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The sum theorem for maximal monotone operators in reflexive Banach spaces revisited

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Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th anniversary.

Abstract. The goal of this note is to present a new shorter proof for the maximal monotonicity of the Minkowski sum of two maximal monotone multi-valued operators defined in a reflexive Banach space under the classical interiority condition involving their domains.

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1. Preliminaries

Recall the following sum rule for maximal monotone operators:

Theorem 1.1. *(Rockafellar* [5, Theorem 1 (a), p. 76]*) Let* $(X, \|\cdot\|)$ *be a reflexive Banach space with topological dual* X^* *and let* $A,\ B : X \rightrightarrows X^*$ *be multi-valued maximal monotone operators from* X *to* X^**. If* $D(A) \cap \operatorname{int} D(B) \neq \emptyset$ *then* $A + B$ *is maximal monotone.*

Here $D(T) := \{x \in X \mid T(x) \neq \emptyset\}$ is the *domain* of $T : X \rightrightarrows X^*$, "$\operatorname{int} S$" denotes the topological interior of $S \subset X$, and $A + B : X \rightrightarrows X^*$ is the *Minkowski sum* of A and B defined by

$$(A + B)(x) := A(x) + B(x) := \{y + v \mid y \in A(x), v \in B(x)\},$$

for $x \in D(A + B) := D(A) \cap D(B)$.

The proof of [5, Theorem 1, p. 76] relies on the use of the duality mapping J of X and the (Minty's style) characterization of maximal monotone operators defined in reflexive Banach spaces. Similar arguments are used in the presence of an improved qualification constraint in a second proof of Theorem 1.1 (see [2, Corollary 3.5, p. 286]). A third proof of the main theorem involves the exact convolution of some specially constructed functions based on the Fitzpatrick functions of A and B (see [10, Corollary

4, p. 1166]). A different proof of Theorem 1.1 is based on the dual-representability $A+B$ in the presence of the qualification constraint (see [8, Remark 1, p. 276]) and the fact that in a reflexive Banach space dual-representability is equivalent to maximal monotonicity (see e.g. [1, Theorem 3.1, p. 2381]). All the previously mentioned proofs make use of the duality mapping J which is characteristic to a normed space.

Our proof relies on the normal cone, is based on full-range characterizations of maximal monotone operators with bounded domain, and uses the representability of sums of representable operators, but, avoids the use of J or the norm. The following intermediary result, is the main ingredient of our argument.

Theorem 1.2. *Let X be a reflexive Banach space, let $T : X \rightrightarrows X^*$ be maximal monotone, and let $C \subset X$ be closed convex and bounded. If $D(T) \cap \operatorname{int} C \neq \emptyset$ then $T + N_C$ is maximal monotone.*

Here N_C denotes the normal cone to C and is defined by $x^* \in N_C(x)$ if, for every $y \in C$, $\langle y - x, x^* \rangle \leq 0$. Here $\langle \cdot, \cdot \rangle$ denotes the *coupling* or *duality product* of $X \times X^*$ and is defined by

$$c(x, x^*) := \langle x, x^* \rangle := x^*(x), \ x \in X, \ x^* \in X^*.$$

An element $z = (x, x^*) \in X \times X^*$ is *monotonically related* (m.r. for short) *to* T if, for every $(a, a^*) \in \operatorname{Graph} T := \{(u, u^*) \in X \times X^* \mid u \in D(T), \ u^* \in T(u)\}$, $\langle x - a, x^* - a^* \rangle \geq 0$.

Recall that a multi-valued operator $T : X \rightrightarrows X^*$ is

• *monotone* if, for every $x_1^* \in T(x_1)$, $x_2^* \in T(x_2)$, $\langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0$.

• *maximal monotone* if every m.r. to T element $z = (x, x^*) \in X \times X^*$ belongs to $\operatorname{Graph} T$.

• *representable* if there is a proper convex $s_X \times w^*-$lower semicontinuous $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that $h \geq c$ and

$$\operatorname{Graph} T = [h = c] := \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}.$$

Here s_X denotes the strong topology of X and w^* stands for the weak-star topology of X^*.

• *NI* if $\varphi_T \geq c$, where φ_T is the Fitzpatrick function of T which is defined by

$$\varphi_T(x, x^*) := \sup\{\langle x - a, a^* \rangle + \langle a, x^* \rangle \mid (a, a^*) \in \operatorname{Graph} T\}, \ (x, x^*) \in X \times X^*. \quad (1.1)$$

2. Proofs of the main result

Proof of Theorem 1.2. The operator $T + N_C$ is representable, which follows from the facts that T, N_C are maximal monotone thus representable and $D(T) \cap \operatorname{int} C \neq \emptyset$ (see e.g. [6, Corollary 5.6, p. 470] or [7, Theorem 16, p. 818]).

We prove that $R(T + N_C) = X^*$ which implies that $T + N_C$ is of NI–type and so it is maximal monotone (see [6, Theorem 3.4, p. 465] or [8, Theorem 1 (ii), (7)]).

It suffices to prove that $0 \in R(T + N_C)$ otherwise we replace T by $T - x^*$ for an arbitrary $x^* \in X^*$.

Consider $F(x,x^*) := \varphi_T(x,x^*) + g(x,x^*)$, with $g(x,x^*) := \iota_C(x) + \sigma_C(-x^*)$, where $\iota_C(x) = 0$, for $x \in C$; $\iota_C(x) = +\infty$, otherwise, and $\sigma_C(x^*) := \sup_{x\in C}\langle x, x^*\rangle$, $x^* \in X^*$.

Then $F \geq 0$ due to $\varphi_T(x,x^*) \geq \langle x, x^*\rangle$ and $\iota_C(x) + \sigma_C(-x^*) \geq -\langle x, x^*\rangle$ (see f.i. [4]). Hence

$$0 \leq \inf_{X\times X^*} F = -(\varphi_T + g)^*(0,0) = -\min_{(x,x^*)\in X\times X^*}\{\psi_T(x,x^*) + g^*(-x^*,-x)\}, \quad (2.1)$$

because C is bounded, g is $s_X \times s_{X^*}-$continuous on $\operatorname{int} C \times X^*$, and X is reflexive (see f.i. [9, Theorem 2.8.7, p. 126]), where s_{X^*} is the strong topology of X^*. Here "min" denotes an infimum that is attained when finite,

$$\psi_T(x,x^*) = \varphi_T^*(x^*,x),\ (x,x^*) \in X \times X^*, \quad (2.2)$$

the convex conjugation being taken with respect to the dual system

$$(X \times X^*, X^* \times X^{**})$$

and, for every $(x,x^*) \in X \times X^*$, $\psi_T(x,x^*) \geq \langle x, x^*\rangle$ because T is monotone (see e.g. [8, (12)]).

From $g^*(x^*,x) = \iota_C(-x) + \sigma_C(x^*)$, $(x,x^*) \in X \times X^*$ and (2.1) there exists $(\bar{x},\bar{x}^*) \in X \times X^*$ such that $\psi_T(\bar{x},\bar{x}^*) + \iota_C(\bar{x}) + \sigma_C(-\bar{x}^*) \leq 0$ which implies that $\iota_C(\bar{x}) + \sigma_C(-\bar{x}^*) = -\langle \bar{x}, \bar{x}^*\rangle$, i.e., $-\bar{x}^* \in N_C(\bar{x})$ and $\psi_T(\bar{x},\bar{x}^*) = \langle \bar{x}, \bar{x}^*\rangle$, that is, $\bar{x}^* \in T(\bar{x})$ since T is representable (see [8, Theorem 1, p. 270]).

Therefore $0 \in (T + N_C)(\bar{x},\bar{x}^*)$ and so $0 \in R(T + N_C)$. □

Proof of Theorem 1.1. First we prove that we can assume without loss of generality that $D(B)$ is bounded. Indeed, assume that the result is true for that case.

Let $z = (x,x^*)$ be m.r. to $A + B$. Take $C \subset X$ closed convex and bounded with $x \in \operatorname{int} C$ and $D(A) \cap \operatorname{int} D(B) \cap \operatorname{int} C \neq \emptyset$ e.g. $C := [x_0, x] + U$, where

$$[x_0, x] := \{tx_0 + (1-t)x \mid 0 \leq t \leq 1\}$$

and U is a closed convex bounded neighborhood of 0, and $x_0 \in D(A) \cap \operatorname{int} D(B)$. Note that z is m.r. to $A + B + N_C = A + (B + N_C)$ which is maximal monotone since, according to Theorem 1.2, $B + N_C$ is maximal monotone, $D(B + N_C)$ is bounded, and $x_0 \in D(A) \cap \operatorname{int} D(B + N_C) \neq \emptyset$. Hence $z \in \operatorname{Graph}(A + B + N_C)$ or $x^* \in (A + B)(x)$ because $N_C(x) = \{0\}$. Therefore $A + B$ is maximal monotone.

It remains to prove that, whenever $D(B)$ is bounded, $R(A + B) = X^*$ or sufficiently $0 \in R(A + B)$ (since $A + B$ is representable, see again [6, Corollary 5.6]).

Let

$$F(x,x^*) := \varphi_A(x,x^*) + \varphi_B(x,-x^*), \quad g(x,x^*) := \varphi_B(x,-x^*), \quad (x,x^*) \in X \times X^*.$$

Since A, B are maximal monotone, for every $(x,x^*) \in X \times X^*$,

$$\min\{\varphi_A(x,x^*), \varphi_B(x,x^*)\} \geq \langle x, x^*\rangle$$

which imply $F \geq 0$ and so

$$0 \leq \inf_{X\times X^*} F = -(\varphi_A + g)^*(0,0) = -\min_{(x,x^*)\in X\times X^*}\{\psi_A(x,x) + \psi_B(x,-x^*)\}, \quad (2.3)$$

because $D(B)$ bounded provides $D(B) \times X^* \subset \operatorname{dom} g$, g is $s_X \times s_{X^*}-$continuous on $\operatorname{int} D(B) \times X^*$, and X is reflexive (see again [9, Theorem 2.8.7, p. 126]). More precisely, for every $(x, x^*) \in D(B) \times X^*$ there is $\overline{x}^* \in B(x)$ and so

$$\begin{aligned}\varphi_B(x, x^*) &:= \sup\{\langle x - b, b^*\rangle + \langle b, x^*\rangle \mid (b, b^*) \in \operatorname{Graph} B\}\\ &\leq \sup\{\langle x - b, \overline{x}^*\rangle + \langle b, x^*\rangle \mid (b, b^*) \in \operatorname{Graph} B\}\\ &\leq \langle x, \overline{x}^*\rangle + \|x^* - \overline{x}^*\| \sup_{b \in D(B)} \|b\| < +\infty.\end{aligned}$$

There exists $(\bar{x}, \bar{x}^*) \in X \times X^*$ such that $\psi_A(\bar{x}, \bar{x}^*) + \psi_B(\bar{x}, -\bar{x}^*) \leq 0$ which implies that $\psi_A(\bar{x}, \bar{x}^*) = \langle \bar{x}, \bar{x}^*\rangle$, $\psi_B(\bar{x}, -\bar{x}^*) = -\langle \bar{x}, \bar{x}^*\rangle$, i.e., $\bar{x}^* \in A(\bar{x})$ and $-\bar{x}^* \in B(\bar{x})$ from which $0 \in R(A + B)$. □

Remark 2.1. Theorem 1.2 still holds if we replace the assumption C bounded with $D(T)$ bounded. In this case an alternate proof of Theorem 1.1 can be performed with $A + N_C$ instead of A and a similar argument as in the current proof.

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