I_{a+}^{1-\beta(1-\alpha)}$ to both sides of (2.20), Lemma 2.9 helps us to obtain

$$^{\varrho}D_{a+}^{a,\beta} x(t) = f\left(t, x(t), \int_a^t h(t,s)x(s)\mathrm{d}s\right) - \frac{^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f(a)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{\beta(1-\alpha)-1}.$$

By Lemma 2.12 it is easy to see that $^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f(a) = 0$. Hence, it reduces to

$$^{\varrho}D_{a+}^{\alpha,\beta} x(t) = f\left(t, x(t), \int_a^t h(t,s)x(s)\mathrm{d}s\right).$$

Hence, the sufficiency is proved. This completes the proof of the lemma. □

3. Existence of solutions

In this section, we state and prove the main results concerning the existence of a solution of NGFIDE (1.8)-(1.9).

By using Krasnoselskii's fixed point theorem we prove the first existence result for NGFIDE (1.8)-(1.9).

Theorem 3.1. *Suppose that:*

(H_{01}) $f:(a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is a function such that* $f(\cdot,x(\cdot),y(\cdot))\in C^{\beta(1-\alpha)}_{1-\gamma,\varrho}[a,T]$ *for any* $x\in C_{1-\gamma,\varrho}[a,T]$ *and there exists a positive constant* $L>0$ *such that for all* $x,y,\bar{x},\bar{y}\in\mathbb{R}$,
$|f(t,x,y)-f(t,\bar{x},\bar{y})|\le L(|x-\bar{x}|+|y-\bar{y}|)$.

(H_{02}) *The constant*

$$\theta=\frac{\Gamma(\gamma)L(1+h_T(T-a))}{\Gamma(\gamma+\alpha)}\left(|K|\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}+\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}\right)$$
$$<1,$$

where K *is as in* (1.11) *and* $h_T=Sup\{|h(t,s)|\,|a\le s\le t\le T\}$.

Then NGFIDE (1.8)-(1.9) *has at least one solution in* $C^{\gamma}_{1-\gamma,\varrho}[a,T]\subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T]$.

Proof. From Lemma 2.16 it is sufficient to prove the existence of a solution for mixed-type integral equation (1.10). Define $N:C_{1-\gamma,\varrho}[a,T]\to C_{1-\gamma,\varrho}[a,T]$ by

$$\begin{aligned}(Nx)(t)=&\frac{K}{\Gamma(\alpha)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j-1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\&\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s\\&+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s.\end{aligned}\tag{3.1}$$

Obviously, the operntor N is well defined. Set $\bar{f}(s)=f(s,0,0)$ and

$$\varpi=\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(|K|\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{a+\gamma-1}+\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}\right)\|\bar{f}\|_{C_{1-\gamma,\varrho}}.$$

Consider

$$B_r=\left\{x\in C_{1-\gamma,\varrho}[a,T]:\|x\|_{C_{1-\gamma,\varrho}}\le r\right\},\quad \text{where } r\ge\frac{\varpi}{1-\theta},\ \theta<1.$$

Now, we subdivide the operator N into two operators P and Q on B_r as follows:

$$(Px)(t) = \frac{K}{\Gamma(\alpha)} \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \times f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s, \tag{3.2}$$

and

$$(Qx)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t s^{\varrho-1} \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s. \tag{3.3}$$

The proof is divided into several steps:

Step 1. For any $x, \bar{x} \in B_r$ we prove $Px + Q\bar{x} \in B_r$. For operator P, multiplying both sides of (3.2) by $\left((t^\varrho - a^\varrho)/\varrho\right)^{1-\gamma}$, we have

$$(Px)(t) \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma} = \frac{K}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \times f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s,$$

then

$$\begin{aligned}
&\left|(Px)(t) \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}\right| \\
&\leq \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \left| f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\right| \mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\
&\times \left(\left| f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right) - f(s,0,0) \right| + \left| f(s,0,0) \right| \right) \mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\
&\times \left(L\left(|x(s)| + h_T \int_a^s |x(\tau)| d\tau\right) + |\bar{f}(s)| \right) \mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \\
&\times \left(\left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma} L(1 + h_T(T-a))|x(s)| + \left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma} |\bar{f}(s)| \right) \mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)} \sum_{j-1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1}
\end{aligned}$$

$$\times\left(L(1+h_T(T-a))\|x\|_{C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{\gamma,\varrho}}\right)\mathrm{d}s$$

$$\leq \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}\mathbf{B}(\alpha,\gamma)\times\left(L(1+h_T(T-a))\|x\|_{C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{1-\gamma,\varrho}}\right),$$

which implies

$$\|Px\|_{C_{1-\gamma,\varrho}}\leq \frac{\Gamma(\gamma)|K|}{\Gamma(\alpha+\gamma)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}$$
$$\times\left(L(1+h_T(T-a))\|x\|_{[C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{1-\gamma,\varrho}}\right). \qquad (3.4)$$

For operator Q,

$$\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}(Qx)(t)-\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}$$
$$\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s, \qquad (3.5)$$

using the same fact that we used in the case of operator P again, we obtain

$$\left|(Qx)(t)\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}$$
$$\times\left|f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\right|\mathrm{d}s$$
$$\leq\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}$$
$$\times\left(L(1+h_T(T-a))|x(s)|+|\bar{f}(s)|\right)\mathrm{d}s$$
$$\leq\frac{\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)}\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}\left(L(1+h_T(T-a))\|x\|_{C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{1-\gamma,\varrho}}\right).$$

This gives

$$\|Qx\|_{C_{1-\gamma,\varrho}}\leq\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}$$
$$\times\left(L(1+h_T(T-a))\|x\|_{C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{1-\gamma,\varrho}}\right). \qquad (3.6)$$

From equations (3.4) and (3.6) for every $x,\bar{x}\in B_r$ we obtain

$$\|Px+Q\bar{x}\|_{C_{1-\gamma,\varrho}}\leq\|Px\|_{C_{1-\gamma,\varrho}}+\|Q\bar{x}\|_{C_{1-\gamma,\varrho}}\leq\theta r+\varpi\leq r$$

which implies that $Px+Q\bar{x}\in B_r$.

Step 2. Now we prove that operator P is a contraction mapping.
Let $x, \bar{x} \in B_r$, for operator P we have,

$$\begin{aligned}
&\Big((Px)(t) - (P\bar{x})(t)\Big)\left(\frac{t^{\varrho} - a^{\varrho}}{\varrho}\right)^{1-\gamma} \\
&= \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} \left(f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\right. \\
&\left. - f\left(s, \bar{x}(s), \int_a^s h(s,\tau)\bar{x}(\tau)d\tau\right)\right)\mathrm{d}s \\
&\le \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} \left(\left|f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\right.\right. \\
&\left.\left. - f\left(s, \bar{x}(s), \int_a^s h(s,\tau)\bar{x}(\tau)d\tau\right)\right|\right)\mathrm{d}s \\
&\le \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_s} s^{\varrho-1} \left(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} L(1 + h_T(T-a))|x(s) - \bar{x}(s)|\mathrm{d}s \\
&\le \frac{L(1 + h_T(T-a))|K|\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1} \|x - \bar{x}\|_{C_{1-\gamma,\varrho}},
\end{aligned}$$

which is

$$\begin{aligned}
\|Px - P\bar{x}\|_{C_{1-\gamma,\varrho}} \le{}& \frac{L(1 + h_T(T-a))|K|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \\
&\sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1} \|x - \bar{x}\|_{C_{1-\gamma,\varrho}} \le \theta \|x - \bar{x}\|_{C_{1-\gamma,\varrho}}.
\end{aligned}$$

Thus, by assumption (H_{02}), operator P is a contraction mapping.

Step 3. Operator Q is compact and continuous.
Since $f \in C_{1-\gamma,\varrho}[a,T]$, by the definition of $C_{1-\gamma,\varrho}[a,T]$, it is obvious that Q is continuous. By Step 1, we have

$$\begin{aligned}
\|Qx\|_{C_{1-\gamma,\varrho}} \le{}& \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \left(\frac{T^{\varrho} - a^{\varrho}}{\varrho}\right)^{\alpha} \\
&\times \left(L(1 + h_T(T-a))\|x\|_{[C_{1-\gamma,\varrho}} + \|\bar{f}\|_{C_{1-\gamma,\varrho}}\right),
\end{aligned}$$

this means Q is uniformly bounded on B_r.

To prove the compactness of Q, for any $0 < a < t_1 < t_2 \leq T$ we have

$$
\begin{aligned}
&\Big|(Qx)(t_1) - (Qx)(t_2)\Big| \\
&\quad = \Bigg|\int_a^{t_1} s^{\varrho-1}\left(\frac{t_1^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \frac{f(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau)}{\Gamma(\alpha)}\mathrm{d}s \\
&\quad - \int_a^{t_2} s^{\varrho-1}\left(\frac{t_2^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \frac{f(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau)}{\Gamma(\alpha)}\mathrm{d}s\Bigg| \\
&\quad \leq \frac{\|f\|_{C_{1-\gamma,\varrho}}}{\Gamma(\alpha)}\Bigg|\int_a^{t_1} s^{\varrho-1}\left(\frac{t_1^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\left(\frac{s^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\mathrm{d}s \\
&\quad - \int_a^{t_2} s^{\varrho-1}\left(\frac{t_2^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\left(\frac{s^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\mathrm{d}s\Bigg| \\
&\quad \leq \frac{\|f\|_{C_{1-\gamma,\varrho}}\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left|\left(\frac{t_1^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1} - \left(\frac{t_2^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}\right|. \qquad (3.7)
\end{aligned}
$$

The right-hand side of inequality (3.7) tends to zero as $t_2 \to t_1$ either $\alpha+\gamma < 1$ or $\alpha+\gamma \geq 1$. Therefore, Q is equicontinuous. Hence, by Arzelà-Ascoli theorem, Q is compact on B_r.

By applying Krasnoselskii's fixed point theorem, NGFIDE (1.8)-(1.9) has at least one solution $x \in C_{1-\gamma,\varrho}[a,T]$. One can easily show that this solution is actually in $C^{\gamma}_{1-\gamma,\varrho}[a,T]$ by repeating the process from the proof of Lemma 2.16. Thus, we complete the proof. □

Now, we will discuss the next existence result by using Schauder fixed point theorem. For this, we consider the following hypothesis:

(H_{11}) $f : (a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot), y(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma,\varrho}[a,T]$ for any $x, y \in C_{1-\gamma,\varrho}[a,T]$, and for all $x, y \in \mathbb{R}$ there exist $L > 0$ and $M \geq 0$ such that

$$|f(t,x,y)| \leq L(|x|+|y|) + M.$$

Theorem 3.2. *Suppose that (H_{11}) and (H_{02}) hold. Then NGFIDE* (1.8)-(1.9) *has at least one solution in* $C^{\gamma}_{1-\gamma,\varrho}[a,T] \subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T]$.

Proof. Let $B_\varepsilon = \left\{x \in C_{1-\gamma,\varrho}[a,T] : \|x\|_{C_{1-\gamma,\varrho}} \leq \varepsilon\right\}$ with $\varepsilon \geq \Omega/(1-\theta)$ for $\theta < 1$, where

$$\Omega = \frac{M|K|}{\Gamma(\alpha+1)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha} + \frac{M}{\Gamma(\alpha+1)}\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha-\gamma+1}.$$

Consider the operator N on B_ε defined in (3.1). We prove the theorem in the following three steps:

Step 1. First we prove that $N\left(B_{\varepsilon}\right) \subset B_{\varepsilon}$. By hypotheses (H_{11}) and (H_{02}), for any $x \in C_{1-\gamma,\varrho}[a,T]$ and $\|x\|_{C_{1-\gamma,\varrho}}$ we have

$$\begin{aligned}
&\left|(Nx)(t)\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}\right| \\
&\leq \left(\frac{L(1+h_T(T-a))\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}\right. \\
&\left.+\frac{L(1+h_T(T-a))\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}\right)\|x\|_{C_{1-\gamma,\varrho}} \\
&+\frac{M}{\Gamma(\alpha+1)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}+\frac{M}{\Gamma(\alpha+1)}\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha-\gamma+1}.
\end{aligned}$$

This is $\|Nx\|_{C_{1-\gamma,\varrho}} \leq \theta\varepsilon+\Omega \leq \varepsilon$, which gives $N\left(B_{\varepsilon}\right) \subset B_{\varepsilon}$.

Next we shall prove that N is completely continuous.

Step 2. N is continuous. Let x_n be a sequence such that $x_n \to x$ in B_{ε}. Then for each $t \in (a,T]$, we have

$$\begin{aligned}
&\left|\left((Nx)\left(x_n\right)-(Nx)(t)\right)\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\right| \\
\leq&\frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\left|f\left(s,x_n(s),\int_a^s h(s,\tau)x_n(\tau)d\tau\right)\right. \\
&\left.-f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\right|\mathrm{d}s \\
&+\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \\
&\times\left|f\left(s,x_n(s),\int_a^s h(s,\tau)x_n(\tau)d\tau\right)-f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\right|\mathrm{d}s \\
\leq&\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(|K|\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}+\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}\right) \\
&\times\left\|f\left(\cdot,x_n(\cdot),\int_a^s h(s,\tau)x_n(\cdot)d\tau\right)-f\left(\cdot,x(\cdot),\int_a^s h(s,\tau)x(\cdot)d\tau\right)\right\|_{C_{1-\gamma,\varrho}},
\end{aligned}$$

this implies

$$\begin{aligned}
\left\|Nx_n-Nx\right\|_{C_{1-\gamma,\varrho}} \leq& \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(|K|\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}+\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}\right) \\
&\times\left\|f\left(\cdot,x_n(\cdot),\int_a^s h(s,\tau)x_n(\tau)d\tau\right)-f\left(\cdot,x(\cdot),\int_a^s h(s,\tau)x(\cdot)d\tau\right)\right\|_{C_{1-\gamma,\alpha}}.
\end{aligned}$$

Thus, N is a continuous operator.

Step 3. Finally, we prove that $N(B_\varepsilon)$ is relatively compact.
Since $N(B_\varepsilon) \subset B_\varepsilon$, it follows that $N(B_\varepsilon)$ is uniformly bounded.

By repeating the same process as in Step 3 in Theorem 3.1 , one can easily prove that N is equicontinuous on B_ε.

As $\alpha \leq \gamma < 1$ and noting (3.7), for any $0 < a < t_1 < t_2 \leq T$ one has

$$\begin{aligned}
&|(Nx)(t_1) - (Nx)(t_2)| \\
&\leq \frac{\|f\|_{C_{1-\gamma,\varrho}}|K|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \\
&\times \left(\left(\frac{t_1^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} - \left(\frac{t_2^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1}\right) + |(Qx)(t_1) - (Qx)(t_2)| \\
&\leq \frac{\|f\|_{C_{1-\gamma,\varrho}}|K|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \left|\frac{t_2^\varrho - t_1^\varrho}{(t_1^\varrho - a^\varrho)(t_2^\varrho - a^\varrho)}\right|^{1-\gamma} \\
&+ \frac{\|f\|_{C_{1-\gamma,\varrho}}\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \left|\left(\frac{t_1^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} - \left(\frac{t_2^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1}\right| \to 0,
\end{aligned}$$

as $t_2 \to t_1$. Thus, Q is equicontinuous.

Hence, $N(B_\varepsilon)$ is an equicontinuous set and therefore $N(B_\varepsilon)$ is relatively compact. As a consequence of Steps 1 to 3 together with Arzelà-Ascoli theorem, we can conclude that $N : B_\varepsilon \to B_\varepsilon$ is completely continuous. By applying Schauder fixed point theorem, we complete the proof. □

4. Example

In this section, we will show the applications of our main results with two examples.

Example 4.1. Consider the nonlocal problem

$$\left({}^\varrho D_{a+}^{\alpha,\beta}\right)x(t) = f(t, x(t), Hx(t)), \ t \in (1, 2], \tag{4.1}$$

$$\left({}^\varrho I_{a+}^{1-\gamma} x\right)(1+) = 2x\left(\frac{5}{3}\right), \quad \gamma = \alpha + \beta(1-\alpha). \tag{4.2}$$

Denoting $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$ gives $\gamma = \frac{7}{8}$. Let $\varrho = \frac{1}{2} > 0$ and set

$$f(t, x(t), Hx(t)) = \left(\frac{t^\varrho - 1}{\varrho}\right)^{-1/16} + \frac{1}{4}\left(\frac{t^\varrho - 1}{\varrho}\right)^{15/16} \sin x(t) + \frac{1}{4}Hx(t),$$

where

$$Hx(t) = \int_1^t \frac{1}{(3+t)^2} x(s) \mathrm{d}s.$$

We can see that

$$\left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{1/8} f(t,x(t),Hx(t)) = \left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{1/16} + \frac{1}{4}\left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{17/16} \sin x(t) + \frac{1}{4}\left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{1/8} Hx(t) \in C[1,2] \tag{4.3}$$

i.e. $f(t,x,Hx(t)) \in C_{1/8,1/2}[1,2]$.
Moreover,

$$|f(t,x,Hx(t)) - f(t,\bar{x},H\bar{x}(t))| \leq \frac{1}{4}\Big(|x-\bar{x}| + |Hx(t) - H\bar{x}(t)|\Big).$$

So, we have $L = \frac{1}{4}$, $h_T = \frac{1}{16}$.
Some elementary computations gives us

$$|K| = \left|\left(\Gamma(0.875) - 2\left(\frac{\left(\frac{5}{3}\right)^{1/2}-1}{\frac{1}{2}}\right)^{-1/8}\right)^{-1}\right| \approx 0.9521 < 1$$

and

$$\begin{aligned}\theta &= \frac{\Gamma(0.875)\frac{1}{4}(1+\frac{1}{16}(2-1))}{4\Gamma(1.625)} \\ &\times \left(|K| \times 2\left(\frac{\left(\frac{5}{3}\right)^{1/2}-1}{\frac{1}{2}}\right)^{5/8} + \left(\frac{2^{1/2}-1}{\frac{1}{2}}\right)^{3/4}\right) \\ &\approx 0.17964219 < 1.\end{aligned}$$

All the assumptions of Theorem 3.1 are satisfied with

$$|K| \approx 0.9521 \text{ and } \theta \approx 0.17964219.$$

Therefore, problem (4.1)-(4.2) has at least one solution in $C_{1/8,1/2}[1,2]$.

Example 4.2. Consider the nonlocal problem

$$\left({}^{\varrho}D^{\alpha,\beta}_{a+}x\right)(t) = f(t,x(t),Hx(t)),\ t \in (1,2], \tag{4.4}$$

$$\left({}^{\varrho}I^{1-\gamma}_{a+}x\right)(1+) = 3x\left(\frac{8}{7}\right) + 2x\left(\frac{4}{3}\right). \tag{4.5}$$

Denote $\alpha = \frac{1}{2}$, $\beta = \frac{3}{4}$ and $\varrho = \frac{1}{2} > 0$. So $\gamma = \frac{7}{6}$ and $\left(\xi_1 = \frac{8}{7}\right) \leq \left(\xi_2 = \frac{4}{3}\right)$. Set

$$f(t,x(t),Hx(t)) = \sin\left(\frac{1}{3}|x(t)|\right) + \frac{1}{3}Hx(t),\ t \in (1,2],$$

where

$$Hx(t) = \int_1^t \frac{1}{(3+t)^2}x(s)\mathrm{d}s.$$

It is easy to see that $f(t,x(t),Hx(t)) \in C_{1/8,1/2}[1,2]$ and

$$|f(t,x,Hx(t))| \leq \frac{1}{3}\Big(|x| + |Hx(t)|\Big).$$

So, we have $L = \frac{1}{3}$, $M = 0$, $h_T = \frac{1}{16}$. Moreover,

$$|K| = \left|\left(\Gamma(0.875) - \left(3\left(\frac{\left(\frac{8}{7}\right)^{1/2}-1}{\frac{1}{2}}\right)^{-1/8} + 2\left(\frac{\left(\frac{4}{3}\right)^{1/2}-1}{\frac{1}{2}}\right)^{-1/8}\right)\right)^{-1}\right| \approx 0.1973 < 1$$

and

$$\theta = \frac{\Gamma(0.875)\frac{1}{3}(1+\frac{1}{16}(2-1))}{3\Gamma(1.375)} \times \left(|K| \times 3\left(\frac{\left(\frac{8}{7}\right)^{1/2}-1}{\frac{1}{2}}\right)^{3/8} + 2\left(\frac{\left(\frac{4}{3}\right)^{1/2}-1}{\frac{1}{2}}\right)^{3/8}\right) \approx 0.2515 < 1$$

With the values of $|K|$ and θ, problem (4.4)-(4.5) satisfies all the conditions of Theorem 3.2. Thus, problem (4.4)-(4.5) has at least one solution in $C_{1/8,1/2}[1,2]$.

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