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New subclasses of bi-univalent functions connected with a q-analogue of convolution based upon the Legendre polynomials

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Abstract. In this paper, we introduce new subclasses of analytic and bi-univalent functions connected with a q-analogue of convolution by using the Legendre polynomials. Furthermore, we find estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses and obtain Fekete-Szegő problem for these subclasses.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ z \in \mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}, \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent functions in $\mathbb{E}$.

If $h \in \mathcal{A}$ is given by

$$h(z) = z + \sum_{k=2}^{\infty} b_k z^k,\ z \in \mathbb{E}, \tag{1.2}$$

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then, the *Hadamard (or convolution) product* of f and h is defined by

$$(f * h)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k, \ z \in \mathbb{E}. \tag{1.3}$$

If f and F are analytic functions in $\mathbb{E}$, we say that f *is subordinate to* F, written $f \prec F$, if there exists a *Schwarz function* w, which is analytic in $\mathbb{E}$, with $w(0) = 0$, and, $|w(z)| < 1$ for all $z \in \mathbb{E}$, such that $f(z) = F(w(z))$, $z \in \mathbb{E}$. Furthermore, if the function F is univalent in $\mathbb{E}$, then we have the following equivalence (see [5] and [17]):

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathbb{E}) \subset F(\mathbb{E}).$$

In [23] Srivastava presented and motivated about brief expository overview of the classical q-analysis versus the so-called (p, q)-analysis with an obviously redundant additional parameter p. We also briefly consider several other families of such extensivelyand widely-investigated linear convolution operators as (for example) the Dziok–Srivastava, Srivastava–Wright and Srivastava–Attiya linear convolution operators, together with their extended and generalized versions. The theory of (p, q)-analysis has important role in many areas of mathematics and physics. Our usages here of the q-calculus and the fractional qcalculus in geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see [1, 14, 15, 21, 22, 26]).

Srivastava [23] made use of various operators of q-calculus and fractional q-calculus and recalling the definition and notations. The q-shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$(\lambda; q)_k = \begin{cases} 1 & k = 0, \\ (1-\lambda)(1-\lambda q) \dots (1-\lambda q^{k-1}) & k \in \mathbb{N}. \end{cases}$$

By using the q-gamma function $\Gamma_q(z)$, we get

$$(q^\lambda; q)_k = \frac{(1-q)^k \ \Gamma_q(\lambda + k)}{\Gamma_q(\lambda)}, \qquad (k \in \mathbb{N}_0),$$

where (see [13])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q; q)_\infty}{(q^z; q)_\infty}, \qquad (|q| < 1).$$

Also, we note that

$$(\lambda; q)_\infty = \prod_{k=0}^{\infty} (1 - \lambda q^k), \qquad (|q| < 1),$$

and, the q-gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z+1) = [z]_q \ \Gamma_q(z),$$

where $[k]_q$ denotes the basic q-number defined as follows

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ 1 + \sum_{j=1}^{k-1} q^j, & k \in \mathbb{N}. \end{cases} \tag{1.4}$$

Using the definition formula (1.4), we have the next two products:

(i) For any non negative integer k, the *q-shifted factorial* is given by

$$[k]_q! := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{n=1}^{k} [n]_q, & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{n=r}^{r+k-1} [n]_q, & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \qquad \text{as } q \to 1^-.$$

Also, we observe that

$$\lim_{q \to 1^-} \left\{ \frac{(q^\lambda; q)_k}{(1-q)^k} \right\} = (\lambda)_k .$$

For $0 < q < 1$, the *q-derivative operator* for $f * h$ is defined by

$$\begin{aligned} D_q(f*h)(z) &= D_q \left[z + \sum_{k=2}^{\infty} a_k b_k z^k \right] \\ &= \frac{(f*h)(z) - (f*h)(qz)}{z(1-q)} \\ &= 1 + \sum_{k=2}^{\infty} [k,q] a_k b_k z^{k-1}, \ z \in \mathbb{E}, \end{aligned}$$

where

$$[k,q] := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j, \qquad [0,q] := 0. \tag{1.5}$$

Using the definition formula (1.5), we will define the next two products:
(i) For any non negative integer k, the *q-shifted factorial* is given by

$$[k,q]! := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{i=1}^{k} [i,q], & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by

$$[r,q]_k := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{i=1}^{k} [r+i-1,q], & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

For $\lambda > -1$ and $0 < q < 1$, El-Deeb et al. [12] defined the linear operator $\mathcal{H}_h^{\lambda,q} : \mathcal{A} \to \mathcal{A}$ as follows

$$\mathcal{H}_h^{\lambda,q} f(z) * \mathcal{M}_{q,\lambda+1}(z) = z\, D_q(f*h)(z), \ z \in \mathbb{E},$$

where the function $\mathcal{M}_{q,\lambda+1}$ is given by

$$\mathcal{M}_{q,\lambda+1}(z) := z + \sum_{k=2}^{\infty} \frac{[\lambda+1,q]_{k-1}}{[k-1,q]!} z^k, \ z \in \mathbb{E}.$$

A simple computation shows that

$$\mathcal{H}_h^{\lambda,q} f(z) := z + \sum_{k=2}^{\infty} \phi_k a_k z^k, \ (\lambda > -1, \ 0 < q < 1, \ z \in \mathbb{E}), \tag{1.6}$$

where

$$\phi_k = \frac{[k,q]!}{[\lambda+1,q]_{k-1}} b_k. \tag{1.7}$$

Remark 1.1. [12] From the definition relation (1.6), we can easily verify that the next relations hold for all $f \in \mathcal{A}$:

(i) $[\lambda+1,q]\mathcal{H}_h^{\lambda,q} f(z) = [\lambda,q]\mathcal{H}_h^{\lambda+1,q} f(z) + q^\lambda \ zD_q\left(\mathcal{H}_h^{\lambda+1,q} f(z)\right), \ z \in \mathbb{E};$

(ii)
$$\lim_{q\to 1^-} \mathcal{H}_h^{\lambda,q} f(z) = \mathcal{H}_h^{\lambda,1} f(z) := \mathcal{I}_h^\lambda f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\lambda+1)_{k-1}} a_k b_k z^k, z \in \mathbb{E}. \tag{1.8}$$

Remark 1.2. By taking special cases b_k in the operator $\mathcal{H}_h^{\lambda,q}$, we obtain

(i) Taking $b_k = \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}$ $(\upsilon > 0)$, we get the operator $\mathcal{N}_{\upsilon,q}^\lambda$ studied by El-Deeb and Bulboaca [8] and El-Deeb [7], as follows:

$$\begin{aligned} \mathcal{N}_{\upsilon,q}^\lambda f(z) &= z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k, \ z \in \mathbb{E}, \\ &= z + \sum_{k=2}^{\infty} \psi_k \ a_k z^k, \quad (\upsilon > 0, \ \lambda > -1, \ 0 < q < 1), \end{aligned} \tag{1.9}$$

where

$$\psi_k = \frac{[k,q]!}{[\lambda+1,q]_{k-1}} \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}; \tag{1.10}$$

(ii) Taking $b_k = \left(\frac{n+1}{n+k}\right)^\delta$ $(\delta > 0, \ n \geq 0)$,we find the operator $\mathcal{N}_{n,1,q}^{\lambda,\delta} = \mathcal{M}_{n,q}^{\lambda,\delta}$ studied by El-Deeb and Bulboaca [9] and Srivastava and El-Deeb [24] as follows:

$$\mathcal{M}_{n,q}^{\lambda,\delta} f(z) := z + \sum_{k=2}^{\infty} \left(\frac{n+1}{n+k}\right)^\delta \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k, \ z \in \mathbb{E}; \tag{1.11}$$

(iii) Taking $b_k = 1$, we have the operator $\mathfrak{J}_q^\lambda$ studied by Arif et al. [2] and Srivastava et al. [27] as follows:

$$\mathfrak{J}_q^\lambda f(z) := z + \sum_{k=2}^{\infty} \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k, \ z \in \mathbb{E}; \tag{1.12}$$

(iv) Taking $b_k = \frac{m^{k-1}}{(k-1)!}e^{-m}$ $(m > 0)$ (see [19]), we get a q-analogue of poission operator $\mathcal{I}_q^{\lambda,m}$ studied by El-Deeb et al. [12] as follows:

$$\mathcal{I}_q^{\lambda,m} f(z) := z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!}e^{-m} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k, \ z \in \mathbb{E}; \tag{1.13}$$

(v) Taking $b_k = \left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m$ $(m \in \mathbb{Z},\ \ell \geq 0,\ \delta \geq 0)$ (see [20]), we get a q-analogue of Prajapat operator $\mathcal{J}_{q,\ell,\delta}^{\lambda,m}$ as follows:

$$\mathcal{J}_{q,\ell,\delta}^{\lambda,m} f(z) := z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k, \ z \in \mathbb{E}; \tag{1.14}$$

(vi) Taking $b_k = \binom{k+m-2}{m-1} \theta^{k-1}(1-\theta)^m$ $(m \geq 1,\ 0 \leq \theta \leq 1)$ (see [10, 11]), we get a q-analogue of Pascal distribution series $\Psi_{q,\theta}^{\lambda,m}$ defined by Srivastava and El-deeb [25] as follows:

$$\Psi_{q,\theta}^{\lambda,m} f(z) := z + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1}\theta^{k-1}(1-\theta)^m \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k, \ z \in \mathbb{E}. \tag{1.15}$$

Definition 1.3. Let $P_k(x)$ be the Legendre polynomials of the first kind of order $k = 0, 1, 2, ...$ for which, the recurrence formula is

$$P_{k+1}(x) = \frac{2k+1}{k+1} x P_k(x) - \frac{k}{k+1} P_{k-1}(x), \tag{1.16}$$

with

$$P_0(x) = 1 \quad \text{and } P_1(x) = x$$

For $|x| < 1$. The generating function for Legendre Polynomials is given by (see [16])

$$G(x,z) = \frac{1}{\sqrt{1-2xz+z^2}} = \sum_{k=0}^{\infty} P_k(x) z^k.$$

The Koebe one quarter theorem (see [6]) proves that the image of $\mathbb{E}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1} satisfied

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{E})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \ldots . \tag{1.17}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{E}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{E}$. Let Σ denote the class of bi-univalent functions in $\mathbb{E}$ given by (1.1). For a brief history and interesting examples in the class Σ (see [3]). Brannan and Taha [4] (see also [28]) introduced certain subclasses of the bi-univalent functions class Σ similar to the familiar subclasses $S^*(\beta)$ and $\mathcal{K}(\beta)$ of starlike and convex functions of order β $(0 \leq \beta < 1)$, respectively (see [3]). Thus, following Brannan and Taha [4] a function $f \in \mathcal{A}$ is said to be in the class $S_{\Sigma}^{*}(\beta)$ of strongly bi-starlike functions of order β $(0 < \beta \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2} \;\; (0 < \beta \leq 1;\; z \in \mathbb{E}) \tag{1.18}$$

and

$$\left|\arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\beta\pi}{2} \;\; (0 < \beta \leq 1;\; w \in \mathbb{E}), \tag{1.19}$$

where h is the extension of f^{-1} to $\mathbb{E}$ is given by (1.17). The classes $S_{\Sigma}^{*}(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$ of bi-starlike functions of order β and bi-convex functions of order β $(0 < \beta \leq 1)$, corresponding to the function classes $S^{*}(\beta)$ and $\mathcal{K}(\beta)$, were also introduced analogously. For each of the function classes $S_{\Sigma}^{*}(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [4] and [28]).

The object of the present paper is to introduce new classes of the function class Σ involving the $q-$analogue of convolution based upon the Legendre polynomials previous defined classes, and find estimates on the coefficients $|a_2|$, and $|a_3|$ for functions in these new subclasses of the function class Σ.

Definition 1.4. Let $\eta \neq 0$ be a complex number and $f(z)$ given by (1.1) and $h(z)$ given by (1.2), then $f(z)$ is said to be in the class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x)$ if the following conditions are satisfied:

$$f \in \Sigma,$$

$$1+\frac{1}{\eta}\left(\frac{\alpha z\; D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)\right)+\alpha D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)}-1\right) \prec G(x,z), \tag{1.20}$$

and

$$1+\frac{1}{\eta}\left(\frac{\alpha w D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)\right)+\alpha D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)}-1\right) \prec G(x,w), \tag{1.21}$$

with $\alpha > 0$, $\lambda > -1;$ $0 < q < 1;$ $\eta \in \mathbb{C}^{*} = \mathbb{C}\backslash\{0\}$, where the function $g = f^{-1}$ is given by (1.17).

Remark 1.5. (i) For $q \to 1^{-}$ we obtain that $\lim\limits_{q\to 1^{-}} \mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x) =: \mathcal{N}_{\Sigma}^{\lambda}(\eta, \alpha, h, x)$, where $\mathcal{N}_{\Sigma}^{\lambda}(\eta, \alpha, h, x)$ represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{I}_h^{\lambda}$ (see (1.8)).
(ii) For $b_k = \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}$ $(\upsilon > 0)$, we obtain the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta, \alpha, \upsilon, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{N}_{\upsilon,q}^{\lambda}$ (see (1.9)).

(iii) For $b_k = \left(\frac{n+1}{n+k}\right)^{\delta}$ $(\delta > 0,\ n \geq 0)$, we obtain the class $\mathcal{I}_{\Sigma}^{q,\lambda}(\eta, \alpha, \delta, n, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{M}_{n,q}^{\lambda,\delta}$ (see (1.11)).

(iv) For $b_k = \frac{m^{k-1}}{(k-1)!}e^{-m}$ $(m > 0)$ we obtain the class $\mathcal{P}_{\Sigma}^{q,\lambda}(\eta, \alpha, m, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{I}_{\lambda,m}^{q}$ (see (1.13)).

(v) For $b_k = \left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m$ $(m \in \mathbb{Z},\ \ell \geq 0,\ \delta \geq 0)$, we obtain the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta, \alpha, m, \ell, \delta, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{J}_{q,\ell,\delta}^{\lambda,m}$ (see (1.14)).

(vi) For $b_k = \binom{k+m-2}{m-1}\theta^{k-1}(1-\theta)^m$ $(m \geq 1,\ 0 < \theta < 1)$, we obtain the class $\Psi_{\Sigma}^{q,\lambda}(\eta, \alpha, m, \theta, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\Psi_{q,\theta}^{\lambda,m}$ (see (1.15)).

The following Lemma will be needed later.

Lemma 1.6. [18, p. 172] *If $w(z) = \sum_{k=1}^{\infty} p_k z^k$ is a Schwarz function for $z \in E$, then*

$$|p_1| \leq 1, \quad |p_k| \leq 1 - |p_1|^2,\ k \geq 1.$$

2. Coefficient bounds for the function class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x)$

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\alpha \geq 0,\ \lambda > -1,\ 0 < q < 1,\ \eta \in \mathbb{C}^*,\ x \in \mathbb{R}$ and h is given by (1.2), the powers are understood as principle values.

Theorem 2.1. *Let the function f given by* (1.1) *belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x)$, then*

$$|a_2| \leq \frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\phi_3 - [\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2\phi_2^2\right|}},$$

and

$$|a_3| \leq \frac{|\eta|\,|x|}{(\alpha(2+q)-1)\,(1+q+q^2)\,\phi_3} + \frac{|\eta|^2 x^2}{(1+q)^2\,(2\alpha-1)^2\,\phi_2^2},$$

where ϕ_k, $k \in \{2, 3\}$, are given by (1.7).

Proof. Since $f \in \mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x)$. Then there exist two analytic functions R and S in $\mathbb{E}$ with $R(0) = S(0) = 0$, and $|R(z)| < 1$, $|S(w)| < 1$ for all $z, w \in \mathbb{E}$ given by

$$R(z) = \sum_{k=1}^{\infty} r_k z^k \qquad \text{and} \qquad S(w) = \sum_{k=1}^{\infty} s_k w^k, \qquad z, w \in \Delta,$$

from Lemma 1.6 we have

$$|r_k| \leq 1 \quad \text{and} \quad |s_k| \leq 1,\ k \in \mathbb{N}. \tag{2.1}$$

In view of (1.20) and (1.21), we get

$$\frac{\alpha z\ D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)\right)+\alpha D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)}-1=\eta\left(G(x,R(z))-1\right), \tag{2.2}$$

and

$$\frac{\alpha wD_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)\right)+\alpha D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)}-1=\eta\left(G(x,S(w))-1\right). \tag{2.3}$$

Since

$$\begin{aligned}&\frac{\alpha z\ D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)\right)+\alpha D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)}-1\\ =\ &(1+q)\left(2\alpha-1\right)\phi_2a_2z\\ +\ &\left[\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3a_3-\left(2\alpha-1\right)(1+q)^2\phi_2^2a_2^2\right]z^2+\ldots,\end{aligned}$$

$$\begin{aligned}&\frac{\alpha wD_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)\right)+wD_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)}-1\\ =\ &-(1+q)\left(2\alpha-1\right)\phi_2a_2w\\ &+\left[\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3\left(2a_2^2-a_3\right)-\left(2\alpha-1\right)(1+q)^2\phi_2^2a_2^2\right]w^2+\ldots,\end{aligned}$$

and

$$\eta\left(G(x,R(z))-1\right)=\eta P_1(x)r_1z+\left(P_1(x)r_2+P_2(x)r_1^2\right)\eta z^2+\ldots,$$
$$\eta\left(G(x,S(w))-1\right)=\eta P_1(x)s_1w+\left(P_1(x)s_2+P_2(x)s_1^2\right)\eta w^2+\ldots.$$

Next, equating the corresponding coefficients of z and w in (2.2) and (2.3), we get

$$(1+q)\left(2\alpha-1\right)\phi_2a_2=\eta P_1(x)r_1, \tag{2.4}$$

$$\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3a_3-\left(2\alpha-1\right)(1+q)^2\phi_2^2a_2^2=\eta P_1(x)r_2+\eta P_2(x)r_1^2 \tag{2.5}$$

$$-\left(1+q\right)\left(2\alpha-1\right)\phi_2a_2=\eta P_1(x)s_1, \tag{2.6}$$

$$(\alpha(2+q)-1)(1+q+q^2)\phi_3(2a_2^2-a_3)-(2\alpha-1)(1+q)^2\phi_2^2a_2^2=\eta P_1(x)s_2+\eta P_2(x)s_1^2. \tag{2.7}$$

From (2.4) and (2.6), we have

$$r_1=-s_1 \tag{2.8}$$

By squaring (2.4) and (2.6), then adding the new relations we get

$$2(1+q)^2\left(2\alpha-1\right)^2a_2^2\phi_2^2=\eta^2P_1^2(x)\left(r_1^2+s_1^2\right). \tag{2.9}$$

If we add (2.5) and (2.7) we obtain

$$2[(\alpha(2+q)-1)(1+q+q^2)\phi_3-(2\alpha-1)(1+q)^2\phi_2^2]a_2^2=\eta P_1(x)(r_2+s_2)+\eta P_2(x)(r_1^2+s_1^2).$$

We can rewrite (2.9) as

$$r_1^2 + s_1^2 = \frac{2(1+q)^2\left(2\alpha-1\right)^2}{\eta^2 P_1^2(x)} a_2^2 \phi_2^2.$$

From above equation, we get

$$2[(\alpha(2+q)-1)(1+q+q^2)\eta P_1^2(x)\phi_3 - [\eta P_1^2(x) + (2\alpha-1)P_2(x)](2\alpha-1)(1+q)^2\phi_2^2]a_2^2$$
$$= \eta^2 P_1^3(x)(r_2+s_2),$$

it follows that

$$a_2^2 = \frac{\eta^2 P_1^3(x)(r_2+s_2)}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta P_1^2(x)\phi_3 - \left(\eta P_1^2(x) + (2\alpha-1)P_2(x)\right)(2\alpha-1)(1+q)^2\phi_2^2\right]}. \tag{2.10}$$

Then taking the absolute value to the above equation and from (1.16) and (2.1), we obtain

$$|a_2| \le \frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\phi_3 - [\eta P_1^2(x) + \frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2\phi_2^2\right|}},$$

which gives the bound for $|a_2|$ as we asserted in our theorem. To find the bound for $|a_3|$. Using (2.5) from (2.7), we have

$$2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3\left(a_3-a_2^2\right) = \eta\left[P_1(x)\left(r_2-s_2\right) + P_2(x)\left(r_1^2-s_1^2\right)\right]. \tag{2.11}$$

Form (2.8), (2.9) and (2.11), we obtain

$$a_3 = \frac{\eta P_1(x)\left(r_2-s_2\right)}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3} + \frac{\eta^2 P_1^2(x)\left(r_1^2+s_1^2\right)}{2(1+q)^2\left(2\alpha-1\right)^2\phi_2^2}. \tag{2.12}$$

Using (1.16) and (2.1), we get

$$|a_3| \le \frac{|\eta|\,|x|}{\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3} + \frac{|\eta|^2 x^2}{(1+q)^2\left(2\alpha-1\right)^2\phi_2^2}.$$ □

In view of Theorem 2.1 we obtain the following results.
Putting $q \to 1^-$ we get the following corollary:

Corollary 2.2. *Let the function f given by* (1.1) *belongs to the class $f \in \mathcal{N}_\Sigma^\lambda(\eta,\alpha,h,x)$, then*

$$|a_2| \le \frac{|\eta|\,|x|\sqrt{x}}{\sqrt{\left|\frac{18(3\alpha-1)\eta x^2 b_3}{(\lambda+1)_2} - 16\left[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)\right]\frac{(2\alpha-1)b_2^2}{(\lambda+1)^2}\right|}},$$

and

$$|a_3| \le \frac{|\eta|\,|x|\left(\lambda+1\right)_2}{18\left(3\alpha-1\right)b_3} + \frac{|\eta|^2\left(x\left(\lambda+1\right)\right)^2}{16\left(2\alpha-1\right)^2 b_2^2}.$$

Considering $b_k = \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}$ $(\upsilon > 0)$, we obtain the following result.

Corollary 2.3. *Let the function f given by (1.1) belongs to the class $f \in \mathcal{B}_{\Sigma}^{q,\lambda}(\eta,\alpha,\upsilon,x)$, then*

$$|a_2| \le \frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\psi_3-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\psi_2^2\right|}},$$

and

$$|a_3| \le \frac{|\eta|\,|x|}{(\alpha(2+q)-1)\,(1+q+q^2)\,\psi_3}+\frac{|\eta|^2\,x^2}{(1+q)^2\,(2\alpha-1)^2\,\psi_2^2}.$$

where ψ_k, $k\in\{2,3\}$, are given by (1.10).

For $b_k=\left(\frac{n+1}{n+k}\right)^{\delta}$ $(\delta>0,\ n\ge 0)$, we obtain the following corollary.

Corollary 2.4. *Let the function f given by (1.1) belongs to the class $f \in \mathcal{I}_{\Sigma}^{q,\lambda}(\eta,\alpha,\delta,n,x)$, then*
$|a_2| \le$

$$\frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\frac{[3,q]!}{[\lambda+1,q]_2}\left(\frac{n+1}{n+3}\right)^{\delta}-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}\left(\frac{n+1}{n+2}\right)^{2\delta}\right|}},$$

and

$$|a_3| \le \frac{|\eta||x|[\lambda+1,q]_2(n+3)^{\delta}}{(\alpha(2+q)-1)(1+q+q^2)[3,q]!(n+1)^{\delta}}+\frac{|\eta|^2(x[\lambda+1,q])^2(n+2)^{2\delta}}{(1+q)^2(2\alpha-1)^2([2,q]!)^2(n+1)^{2\delta}}.$$

If we take $b_k=\frac{m^{k-1}}{(k-1)!}e^{-m}$ $(m>0)$ we get the following special case.

Corollary 2.5. *Let the function f given by (1.1) belongs to the class $f \in \mathcal{P}_{\Sigma}^{q,\lambda}(\eta,\alpha,m,x)$, then*
$|a_2| \le$

$$\frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\frac{[3,q]!}{2[\lambda+1,q]_2}m^2e^{-m}-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}m^2e^{-2m}\right|}},$$

and

$$|a_3| \le \frac{2|\eta||x|[\lambda+1,q]_2}{(\alpha(2+q)-1)(1+q+q^2)[3,q]!m^2e^{-m}}+\frac{|\eta|^2x^2([\lambda+1,q])^2}{(1+q)^2(2\alpha-1)^2([2,q]!)^2m^2e^{-2m}}.$$

Putting $b_k=\left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m$ $(m\in\mathbb{Z},\ \ell\ge 0,\ \delta\ge 0)$ we get the following result.

Corollary 2.6. *Let the function f given by (1.1) belongs to the class $f \in \mathcal{B}_{\Sigma}^{q,\lambda}(\eta,\alpha,m,\ell,\delta,x)$, then*
$|a_2| \le$

$$\frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\frac{[3,q]!}{[\lambda+1,q]_2}[\frac{1+\ell+2\delta}{1+\ell}]^m-[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}[\frac{1+\ell+\delta}{1+\ell}]^{2m}\right|}},$$

and

$$|a_3| \le \frac{|\eta||x|[\lambda+1,q]_2[1+\ell]^m}{(\alpha(2+q)-1)(1+q+q^2)[3,q]![1+\ell+2\delta]^m}+\frac{|\eta|^2x^2([\lambda+1,q])^2[1+\ell]^{2m}}{(1+q)^2(2\alpha-1)^2([2,q]!)^2[1+\ell+\delta]^{2m}}.$$

For $b_k=\binom{k+m-2}{m-1}\theta^{k-1}(1-\theta)^m$ $(m\ge 1,\ 0<\theta<1)$, we obtain the following corollary.

Corollary 2.7. *Let the function f given by (1.1) belongs to the class $f \in \Psi_{\Sigma}^{q,\lambda}(\eta, \alpha, m, \theta, x)$, then*

$$|a_2| \leq \frac{|\eta||x|\sqrt{x}}{\sqrt{A}},$$

where

$$A = \Big|(\alpha(2+q)-1)(1+q+q^2)\eta x^2 \frac{[3,q]!}{2[\lambda+1,q]_2} m(m+1)\theta^2(1-\theta)^m$$
$$-[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2 \frac{([2,q]!)^2}{([\lambda+1,q])^2} m^2\theta^2(1-\theta)^{2m}\Big|$$

and

$$|a_3| \leq \frac{2|\eta||x|[\lambda+1,q]_2}{(\alpha(2+q)-1)(1+q+q^2)[3,q]!m(m+1)\theta^2(1-\theta)^m} + \frac{|\eta|^2x^2([\lambda+1,q])^2}{(1+q)^2(2\alpha-1)^2m^2\theta^2(1-\theta)^{2m}([2,q]!)^2}.$$

3. Fekete-Szegő problem for the function class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta;\alpha,h;x)$

Theorem 3.1. *If the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta,\alpha,h,x)$, and $\eta \in \mathbb{C}^*$, then*

$$\left|a_3 - \mu a_2^2\right| \leq |\eta||x| \left(|K+L| + |K-L|\right), \tag{3.1}$$

where

$$K = \frac{(1-\mu)\eta x^2}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta x^2\phi_3 - \left[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\phi_2^2\right]}, \tag{3.2}$$

and

$$L = \frac{1}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3},$$

where $\mu \in \mathbb{C}$, and ϕ_k, $k \in \{2,3\}$, are given by (1.7).

Proof. If $f \in \mathcal{F}_{\Sigma}^{q,\lambda}(\eta,\alpha,h,x)$. As in the proof of Theorem 2.1, from (2.8) and (2.11), we have

$$a_3 - a_2^2 = \frac{\eta P_1(x)\left(r_2 - s_2\right)}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3}, \tag{3.3}$$

and multiplying (2.10) by $(1-\mu)$ we get

$$(1-\mu)\, a_2^2 = \frac{(1-\mu)\eta^2 P_1^3(x)(r_2+s_2)}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta P_1^2(x)\phi_3 - \left[\eta P_1^2(x) + (2\alpha-1)P_2(x)\right](2\alpha-1)(1+q)^2\phi_2^2\right]}. \tag{3.4}$$

Adding (3.3) and (3.4) leads to

$$a_3 - \mu a_2^2 = \eta h_2\left[(K+L)\, r_2 + (K-L)\, s_2\right], \tag{3.5}$$

where K and L are given by (3.2), and taking the absolute value of (3.5), from (2.1) we obtain the inequality (3.1). The proof is complete. □

Remark 3.2. A simple computation shows that the inequality $|K| \leq L$ is equivalent to

$$|\mu - 1| \leq \left|1 - \frac{\left[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)\right]\left(2\alpha-1\right)(1+q)^2\phi_2^2}{\eta x^2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3}\right|,$$

therefore, from Theorem 3.1 we get the next result. If the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta;\alpha,h;x)$, and $\eta\in\mathbb{C}^*$, then

$$\left|a_3-\mu a_2^2\right|\leq\frac{\eta x}{\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3},$$

where $\mu\in\mathbb{C}$, with

$$|\mu-1|\leq\left|1-\frac{\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\phi_2^2}{\eta x^2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3}\right|,$$

and ϕ_k, $k\in\{2,3\}$, are given by (1.7).

We conclude our result with the following consequence of Theorem 3.1. Putting $q\to1^-$, we obtain the following corollary.

Corollary 3.3. *If the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta;\alpha,h;x)$, and $\mu\in\mathbb{C}$, $\eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\leq|\eta||x|\left(|K+L|+|K-L|\right),$$

where

$$K=\frac{(1-\mu)\,\eta x^2}{\frac{36(3\alpha-1)\eta x^2b_3}{(\lambda+1)_2}-32\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right]\frac{(2\alpha-1)b_2^2}{(\lambda+1)^2}},$$

and

$$L=\frac{\eta x\,(\lambda+1)_2}{36\,(3\alpha-1)\,b_3}.$$

If we put $b_k=\frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}$ $(\upsilon>0)$, we obtain the following result.

Corollary 3.4. *If the function f given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta,\alpha,\upsilon,x)$, and $\eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\leq|\eta||x|\left(|K+L|+|K-L|\right),$$

where

$$K=\frac{(1-\mu)\eta x^2}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta x^2\psi_3-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\psi_2^2\right]},$$

and

$$L=\frac{1}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\psi_3},$$

where $\mu\in\mathbb{C}$, and ψ_k, $k\in\{2,3\}$, are given by (1.10).

Considering $b_k=\left(\frac{n+1}{n+k}\right)^{\delta}$ $(\delta>0,\ n\geq0)$, we get the following corollary.

Corollary 3.5. *If the function f given by (1.1) belongs to the class $\mathcal{I}_{\Sigma}^{q,\lambda}(\eta,\alpha,\delta,n,x)$, and $\mu\in\mathbb{C}$, $\eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\leq|\eta||x|\left(|K+L|+|K-L|\right),$$

where

$$K=\frac{(1-\mu)\eta x^2}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta x^2\frac{[3,q]!}{[\lambda+1,q]_2}\left(\frac{n+1}{n+3}\right)^{\delta}-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}\left(\frac{n+1}{n+2}\right)^{2\delta}\right]},$$

and

$$L=\frac{[\lambda+1,q]_2\,(n+3)^{\delta}}{2\,(\alpha(2+q)-1)\,(1+q+q^2)\,[3,q]!\,(n+1)^{\delta}}.$$

If we take $b_k=\frac{m^{k-1}}{(k-1)!}e^{-m}$ $(m>0)$, we get the following case.

Corollary 3.6. *If the function f given by (1.1) belongs to the class $\mathcal{P}_{\Sigma}^{q,\lambda}(\eta,\alpha,m,x)$, and $\mu\in\mathbb{C},\ \eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\le|\eta||x|\left(|K+L|+|K-L|\right),$$

where

$$K=\frac{(1-\mu)\eta x^2}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta x^2\frac{[3,q]!}{2[\lambda+1,q]_2}m^2e^{-m}-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}m^2e^{-2m}\right]},$$

and

$$L=\frac{[\lambda+1,q]_2}{(\alpha(2+q)-1)\,(1+q+q^2)\,[3,q]!m^2e^{-m}}.$$

Putting $b_k=\left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m$ $(m\in\mathbb{Z},\ \ell\ge 0,\ \delta\ge 0)$, we obtain the following result.

Corollary 3.7. *If the function f given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta,\alpha,m,\ell,\delta,x)$, and $\mu\in\mathbb{C},\ \eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\le|\eta||x|\left(|K+L|+|K-L|\right),$$

where

$$K=\frac{(1-\mu)\,\eta x^2}{B},$$

where

$$B=2\left[(\alpha(2+q)-1)\left(1+q+q^2\right)\eta x^2\frac{[3,q]!}{[\lambda+1,q]_2}\left[\frac{1+\ell+2\delta}{1+\ell}\right]^m\right.$$
$$\left.-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)\,(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}\left[\frac{1+\ell+\delta}{1+\ell}\right]^{2m}\right]$$

and

$$L=\frac{[\lambda+1,q]_2\,[1+\ell]^m}{2\,(\alpha(2+q)-1)\,(1+q+q^2)\,[3,q]!\,[1+\ell+2\delta]^m}.$$

For $b_k=\binom{k+m-2}{m-1}\theta^{k-1}(1-\theta)^m$ $(m\ge 1,\ 0<\theta<1)$, we get the following special case.

Corollary 3.8. *If the function f given by (1.1) belongs to the class $\Psi_{\Sigma}^{q,\lambda}(\eta,\alpha,m,\theta,x)$, and $\mu\in\mathbb{C},\ \eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\le|\eta||x|\left(|K+L|+|K-L|\right),$$

where

$$K=\frac{(1-\mu)\eta x^2}{C},$$

where

$$C = 2[(\alpha(2+q)-1)(1+q+q^2)\eta x^2 \frac{[3,q]!}{2[\lambda+1,q]_2} m(m+1)\theta^2(1-\theta)^m$$
$$- [\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2 \frac{([2,q]!)^2}{([\lambda+1,q])^2} m^2\theta^2(1-\theta)^{2m}]$$

and

$$L = \frac{[\lambda+1,q]_2}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)[3,q]!m(m+1)\theta^2\left(1-\theta\right)^m},$$

Now, the following examples are presented here to illustrate our results. For $\eta = 1$ and $\alpha = 1$. Therefore, from Theorem 2.1 and Theorem 3.1.

Example 3.9. Let the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(1;1,h;x)$, then

$$|a_2| \leq \frac{|x|\sqrt{x}}{\sqrt{\left|(1+q)\left(1+q+q^2\right)x^2\phi_3 - \frac{1}{2}(5x^2-1)(1+q)^2\phi_2^2\right|}},$$

$$|a_3| \leq \frac{|x|}{(1+q)\left(1+q+q^2\right)\phi_3} + \frac{x^2}{(1+q)^2\phi_2^2},$$

and

$$\left|a_3 - \mu a_2^2\right| \leq |x|\left(|K+L| + |K-L|\right),$$

with

$$K = \frac{(1-\mu)\,x^3}{2\left[(1+q)\left(1+q+q^2\right)x^2\phi_3 - \frac{1}{2}(5x^2-1)(1+q)^2\phi_2^2\right]},$$

and

$$L = \frac{x}{2\left(1+q\right)\left(1+q+q^2\right)\phi_3},$$

where $\mu \in \mathbb{C}$ and ϕ_k, $k \in \{2,3\}$, are given by (1.7).

For $\eta = 1$ and $\alpha = 0$. Therefore, from Theorem 2.1 and Theorem 3.1.

Example 3.10. Let the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(1;0,h;x)$, then

$$|a_2| \leq \frac{|x|\sqrt{x}}{\sqrt{\left|\left[-\left(1+q+q^2\right)x^2\phi_3 + \frac{1}{2}(1-x^2)(1+q)^2\phi_2^2\right]\right|}},$$

$$|a_3| \leq -\frac{|x|}{\left(1+q+q^2\right)\phi_3} + \frac{x^2}{(1+q)^2\phi_2^2},$$

and

$$\left|a_3 - \mu a_2^2\right| \leq |x|\left(|K+L| + |K-L|\right),$$

with

$$K = \frac{(1-\mu)\,x^3}{2\left[-\left(1+q+q^2\right)x^2\phi_3 + \frac{1}{2}(1-x^2)(1+q)^2\phi_2^2\right]},$$

and

$$L = -\frac{x}{2\left(1+q+q^2\right)\phi_3},$$

where $\mu \in \mathbb{C}$ and ϕ_k, $k \in \{2,3\}$, are given by (1.7).

Remark 3.11. We mention that all the above estimations for the coefficients $|a_2|$, $|a_3|$, and Fekete-Szegő problem for the function class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta;\alpha,h;x)$ are not sharp. To find the sharp upper bounds for the above functionals remains an interesting open problem, as well as those for $|a_n|$, $n \geq 4$.

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