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New integral inequalities involving generalized Riemann-Liouville fractional operators

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Abstract. In this paper, using a generalized operator of the Riemann-Liouville type, defined and studied in a previous work, several integral inequalities for synchronous functions are established.

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1. Introduction

One of the most developed mathematical areas in the last 20 years is that of Integral Inequalities, associated with different functional notions: convex, synchronous functions among other, within the framework of Riemann, fractional and generalized integral operators. A detail that we want to point out is the fact of the appearance in recent years of various integral operators, natural extensions of the fractional integral of Riemann-Liouville, this together with the attention received by Integral Inequalities, make more and more researchers and research is devoted to this topic. To get a more complete idea in this regard, we recommend consulting the works [1, 2, 6, 8, 10, 11, 12, 13, 16] and the references cited therein.

In this direction, one of the most fruitful notions is the following (see [4]).

Definition 1.1. If χ and ψ are two integrable functions on $[a, b]$, they are synchronous on $[a_1, a_2]$ if $(\chi(x) - \chi(y))(\psi(x) - \psi(y)) \geq 0$, for any $x, y \in [a_1, a_2]$.

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Various integral inequalities have been obtained using this notion, within the framework of different integral operators (see [4], [7], [9], [15], [18], [20], and the references cited there).

Throughout the work we use the functions Γ (see [17, 19, 22, 23]) and Γ_k (cf. defined by [5]):

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau}\, d\tau, \quad \Re(z) > 0, \tag{1.1}$$

$$\Gamma_k(z) = \int_0^\infty \tau^{z-1} e^{-\tau^k/k}\, d\tau, k > 0. \tag{1.2}$$

It is clear that if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$. As well, we define the k-beta function as follows

$$B_k(u,v) = \frac{1}{k}\int_0^1 \tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1} d\tau,$$

notice that $B_k(u,v) = \frac{1}{k}B(\frac{u}{k},\frac{v}{k})$ and $B_k(u,v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$.

In [7] the following fractional integral operator of the Riemann-Liouville type is defined.

Definition 1.2. The k-generalized fractional Riemann-Liouville integral of order α with $\alpha \in \mathbb{R}$, and $s \neq -1$ of an integrable function $\chi(u)$ on $[0,\infty)$, are given as follows:

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\chi(u) = \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{u} \frac{F(\tau,s)\chi(\tau)d\tau}{[\mathbb{F}(u,\tau)]^{1-\frac{\alpha}{k}}}, \tag{1.3}$$

with $F(\tau,0) = 1$ and $\mathbb{F}(u,\tau) = \int_\tau^u F(\theta,s)d\theta$.

Remark 1.3. In the aforementioned paper, the main properties of this operator (boundedness, conmutatity, etc.) and various inequalities associated with it were studied.

Remark 1.4. If in Definition 1.2 we consider the kernel $F(t,s) = 1$ and $k = 1$, we obtain the classic fractional Integral Riemann-Liouville, used in the work [4]; in the case of the same kernel but $k \neq 1$ then the k-fractional integral of the Riemann-Liouville type of [14] is obtained (see also [15, 18, 20]). If, on the contrary, we consider the kernel $F(t,s) = t^s$, we obtain the (k;s)-Riemann-Liouville fractional integral of [21]. In the case of taking the kernel as $F(t,s) = h'(t)$, we obtain the (k;h)-Riemann-Liouville integral fractional used in [9]. It is clear then, that the results obtained in our work, generalize those of the works mentioned before.

The main purpose of this paper, using the generalized fractional integral operator of the Riemann-Liouville type of Definition 1.2, is to establish several integral inequalities, which contain as particular cases, several of those reported in the literature.

2. Main results

Our first fundamental result is the following.

Theorem 2.1. *Let φ,ψ be two synchronous functions on $[0,\infty)$ and let $f,\phi,\omega\geq 0$, then for all $\tau>a_1\geq 0$, $\alpha>0$, and $s\neq -1$, we have*

$$\begin{aligned}
&2\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi\psi)(\tau)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau)\Big]\\
&\qquad +2\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau)\\
&\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\psi)(\tau)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau)\Big]\\
&+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\psi)(\tau)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\Big]\\
&+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\phi)(\tau)\Big].
\end{aligned}\tag{2.1}$$

To prove the previous Theorem, we need the following lemma.

Lemma 2.2. *Let φ,ψ be two synchronous functions on $[0,\infty)$ and let $h,g\geq 0$, then for all $\tau>a_1\geq 0$, $\alpha>0$, and $s\neq -1$, we have the following inequality*

$$\begin{aligned}
{}^sJ^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\varphi\psi)(\tau)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau)\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\psi)(\tau)\\
+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau).
\end{aligned}\tag{2.2}$$

Proof. Since φ,ψ are synchronous functions on $[0,\infty)$, then for all $u,v\geq 0$, we have

$$\varphi(u)\psi(u)+\varphi(v)\psi(v)\geq\varphi(u)\psi(v)+\varphi(v)\psi(u).\tag{2.3}$$

Thus, if we multiple both sides of (2.3) by $\frac{F(u,s)h(u)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, and then we integrate the resulting inequality with respect to u over (a_1,τ), it holds that

$${}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau)+\varphi(v)\psi(v)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\geq\psi(v)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau)+\varphi(v)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau).\tag{2.4}$$

Now, multiplying both sides of (2.4) by $\frac{F(v,s)g(v)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,v)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to v over (a_1,τ), we get

$$\begin{aligned}
{}^sJ^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\varphi\psi)(\tau)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau)\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\psi)(\tau)\\
+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau).
\end{aligned}$$

Thus, we conclude the result. □

Remark 2.3. If we consider the kernel $F(t,s)=1$ and $k=1$ of this result, we obtain the Lemma 3 of [4], in the case that $F(t,s)=h'(t)$, this result covers Lemma 1 of [9].

Let us now prove the Theorem 2.1.

Proof. By setting $h = f$ and $g = \phi$ in (2.2), then we multiple the resulting inequality by ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)$, we get

$$\begin{aligned}&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau)\right]\\ &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau)\right]. \quad (2.5)\end{aligned}$$

By setting $h = \omega$ and $g = \phi$ in (2.2), and multiplying the result by ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)$, we get

$$\begin{aligned}&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau)\right]\\ &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\right]. \quad (2.6)\end{aligned}$$

Now, using the same idea, we put $h = \omega$ and $g = f$, and then we multiple the result by ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)$, we find

$$\begin{aligned}&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau)\right]\\ &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\right]. \quad (2.7)\end{aligned}$$

Finally, by adding the inequalities (2.5), (2.6) and (2.7) we obtain the result. □

Remark 2.4. If we take the kernel $F(t, s) = 1$ and $k = 1$ we obtain Theorem 2 of [4], and if $F(t, s) = h'(t)$, this result reduces to Theorem 3 of [9].

The next Lemma will be from very useful for proving the last theorem.

Lemma 2.5. *Let φ, ψ be two synchronous functions on $[0, \infty)$ and let $h, g \geq 0$, then for all $\tau > a_1 \geq 0$, $\alpha, \beta > 0$, and $s \neq -1$, we have the following inequality*

$$\begin{aligned}{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\varphi\psi)(\tau) + {}^{s}J^{\frac{\beta}{k}}_{F,a_1}g(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau) &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau)\ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\psi)(\tau)\\ &+ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau). \quad (2.8)\end{aligned}$$

Proof. Multiplying both sides of (2.4) by $\frac{F(v,s)g(v)}{k\Gamma_k(\beta)[\mathbb{F}(\tau,v)]^{1-\frac{\beta}{k}}}$, then we integrate the resulting inequality with respect to v over (a_1, τ), we conclude that

$$\begin{aligned}{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\varphi\psi)(\tau) + {}^{s}J^{\frac{\beta}{k}}_{F,a_1}g(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau) &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau)\ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\psi)(\tau)\\ &+ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau).\end{aligned}$$

Thus, the Lemma is proved. □

Remark 2.6. If we take the kernel $F(t, s) = 1$ and $k = 1$ we obtain Lemma 6 of [4], and if $F(t, s) = h'(t)$, this result reduces to Lemma 2 of [9].

Theorem 2.7. *Let φ, ψ be two synchronous functions on $[0, \infty)$ and let $f, \phi, \omega \geq 0$, then for all $\tau > a_1 \geq 0$, $\alpha, \beta > 0$, and $s \neq -1$, we get*

$$\begin{aligned}
&{}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\varphi\psi)(\tau) + 2\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) \\
&\qquad + {}^sJ^{\frac{\beta}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau)\Big] \\
&+ \Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}\phi(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big]\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau) \\
&\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau)\Big] \\
&+ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\Big] \\
&+ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\phi)(\tau)\Big].
\end{aligned} \tag{2.9}$$

Proof. Substituting in (2.8): $h = f$ and $g = \phi$,then we multiple the resulting inequality by ${}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)$, we have

$$\begin{aligned}
&{}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau)\Big] \\
&\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau)\Big].
\end{aligned} \tag{2.10}$$

Replacing again in (2.8): $h = \omega$ and $g = \phi$, and multiplying the result by ${}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)$, we get

$$\begin{aligned}
&{}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau)\Big] \\
&\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\Big].
\end{aligned} \tag{2.11}$$

Now, by using the same idea, we put $h = \omega$ and $g = f$, and then we multiply the result by ${}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)$, we conclude that

$$\begin{aligned}
&{}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\varphi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau)\Big] \\
&\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\Big].
\end{aligned} \tag{2.12}$$

Finally, by adding the inequalities (2.10), (2.11) and (2.12) we obtain (2.9). □

Remark 2.8. In the case of the classical Riemann-Liouville Integral, this result extends the Theorem 4 of [4], and if we use the (k; h) -Riemann-Liouville integral fractional, we obtain the Theorem 4 of [9].

Remark 2.9. Remark 2 and Remark 3 of [9], and Remark 7 of [4] are true for the general kernel used in our work.

3. Conclusions

In our work we obtained several generalized integral inequalities, which contain, as a particular case, some of those known in the literature, for example, if in Theorem 2.7 we consider $\alpha = \beta = 1$, $k = 1$ and $F(t, s) = 1$, we obtain the well-known Chebishev Inequality (see [3]). In the same direction, we can add that one of the strengths of our results lies in the fact that by suitably choosing of F, i.e., if we consider kernels different than those indicated in the various remarks, one can further easily obtain additional integral inequalities involving the various types of fractional integral operators from our main results.

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