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Expansion-compression fixed point theorem of Leggett-Williams type for the sum of two operators and applications for some classes of BVPs

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Abstract. The purpose of this work is to establish an extension of a Leggett-Williams type expansion-compression fixed point theorem for a sum of two operators. As illustration, our approach is applied to prove the existence of non trivial nonnegative solutions for two-point BVPs and three-point BVPs.

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1. Introduction

For applicability reasons, we often search for existence and localization of positive fixed points which may represent positive solutions for various nonlinear problems posed in a Banach space.

One of the main results in fixed point theory is the cone expansion and compression theorem proved by Krasnosel'skii in 1964 (see, e.g., [8, 14, 15]). It represents a powerful existence tool in studying operator equations and showing existence of nonnegative solutions to various boundary value problems. Then, many researchers have been intersted in the extension of the above theorem in various directions (see, e.g., [1, 6, 7, 9, 16, 18, 19]).

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Throughout this paper, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$. Let Ψ and δ be nonnegative continuous functionals on $\mathcal{P}$; then, for positive real numbers a and b, we define the sets:

$$\mathcal{P}(\Psi, b) = \{x \in \mathcal{P} : \Psi(x) \leq b\},$$

and

$$\mathcal{P}(\Psi, \delta, a, b) = \{x \in \mathcal{P} : a \leq \Psi(x) \text{ and } \delta(x) \leq b\}.$$

Krasnosel'skii type compression-expansion fixed point theorems gives us fixed points localized in a conical shell of the form $\{x \in \mathcal{P} : a \leq \|x\| \leq b\}$, where $a, b \in (0, \infty)$, while with the Leggett-Williams type they are localized in a conical shell of the form $\mathcal{P}(\alpha, \beta, a, b)$, where α is a concave nonnegative functional and β a convex nonnegative functional.

The original Leggett-Williams fixed point theorem (see [17, Theorem 3.2]) discuss the existence of at least one fixed point in a conical shell of the form $\{x \in \mathcal{P} : a \leq \alpha(x) \text{ and } \|x\| \leq b\}$, where $a, b \in (0, +\infty)$. Noting that this result has been widely extended in many directions, (see, e.g., [2, 3, 10, 11, 17]). In [2, Theorem 4.1], Anderson et al. have discussed the existence of at least one solution in $\mathcal{P}(\beta, \alpha, r, R)$ or in $\mathcal{P}(\alpha, \beta, r, R)$ for the nonlinear operational equation

$$Ax = x \tag{1.1}$$

where A is a completely continuous nonlinear map acting in $\mathcal{P}$, α is a nonnegative continuous concave functional on $\mathcal{P}$ and β is a nonnegative continuous convex functional on $\mathcal{P}$. In this result, the authors have used techniques similar to those of Leggett-Williams that require only subsets of both boundaries to be mapped inward and outward, respectively. They thus provide more general results than those obtained by using the Krasnosel'skii's cone compression and expansion one. Noting that, in [2], the authors provided more general results than those obtained in [1, 4, 11, 12, 17, 19] for completely continuous mappings.

In this paper, we use the fixed point index theory developed in [6] to generalize the main result of [2] for the sum $T+F$ where T is an expansive mapping with constant $h > 1$ and $I - F$ is a k-set contraction with $k < h$. The concept of set contraction is related to that of the Kuratowski measure of noncompactness (see [5, 13]).

The paper is organized as follows. In Section 2 we give some auxiliary results used for the proof of the main result. In Section 3, we present our main result. As application, the existence of non trivial nonnegative solution for two-point BVPs and three-point BVPs are considered in Section 4.

2. Auxiliary results

Let Ω be any subset of $\mathcal{P}$, and U be a bounded open subset of $\mathcal{P}$.
Consider $T : \Omega \to E$ an expansive mapping with constant $h > 1$, and $I - F : \overline{U} \to E$ a k-set contraction with $0 \leq k < h$. So, the operator T^{-1} is $\frac{1}{h}$-Lipschtzian on $T(\Omega)$. Assume that

$$(I - F)\left(\overline{U}\right) \subset T(\Omega).$$

Then the mapping $T^{-1}(I-F):\overline{U}\to\mathcal{P}$ is a strict $\frac{k}{h}$-set contraction.
Hence, by Djebali et al. in [6], the fixed point index of the sum $T+F$ on $U\cap\Omega$ with respect to the cone $\mathcal{P}$, noted $i_*\left(T+F,U\cap\Omega,\mathcal{P}\right)$, is well defined.

The proof of our theorical result invokes the following main properties of the fixed point index i_*.

(i). *(Normalization)* If $U=\mathcal{P}(\Psi,R),\ 0\in\Omega$, and $(I-F)x=z_0$ for all $x\in\overline{U}$, where $z_0\in\mathcal{P}$, Ψ is a nonnegative continuous functionals on $\mathcal{P}$ satisfying $\Psi(x)\leq\|x\|$ for all $x\in\mathcal{P}$ and $\|z_0-T0\|<hR$, then

$$i_*\left(T+F,U\cap\Omega,\mathcal{P}\right)=1.$$

(ii). *(Additivity)* For any pair of disjoint open subsets $U_1,U_2\subset U$ such that $T+F$ has no fixed point on $(\overline{U}\setminus(U_1\cup U_2))\cap\Omega$, we have

$$i_*(T+F,U\cap\Omega,\mathcal{P})=i_*(T+F,U_1\cap\Omega,\mathcal{P})+i_*(T+F,U_2\cap\Omega,\mathcal{P}).$$

(iii). *(Homotopy invariance)* The fixed point index $i_*(T+H(.,t),U\cap\Omega,\mathcal{P})$ does not depend on the parameter $t\in[0,1]$, where

- **(a).** $(I-H):[0,1]\times\overline{U}\to E$ is continuous and $H(t,x)$ is uniformly continuous in t with respect to $x\in\overline{U}$,
- **(b).** $(I-H)([0,1]\times\overline{U})\subset T(\Omega)$,
- **(c).** $(I-H(t,.)):\overline{U}\to E$ is a ℓ-set contraction with $0\leq\ell<h$ for all $t\in[0,1]$,
- **(d).** $Tx+H(t,x)\neq x$ for all $t\in[0,1]$ and $x\in\partial U\cap\Omega$.

(iv). *(Solvability)* If $i_*(T+F,U\cap\Omega,\mathcal{P})\neq 0$, then $T+F$ has a fixed point in $U\cap\Omega$.

For proof and more details see [6, Theorem 3.1].

3. Main result

Let Ω be a subset of $\mathcal{P}$ such that $0\in\Omega$. We consider the nonlinear equation

$$Tx+Fx=x, \tag{3.1}$$

where $T:\Omega\to E$ an expansive mapping with constant $h>1$, and $I-F:\mathcal{P}\to E$ a k-set contraction with $0\leq k<h$.

In what follows, we will establish an extension of [2, Theorem 4.1], which guarantees the existence of at least one non trivial nonnegative solution of equation (3.1).

Theorem 3.1. *Let α be a nonnegative continuous concave functional on $\mathcal{P}$ and β be a nonnegative continuous convex functional on $\mathcal{P}$ with $\beta(x)\leq\|x\|$ for all $x\in\mathcal{P}$. Assume that there exists nonnegative numbers a, b, c, d and $z_0\in\mathcal{P}$ such that $\|T0\|<h\min(b,d)$ and $\alpha(T^{-1}z_0)>\max(a,c)$.*
Suppose that:

- **(A1).** *if $x\in\mathcal{P}$ with $\beta(x)=b$, then $\alpha(Tx+x)\geq a$;*
- **(A2).** *if $x\in\mathcal{P}$ with $\beta(x)=b$ and $\alpha(x)\geq a$, then $\beta(Tx+Fx)<b$ and $\beta(Tx+x)\leq b$;*
- **(A3).** *if $x\in\mathcal{P}$ with $\beta(x)=b$ and $\alpha(Tx+Fx)<a$, then $\beta(Tx+Fx)<b$ and $\beta(Tx+x)\leq b$;*
- **(A4).** *if $x\in\mathcal{P}$ with $\alpha(x)=c$, then $\beta(Tx+x-z_0)\leq d$;*

(A5). *if $x \in \mathcal{P}$ with $\alpha(x) = c$ and $\beta(x) \leq d$, then $\alpha(Tx + Fx) > c$ and $\alpha(Tx + x - z_0) \geq c$;*

(A6). *if $x \in \mathcal{P}$ with $\alpha(x) = c$ and $\beta(Tx + Fx) > d$, then $\alpha(Tx + Fx) > c$ and $\alpha(Tx + x - z_0) \geq c$.*

Then,

1. *(Expansive form) $T + F$ has a fixed point x^* in $\mathcal{P}(\beta, \alpha, b, c) \cap \Omega$ if*

   (H1). *$a < c$, $b < d$, $\{x \in \mathcal{P} : b < \beta(x)$ and $\alpha(x) < c\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$, $\mathcal{P}(\beta, b) \cap \Omega \neq \emptyset$ and $\mathcal{P}(\alpha, c)$ is bounded and*

$$t(I - F)(\mathcal{P}(\beta, b)) \subset T(\Omega), \text{ for all } t \in [0, 1], \tag{3.2}$$

$$t(I - F)(\mathcal{P}(\alpha, c)) + (1 - t)z_0 \subset T(\Omega), \text{ for all } t \in [0, 1]. \tag{3.3}$$

2. *(Compressive form) $T + F$ has a fixed point x^* in $\mathcal{P}(\alpha, \beta, a, d) \cap \Omega$ if*

   (H2). *$c < a$, $d < b$, $\{x \in \mathcal{P} : a < \alpha(x)$ and $\beta(x) < d\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\alpha, a) \subset \mathcal{P}(\beta, d)$, $\mathcal{P}(\alpha, a) \cap \Omega \neq \emptyset$, and $\mathcal{P}(\beta, d)$ is bounded and*

$$t(I - F)(\mathcal{P}(\beta, d)) \subset T(\Omega), \text{ for all } t \in [0, 1], \tag{3.4}$$

$$t(I - F)(\mathcal{P}(\alpha, a)) + (1 - t)z_0 \subset T(\Omega), \text{ for all } t \in [0, 1]. \tag{3.5}$$

Proof. We will prove the expansion form. The proof of the compression form is nearly identical.
If we list

$$U = \{x \in \mathcal{P} : \beta(x) < b\}, \tag{3.6}$$

$$V = \{x \in \mathcal{P} : \alpha(x) < c\}, \tag{3.7}$$

then, the interior of $V - U$ is given by

$$W = (V - U)^o = \{x \in \mathcal{P} : \; b < \beta(x) \text{ and } \alpha(x) < c\}.$$

Thus U, V and W are bounded (they are subsets of V which is bounded by condition $(H1)$), not empty (by condition $(H1)$) and open subsets of $\mathcal{P}$. To prove the existence of a fixed point for the sum $T + F$ in $\mathcal{P}(\beta, \alpha, b, c) \cap \Omega$, it is enough for us to show that $i_*(T + F, W \cap \Omega, \mathcal{P}) \neq 0$ since W is the interior of $\mathcal{P}(\beta, \alpha, b, c)$.

Claim 1. $Tx + Fx \neq x$ for all $x \in \partial U \cap \Omega$.
Let $x_0 \in \partial U \cap \Omega$, then $\beta(x_0) = b$. Suppose that $x_0 = Tx_0 + Fx_0$, then $\beta(Tx_0 + Fx_0) = b$. If $\alpha(x_0) \geq a$, then $\beta(Tx_0 + Fx_0) < b$ by condition $(A2)$, and if $\alpha(x_0) < a$, then $\alpha(Tx_0 + Fx_0) < a$, then $\beta(Tx_0 + Fx_0) < b$ by condition $(A3)$. This is a contradiction. Thus we have $Tx + Fx \neq x$ for all $x \in \partial U \cap \Omega$.

Claim 2. $Tx + Fx \neq x$ for all $x \in \partial V \cap \Omega$.
Let $x_1 \in \partial V \cap \Omega$, then $\alpha(x_1) = c$. Suppose that $x_1 = Tx_1 + Fx_1$, then $\alpha(Tx_1 + Fx_1) = c$. If $\beta(x_1) \leq d$, then $\alpha(Tx_1 + Fx_1) > c$ by condition $(A5)$, and if $\beta(x_1) > d$, then $\beta(Tx_1 + Fx_1) > d$, then $\alpha(Tx_1 + Fx_1) > c$ by condition $(A6)$.
This is a contradiction. Thus we have $Tx + Fx \neq x$ for all $x \in \partial V \cap \Omega$.

Claim 3. Let $H_1 : [0, 1] \times \overline{U} \to E$ be defined by

$$H_1(t, x) = tFx + (1 - t)\, x.$$

Clearly H_1 is uniformly continuous in t with respect to $x \in \overline{U}$ and $(I - H_1)$ is continuous, and from (3.2) we easily see that $(I - H_1([0,1] \times \overline{U})) \subset T(\Omega)$. Moreover $(I - H_1(t,.)) : \overline{U} \to E$ is a k-set contraction for all $t \in [0,1]$ and $Tx + H_1(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U \cap \Omega$. Otherwise, there would exists $(t_2, x_2) \in [0,1] \times \partial U \cap \Omega$ such that $Tx_2 + H_1(t_2, x_2) = x_2$. Since $x_2 \in \partial U$, $\beta(x_2) = b$. Either $\alpha(Tx_2 + Fx_2) < a$ or $\alpha(Tx_2 + Fx_2) \geq a$.

Case (1): If $\alpha(Tx_2 + Fx_2) < a$, the convexity of β and the condition $(A3)$ lead

$$\begin{array}{rcl} b = \beta(x_2) & = & \beta\left(Tx_2 + H_1(t_2, x_2)\right) \\ & = & \beta\left(Tx_2 + t_2 Fx_2 + (1-t_2)x_2\right) \\ & \leq & t_2 \beta\left(Tx_2 + Fx_2\right) + (1-t_2)\,\beta(Tx_2 + x_2) \\ & < & b, \end{array}$$

which is a contradiction.
Case (2): If $\alpha(Tx_2 + Fx_2) \geq a$, from the concavity of α and the condition $(A1)$, we obtain $\alpha(x_2) \geq a$. Indeed,

$$\begin{array}{rcl} \alpha(x_2) & = & \alpha\left(Tx_2 + H_1(t_2, x_2)\right) \\ & \geq & t_2 \alpha\left(Tx_2 + Fx_2\right) + (1-t_2)\,\alpha(Tx_2 + x_2) \\ & \geq & a, \end{array}$$

and thus by condition $(A2)$, we have $\beta(Tx_2 + Fx_2) < b$ and $\beta(Tx_2 + x_2) < b$, which is the same contradiction we arrived at in the previous case.
Being $T^{-1}0 \in U$ (we have $h\beta(T^{-1}0) \leq h\|T^{-1}0\| \leq \|T0\| < hb$), the homotopy invariance property (iii) and the normality property (i) of the fixed point index i_* lead

$$i_*(T + F, U \cap \Omega, \mathcal{P}) = i_*(T + I, U \cap \Omega, \mathcal{P}) = 1.$$

Claim 4. Let $H_2 : [0,1] \times \overline{V} \to E$ be defined by

$$H_2(t,x) = tFx + (1-t)(x - z_0).$$

Clearly H_2 is uniformly continuous in t with respect to $x \in \overline{V}$ and $(I - H_2)$ is continuous, and from (3.3) we easily see that $(I - H_2([0,1] \times \overline{V})) \subset T(\Omega)$. Moreover $I - H_2(t,.) : \overline{V} \to E$ is a k-set contraction for all $t \in [0,1]$ and $Tx + H_2(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial V \cap \Omega$. Otherwise, there would exists $(t_3, x_3) \in [0,1] \times \partial V \cap \Omega$ such that $H_2(t_3, x_3) = x_3$. Since $x_3 \in \partial V$ we have that $\alpha(x_3) = c$. Either $\beta(Tx_3 + Fx_3) \leq d$ or $\beta(Tx_3 + Fx_3) > d$.

Case (1): If $\beta(Tx_3 + Fx_3) > d$. the concavity of α and the condition $(A6)$ lead

$$\begin{array}{rcl} c = \alpha(x_3) & = & \alpha(Tx_3 + H_2(t_3, x_3)) \\ & = & \alpha(Tx_3 + t_3 Fx_3 + (1-t_3)(x_3 - z_0)) \\ & \geq & t_3 \alpha(Tx_3 + Fx_3) + t_3 \alpha(Tx_3 + x_3 - z_0) \\ & > & c. \end{array}$$

This is a contradiction.

Case (2): If $\beta(Tx_3 + Fx_3) \le d$, from the convexity of β and the condition $(A4)$, we obtain $\beta(x_3) \le d$. Indeed,

$$\begin{aligned} \beta(x_3) &= \beta(Tx_3 + H_2(t_3, x_3)) \\ &\le t_3\beta(Tx_3 + Fx_3) + (1 - t_3)\beta(Tx_3 + x_3 - z_0) \\ &\le d, \end{aligned}$$

and thus by condition $(A5)$, we have $\alpha(Tx_3+Fx_3) > c$, which is the same contradiction we arrived at in the previous case.
The homotopy invariance property (iii) of the fixed index i_* yields

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + I - z_0, V \cap \Omega, \mathcal{P}),$$

and by the solvability property (iv) of the index i_* (since $T^{-1}z_0 \notin V$ the index cannot be nonzero) we have

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + I - z_0, V \cap \Omega, \mathcal{P}) = 0.$$

Since U and W are disjoint open subsets of V and $T + F$ has no fixed points in $\overline{V} - (U \cup W)$ (by claims 1 and 2), from the additivity property (ii) of the index i_*, we deduce

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + F, U \cap \Omega, \mathcal{P}) + i_*(T + F, W \cap \Omega, \mathcal{P}).$$

Consequently, we have

$$i(T + F, W \cap \Omega, \mathcal{P}) = -1,$$

and thus by the solvability property (iv) of the fixed point index i_*, the sum $T + F$ has a fixed point $x^* \in W \subset \mathcal{P}(\beta, \alpha, b, c) \cap \Omega$. □

4. Applications

In this section we will apply our main result Theorem 3.1 for two-point BVPs and for three-point BVPs and will show that, using Theorem 3.1, some well-known results can be enriched.

4.1. A Three-Point BVP

In this subsection, we will investigate the three-point BVP

$$\begin{aligned} &y'' + f(t, y) = 0, \quad t \in (0, 1), \\ &y(0) = ky(\eta), \quad y(1) = 0, \end{aligned} \tag{4.1}$$

where

(B1). $f \in \mathcal{C}([0, 1] \times \mathbb{R}^+)$, $0 < \widetilde{A} \le f(t, u) \le A$, $t \in [0, 1]$, $u \in [0, \infty)$, for some positive constants $A \ge \widetilde{A}$.

(B2). $\eta \in (0, 1)$, $k > 0$, $k(1 - \eta) < 1$, $B = \frac{1+k\eta}{1-k(1-\eta)}$, $\epsilon \in (1, 2)$, $c = 0$ and there exist $a, b, d, z_0 > 0$ so that $z_0 = a$ and

$$a < d < b, \quad 2z_0 < \epsilon d, \quad (\epsilon - 1)b + 2z_0 < \tfrac{d}{2},$$

$$(\epsilon - 1)b + \epsilon AB < d, \quad a < \tfrac{\epsilon AB+2z_0}{\epsilon} \le d.$$

After the proof of the main result in this subsection, we will give an example for a function f and constants A, $\widetilde{A}$, B, η, k, a, b, d, ϵ, z_0 which satisfy $(B1)$ and $(B2)$. We will investigate the BVP (4.1) for existence of at least one non trivial nonnegative solution. Our main result is as follows.

Theorem 4.1. *Suppose* $(B1)$ *and* $(B2)$. *Then the BVP* (4.1) *has at least one non trivial nonnegative solution* $y \in \mathcal{C}^2([0,1])$.

To prove our main result, we will use Theorem 3.1.

In [20] the BVP (4.1) is investigated when the function f satisfies the following conditions

(B3). $f(t,u)$ is nonnegative and continuous on $(0,1)\times[0,\infty)$, $f(t,u)$ is monotone increasing on u for fixed $t \in (0,1)$, there exists $q \in (0,1)$ such that

$$f(t,ru) > r^q f(t,u), \quad 0<r<1, \quad (t,u)\in(0,1)\times[0,\infty),$$

and in [20] it is proved that the BVP (4.1) has a unique solution $u \in \mathcal{C}([0,1])\bigcap\mathcal{C}^2((0,1))$. We will note that there are cases for the function f for which we can apply Theorem 4.1 and we can not apply Theorem 4.1 in [20] and conversely. For example, if $f(t,u) = 1+\frac{1}{1+u}$, $t,u\in[0,\infty)$, then it is bounded below and above and we can apply Theorem 4.1. At the same time, it is decreasing with respect to u for $t,u\in[0,\infty)$ and we can not apply Theorem 4.1 in [20]. If $f(t,u)=\sum\limits_{j=1}^{m} a_j(t)u^{\alpha_j}$, where $a_j\in\mathcal{C}([0,\infty))$ are nonnegative functions and $\alpha_j\in(0,1)$, $j\in\{1,\dots,m\}$, as it is shown in [20], it satisfies $(B3)$. On the other hand, it is unbounded above and we can not apply Theorem 4.1. Thus, our result Theorem 4.1 and Theorem 4.1 in [20] are complementary.

4.1.1. Proof of Theorem 4.1. Set

$$H(t,s)=\begin{cases} s(1-t), & 0\le s\le t\le 1,\\ \\ t(1-s), & 0\le t\le s\le 1,\end{cases}$$

and

$$G(t,s)=H(t,s)+\frac{k(1-t)}{1-k(1-\eta)}H(\eta,s), \quad t,s\in[0,1].$$

Note that $0\le H(t,s)\le 1$, $t,s\in[0,1]$. Hence,

$$0\le G(t,s)\le 1+\frac{k}{1-k(1-\eta)}=\frac{1-k+k\eta+k}{1-k(1-\eta)}=\frac{1+k\eta}{1-k(1-\eta)}=B,$$

$t,s\in[0,1]$. Moreover, for $t,s\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]$, we have

$$H(t,s)\ge\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)$$

and

$$G(t,s)\ge H(t,s)\ge\frac{\eta}{3}\left(1-\frac{\eta}{2}\right).$$

Next,

$$H_t(t,s) = \begin{cases} -s, & 0 \le s \le t \le 1, \\ 1-s, & 0 \le t \le s \le 1. \end{cases}$$

Hence, $|H_t(t,s)| \le 1$, $t, s \in [0,1]$, and

$$\begin{aligned} |G_t(t,s)| &= \left| H_t(t,s) - \frac{k}{1-k(1-\eta)} H(\eta,s) \right| \\ &\le |H_t(t,s)| + \frac{k}{1-k(1-\eta)} H(\eta,s) \\ &\le 1 + \frac{k}{1-k(1-\eta)} = \frac{1+k\eta}{1-k(1-\eta)} = B, \quad t,s \in [0,1]. \end{aligned}$$

Let $E = \mathcal{C}([0,1])$ be endowed with the maximum norm

$$\|y\| = \max_{t\in[0,1]} |y(t)|.$$

On E, define

$$\alpha(y) = \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} |y(t)| + z_0, \quad \beta(y) = \max_{t\in[0,1]} |y(t)|.$$

In [20] it is proved that the solution of the BVP (4.1) can be expressed in the following form

$$y(t) = \int_0^1 G(t,s) f(s,y(s)) ds, \quad t \in [0,1].$$

Set

$$k_1 = \frac{\min\left\{\epsilon \frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right\}}{d\epsilon}.$$

Define

$$\begin{aligned} \mathcal{P} &= \{y \in E : y(t) \ge 0, \quad t \in [0,1], \quad \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) \ge k_1 \max_{t\in[0,1]} y(t)\}, \\ \Omega &= \{y \in \mathcal{P} : \|y\| \le \frac{2z_0 + \epsilon AB}{\epsilon}\}. \end{aligned}$$

Note that $0 \in \Omega$ and $\Omega \subset \mathcal{P}$. For $y \in \mathcal{P}$, define the operators

$$\begin{aligned} Ty(t) &= -\epsilon y(t) + 2z_0, \\ Fy(t) &= y(t) - 2z_0 + \epsilon \int_0^1 G(t,s) f(s,y(s)) ds, \quad t \in [0,1]. \end{aligned}$$

Note that if $y \in \mathcal{P}$ is a fixed point of the operator $T+F$, then it is a solution to the BVP (4.1). Next, if $y \in \mathcal{P}$ and $\beta(y) \le b$, we have

$$\begin{aligned} |Ty(t)+y(t)| &\le (\epsilon-1)y(t)+2z_0 \\ &\le (\epsilon-1)b+2z_0 \\ &< \frac{d}{2}, \quad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} |Ty(t)+Fy(t)| &= \left|-(\epsilon-1)y(t)+\epsilon\int_0^1 G(t,s)f(s,y(s))ds\right| \\ &\le (\epsilon-1)y(t)+\epsilon\int_0^1 G(t,s)f(s,y(s))ds \\ &\le (\epsilon-1)b+\epsilon A\int_0^1 G(t,s)ds \\ &\le (\epsilon-1)b+\epsilon AB \\ &< d. \end{aligned}$$

Therefore, if $y \in \mathcal{P}$ and $\beta(y) \le b$, we have

$$\beta(Ty+y) < d \tag{4.2}$$

and

$$\beta(Ty+Fy) < d. \tag{4.3}$$

For $y, z \in \mathcal{P}$, we have

$$|Ty(t)-Tz(t)| = \epsilon|y(t)-z(t)|, \quad t \in [0,1].$$

Hence,

$$\|Ty-Tz\| = \epsilon\|y-z\|.$$

Thus, $T : \mathcal{P} \to E$ is an expansive operator with constant $h=\epsilon$.
Let now, $y \in \mathcal{P}$. Then

$$\begin{aligned} |(I-F)y(t))| &= \epsilon\left|\int_0^1 G(t,s)f(s,y(s))ds\right| \\ &\le \epsilon A\int_0^1 G(t,s)ds \\ &\le \epsilon AB, \quad t \in [0,1], \end{aligned}$$

whereupon

$$\|(I-F)y\| \le \epsilon AB$$

and $I-F:\mathcal{P}\to E$ is uniformly bounded. Moreover,

$$\begin{aligned}\left|\frac{d}{dt}(I-F)y(t)\right| &= \left|\int_0^1 G_t(t,s)f(s,y(s))ds\right|\\ &\le \int_0^1 |G_t(t,s)|f(s,y(s))ds\\ &\le AB, \quad t\in[0,1].\end{aligned}$$

Consequently, $I-F:\mathcal{P}\to E$ is completely continuous. Then $I-F:\mathcal{P}\to E$ is a 0-set contraction.
Note that

$$\|T0\| = 2z_0 < \epsilon\min\{b,d\}.$$

For $y\in E$, we have

$$T^{-1}y = -\frac{y-2z_0}{\epsilon}.$$

Hence,

$$\alpha\left(T^{-1}z_0\right) = \alpha\left(\frac{z_0}{\epsilon}\right) = \frac{z_0}{\epsilon}+z_0 > \max\{a,c\}.$$

Suppose that $y\in\mathcal{P}$ with $\beta(y)=b$. Then

$$\alpha(Ty+y) = \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} |Ty(t)+y(t)| + z_0 \ge z_0 = a.$$

Consequently $(A1)$ holds.
Now, we take $y\in\mathcal{P}$ with $\beta(y)=b$, $\alpha(y)\ge a$. Then, using $d<b$, (4.2) and (4.3), we obtain

$$\beta(Ty+y) < b \quad \text{and} \quad \beta(Ty+Fy) < b.$$

Consequently $(A2)$ holds.
Observe that, if $y\in\mathcal{P}$, $\beta(y)=b$ and $\alpha(Ty+Fy)<a$, using $d<b$ and (4.2), (4.3), we find

$$\beta(Ty+Fy) < b \quad \text{and} \quad \beta(Ty+y) < b.$$

Thus, $(A3)$ holds.
Since $c=0$ and $\alpha(y)>0$ for any $y\in\mathcal{P}$, the case $\alpha(y)=c$ is impossible.
Let now, $a_1\in\left(a,\frac{\epsilon AB+z_0}{\epsilon}\right)$ be arbitrarily chosen. Then

$$\alpha(a_1) = a_1 + z_0 > a$$

and

$$\beta(a_1) = a_1 < \frac{\epsilon AB + 2z_0}{\epsilon} \le d.$$

Therefore

$$\{y\in\mathcal{P} : a<\alpha(y) \quad \text{and} \quad \beta(y)<d\}\cap\Omega \ne \emptyset.$$

Let $y\in\mathcal{P}(\alpha,a)$. Then $y\in\mathcal{P}$ and $\alpha(y)\le a$. Hence,

$$a \ge \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) + z_0 = \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) + a.$$

Therefore $\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) = 0$ and using the definition of the cone $\mathcal{P}$, we find

$$\beta(y) = \max_{t\in[0,1]} y(t) \le \frac{1}{k_1} \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) = 0 \le d.$$

Thus, $y \in \mathcal{P}(\beta, d)$ and $\mathcal{P}(\alpha, a) \subset \mathcal{P}(\beta, d)$.
Since $0 \in \mathcal{P}(\alpha, a)$, we have $\mathcal{P}(\alpha, a) \cap \Omega \neq \emptyset$.
Note that $\mathcal{P}(\beta, d)$ is bounded.
Let $\lambda \in [0, 1]$ is fixed and $u \in \mathcal{P}(\alpha, a)$ is arbitrarily chosen. Then $\beta(u) \le d < b$. Set

$$v(t) = \frac{\lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds + (1-\lambda)z_0}{\epsilon}, \quad t \in [0,1].$$

We have that $v(t) \ge 0$, $t \in [0, 1]$, and

$$v(t) \le \frac{\epsilon AB + z_0}{\epsilon} \le d, \quad t \in [0,1],$$

and

$$\|v\| \le \frac{\epsilon AB + z_0}{\epsilon} \le d.$$

$$\begin{aligned}
\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} v(t) \quad &\ge \quad \frac{\lambda\epsilon \int_{\frac{\eta}{3}}^{\frac{\eta}{2}} \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} G(t,s)f(s,u(s))ds + (1-\lambda)z_0}{\epsilon} \\
&\ge \quad \frac{\lambda\epsilon\left(\frac{\eta}{2}-\frac{\eta}{3}\right)\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)\widetilde{A} + (1-\lambda)z_0}{\epsilon} \\
&\ge \quad \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right\}}{\epsilon} \\
&= \quad \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right\}}{d\epsilon} d \\
&\ge \quad k_1 \max_{t\in[0,1]} v(t).
\end{aligned}$$

Thus, $v \in \Omega$. Next,

$$\begin{aligned}
\lambda(I-F)u(t) + (1-\lambda)z_0 \quad &= \quad 2\lambda z_0 - \lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds + z_0 - \lambda z_0 \\
&= \quad -\lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds + (1+\lambda)z_0 \\
&= \quad -\epsilon\frac{\lambda\epsilon\int_0^1 G(t,s)f(s,u(s))ds + (1-\lambda)z_0}{\epsilon} + 2z_0 \\
&= \quad Tv(t), \quad t \in [0,1].
\end{aligned}$$

Therefore

$$\lambda(I-F)(\mathcal{P}(\alpha,a))+(1-\lambda)z_0\subset T(\Omega).$$

Let $\lambda\in[0,1]$ be fixed and $u\in\mathcal{P}(\beta,d)$ be arbitrarily chosen. Take

$$w(t)=\frac{2(1-\lambda)z_0+\lambda\epsilon\int_0^1 G(t,s)f(s,u(s))ds}{\epsilon},\quad t\in[0,1].$$

We have $v(t)\geq 0$, $t\in[0,1]$, and

$$w(t)\leq\frac{\epsilon AB+2z_0}{\epsilon}\leq d,\quad t\in[0,1].$$

Moreover,

$$\begin{aligned}
\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} w(t) &\geq \frac{\lambda\epsilon\int_{\frac{\eta}{3}}^{\frac{\eta}{2}}\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} G(t,s)f(s,u(s))ds+2(1-\lambda)z_0}{\epsilon}\\
&\geq \frac{\lambda\epsilon\left(\frac{\eta}{2}-\frac{\eta}{3}\right)\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)\widetilde{A}+(1-\lambda)z_0}{\epsilon}\\
&\geq \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A},z_0\right\}}{\epsilon}\\
&= \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A},z_0\right\}}{d\epsilon}d\\
&\geq k_1\max_{t\in[0,1]} w(t).
\end{aligned}$$

Therefore $w\in\Omega$. Also,

$$\begin{aligned}
\lambda(I-F)u(t) &= \lambda\left(2z_0-\epsilon\int_0^1 G(t,s)f(s,u(s))ds\right)\\
&= -\epsilon\frac{\epsilon\int_0^1 G(t,s)f(s,u(s))ds+2(1-\lambda)z_0}{\epsilon}+2z_0\\
&= -\epsilon w(t)+2z_0\\
&= Tw(t),\quad t\in[0,1].
\end{aligned}$$

Therefore

$$\lambda(I-F)(\mathcal{P}(\beta,d))\subset T(\Omega).$$

By Theorem 3.1, it follows that the BVP (4.1) has at least one solution in $\{y\in\mathcal{P}:a<\alpha(y)\text{ and }\beta(y)<d\}\cap\Omega\subset P(\alpha,\beta,a,d)\cap\Omega$.

4.1.2. An Example. Consider the BVP

$$\begin{aligned} y'' + \tfrac{1}{300(1+t^2)(1+y)} + \tfrac{1}{300} &= 0, \quad t \in (0,1), \\ y(0) = y\left(\tfrac{1}{2}\right), \quad y(1) &= 0. \end{aligned} \tag{4.4}$$

Here

$$f(t,y) = \frac{1}{300(1+t^2)(1+y)} + \frac{1}{300}, \quad t \in (0,1), \quad y \in [0,\infty), \quad k = 1, \quad \eta = \frac{1}{2}.$$

Note that for the function f we can not apply Theorem 4.1 in [20] because it is a decreasing function with respect to y for $t, y \in [0,\infty)$. Take the constants

$$\epsilon = \tfrac{41}{40}, \quad B = 3, \quad A = \tfrac{1}{123}, \quad \widetilde{A} = \tfrac{1}{300}, \quad b = 1, \quad d = \tfrac{1}{2},$$

$$z_0 = \tfrac{1}{400}, \quad a = \tfrac{1}{400}.$$

We have

$$a < d < b, \quad 2z_0 = 2a = \frac{1}{200} < \frac{41}{80} = \epsilon d,$$

$$(\epsilon - 1)b + 2z_0 = \frac{1}{40} + \frac{1}{200} = \frac{3}{100} < \frac{1}{4} = \frac{d}{2},$$

$$(\epsilon - 1)b + \epsilon AB = \frac{1}{40} + \frac{41}{40} \cdot \frac{3}{123} = \frac{1}{40} + \frac{1}{40} = \frac{1}{20} < \frac{1}{2} = d,$$

$$\frac{1}{400} = a < \frac{\epsilon AB + 2z_0}{\epsilon} = \frac{40}{41} \cdot \left(\frac{41}{40} \cdot \frac{3}{123} + \frac{1}{200}\right) < \frac{1}{2} = d.$$

Thus, $(B2)$ holds. Next, $f \in \mathcal{C}([0,1] \times \mathbb{R}^+)$ and

$$\frac{1}{300} \le f(t,y) = \frac{1}{300(1+t^2)(1+y)} + \frac{1}{300} \le \frac{1}{150} \le \frac{1}{123} = A,$$

i.e., $(B1)$ holds. By Theorem 3.1, it follows that the BVP (4.4) has at least one nonnegative solution.

4.2. A Two-Point BVP

In this subsection, we will investigate the following BVP

$$\begin{aligned} x''(t) + g(x(t)) &= 0, \quad t \in (0,1), \\ x(0) = 0 &= x'(1), \end{aligned} \tag{4.5}$$

where

(C1). $g \in \mathcal{C}(\mathbb{R}^+)$, $0 < \widetilde{A}_1 \le g(x) \le A_1$, $x \in [0,\infty)$, for some positive constants $A_1 \ge \widetilde{A}_1$.

(C2). The nonnegative constants z_1, a_1, b_1, c_1, d_1, ϵ_1 satisfy

$$\epsilon_1 \in (1,2), \quad (\epsilon_1 - 1)b_1 + 2z_1 < \frac{d_1}{2}, \quad (\epsilon_1 - 1)b_1 + \epsilon_1 A_1 < d_1,$$

$$c_1 = 0, \quad 2z_1 < \epsilon_1 \min\{b_1, d_1\}, \quad \frac{z_1}{\epsilon_1} + z_1 > \max\{a_1, c_1\}, \quad z_1 = a_1,$$

$$a_1 < d_1 < b_1, \quad a_1 < \frac{\epsilon_1 A_1 + 2z_1}{\epsilon_1} \le d_1.$$

Our main result in this subsection is as follows.

Theorem 4.2. *Suppose* $(C1)$ *and* $(C2)$. *Then the BVP* (4.5) *has at least one non trivial nonnegative solution.*

The BVP (4.5) is investigated in [2] under the following conditions

(C1.1). $\tau \in (0,1)$ is fixed, b and c are positive constants with $3b \leq c$, $g : [0,\infty) \to [0,\infty)$ is a continuous function such that

1. $g(w) > \frac{c}{\tau(1-\tau)}, \quad w \in \left[c, \frac{c}{\tau}\right]$,
2. g is decreasing on $[a, b\tau]$ with $g(b\tau) \geq g(w)$ for $w \in [b\tau, b]$.
3. $\int_0^\tau sg(s)ds \leq \frac{2b-g(b\tau)(1-\tau^2)}{2}$,

and it is proved that the BVP (4.5) has at least one nonnegative solution. Note that there are cases for the function g for which we can apply Theorem 4.2 and we can not apply Theorem 5.1 in [2] and conversely. For instance, if $g(x) = \frac{x}{1+x} + 1$, $x \in [0,\infty)$, then it is bounded above and below and we can apply Theorem 4.2. On the other hand, g is an increasing function on $[0,\infty)$ and we can not apply Theorem 5.1 in [2]. If $g(x) = \frac{1}{\sqrt{x}} + e^{x-2}$, $x \in (0,\infty)$, as it is shown in [2], we can apply for it Theorem 5.1 in [2]. Since it is unbounded above, we can not apply Theorem 4.2. Therefore our main result Theorem 3.1 and the main result Theorem 4.1 in [2] are complementary.

After the proof of Theorem 4.2, we will give an example for a function g and constants A_1, $\widetilde{A}_1$, z_1, a_1, b_1, c_1, d_1, ϵ_1 that satisfy $(C1)$ and $(C2)$.

4.2.1. Proof of Theorem 4.2. Let $E = \mathcal{C}([0,1])$ be endowed with the maximum norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

Define

$$G_1(t,s) = \min\{t,s\}, \quad (t,s) \in [0,1]\times[0,1].$$

Note that

$$0 \leq G_1(t,s) \leq 1, \quad (t,s) \in [0,1]\times[0,1],$$

and

$$G_1(t,s) \geq \frac{1}{3}, \quad t,s \in \left[\frac{1}{3}, \frac{1}{2}\right].$$

On E, define the following functionals

$$\alpha_1(x) = \min_{t\in[0,1]} |x(t)| + z_1, \quad \beta_1(x) = \max_{t\in[0,1]} |x(t)|.$$

In [2] it is proved that the solution of the BVP (4.5) can be represented in the form

$$x(t) = \int_0^1 G_1(t,s)g(x(s))ds, \quad t \in [0,1].$$

Set

$$k_2 = \frac{\min\left\{\frac{\epsilon_1 \widetilde{A}_1}{3}, z_1\right\}}{d_1\epsilon_1}.$$

Define

$$\begin{aligned}\mathcal{P}_1 &= \{x \in E : x(t) \geq 0, \quad t \in [0,1], \quad \min_{t \in \left[\frac{1}{3}, \frac{1}{2}\right]} x(t) \geq k_2 \max_{t\in[0,1]} x(t)\},\\ \Omega_1 &= \left\{x \in \mathcal{P}_1 : \|x\| \leq \frac{2z_1 + \epsilon_1 A_1}{\epsilon_1}\right\}.\end{aligned}$$

Note that $0 \in \Omega_1$ and $\Omega_1 \subset \mathcal{P}_1$. For $x \in \mathcal{P}_1$, define the following operators.

$$\begin{aligned}T_1 x(t) &= -\epsilon_1 x(t) + 2z_1,\\ F_1 x(t) &= x(t) - 2z_0 + \epsilon_1 \int_0^1 G_1(t,s) g(x(s)) ds, \quad t \in [0,1].\end{aligned}$$

Now, the proof of Theorem 4.2 follows similar arguments to those in the proof of Theorem 4.1.

4.2.2. An Example. Consider the BVP

$$\begin{aligned}x''(t) + \tfrac{x(t)}{400(1+x(t))} + \tfrac{1}{400} &= 0, \quad t \in (0,1),\\ x(0) = 0 &= x'(1).\end{aligned} \tag{4.6}$$

Here

$$g(x) = \frac{x}{400(1+x)} + \frac{1}{400}, \quad x \in [0, \infty).$$

Note that the function g is an increasing function on $[0,\infty)$ and then we can not apply Theorem 5.1 in [2]. Take

$$\begin{aligned}\epsilon_1 &= \frac{41}{40}, \quad A_1 = \frac{1}{123}, \quad \widetilde{A}_1 = \frac{1}{400}, \quad b_1 = 1, \quad d_1 = \frac{1}{2},\\ z_1 &= \frac{1}{400}, \quad a_1 = \frac{1}{400}, \quad c_1 = 0.\end{aligned}$$

Then, $\epsilon_1 > 1$ and

$$(\epsilon_1 - 1)b_1 + 2z_1 = \frac{1}{40} + \frac{1}{200} < \frac{1}{4} = \frac{d_1}{2},$$

$$(\epsilon_1 - 1)b_1 + \epsilon_1 A_1 = \frac{1}{40} + \frac{41}{40} \cdot \frac{1}{123} = \frac{1}{40} + \frac{1}{120} < \frac{1}{2} = d_1,$$

$$\epsilon_1 \min\{b_1, d_1\} = \frac{41}{40} \cdot \frac{1}{2} = \frac{41}{80} > \frac{1}{200} = 2z_1,$$

$$\frac{z_1}{\epsilon_1} + z_1 = \frac{\frac{1}{400}}{\frac{41}{40}} = \frac{1}{410} + \frac{1}{400} > \frac{1}{400} = \max\{a_1, c_1\},$$

$$a_1 < d_1 < b_1,$$

$$\begin{aligned}a_1 = \frac{1}{400} < \frac{\epsilon_1 A_1 + 2z_1}{\epsilon_1} &= \frac{\frac{41}{40} \cdot \frac{1}{123} + \frac{1}{200}}{\frac{41}{40}} = \frac{\frac{1}{120} + \frac{1}{200}}{\frac{41}{40}}\\ &= \frac{\frac{1}{3} + \frac{1}{5}}{41} = \frac{8}{615} < \frac{1}{2} = d_1.\end{aligned}$$

Thus, $(C2)$ holds. Next,

$$\frac{1}{400} \leq g(x) \leq \frac{1}{200}, \quad x \in [0, \infty).$$

So, $(C1)$ holds. Hence, applying Theorem 4.2, we conclude that the BVP (4.6) has at least one nonnegative solution.

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